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Volume 5: Selected Topics

C. Constantinescu

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C*-Algebras

Volume 5: Selected Topics

Corneliu Constantinescu
Departement Mathematik, ETH Zürich
CH-8092 Zürich
Switzerland

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Preface

Functional analysis plays an important role in the program of studies at the Swiss Federal Institute of Technology. At present, courses entitled Functional Analysis I and II are taken during the fifth and sixth semester, respectively. I have taught these courses several times and after a while typewritten lecture notes resulted that were distributed to the students. During the academic year 1987/88, I was fortunate enough to have an eager enthusiastic group of students that I had already encountered previously in other lecture courses. These students wanted to learn more in the area and asked me to design a continuation of the courses. Accordingly, I proceeded during the academic year, following, with a series of special lectures, Functional Analysis III and IV, for which I again distributed typewritten lecture notes. At the end I found that there had accumulated a mass of textual material, and I asked myself if I should not publish it in the form of a book. Unfortunately, I realized that the two special lecture series (they had been given only once) had been badly organized and contained material that should have been included in the first two portions. And so I came to the conclusion that I should write everything anew – and if at all – then preferably in English. Little did I realize what I was letting myself in for! The number of pages grew almost imperceptibly and at the end it had more than doubled. Also, the English language turned out to be a stumbling block for me; I would like to take this opportunity to thank Prof. Imre Bokor and Prof. Edgar Reich for their help in this regard. Above all I must thank Mrs. Barbara Aquilino, who wrote, first a WordMARC™, and then a LaTeX™ version with great competence, angelic patience, and utter devotion, in spite of illness. My thanks also go to the Swiss Federal Institute of Technology that generously provided the infrastructure for this extensive enterprise and to my colleagues who showed their understanding for it.

Corneliu Constantinescu

Table of Contents of Volume 5

Introduction xix

6 Selected Chapters of C^*–Algebras 3
6.1 $\mathcal{L}^p$–Spaces 3
6.1.1 Characteristic Families of Eigenvalues 3
6.1.2 Characteristic Sequences 10
6.1.3 Properties of the $\mathcal{L}^p$–spaces 21
6.1.4 Hilbert–Schmidt Operators 46
6.1.5 The Trace 56
6.1.6 Duals of $\mathcal{L}^p$–spaces 72
6.1.7 Exterior Multiplication and $\mathcal{L}^p$–Spaces 79
6.1.8 The Canonical Projection of the Tridual of $\mathcal{K}$ 102
6.1.9 Integral Operators on Hilbert Spaces 116
6.2 Selfadjoint Linear Differential Equations 124
6.2.1 Selfadjoint Boundary Value Problems 125
6.2.2 The Regular Sturm–Liouville Theory 139
6.2.3 Selfadjoint Linear Differential Equations on $\mathbf{T}$ 150
6.2.4 Associated Parabolic and Hyperbolic Evolution Equations 153
6.2.5 Selfadjoint Linear Partial Differential Equations 184
6.2.6 Associated Parabolic and Hyperbolic Evolution Equations 192
6.3 Von Neumann Algebras 202
6.3.1 The Strong Topology 203
6.3.2 Bidual of a C^*–algebra 218
6.3.3 Extension of the Functional Calculus 263
6.3.4 Von Neumann-Algebras 283
6.3.5 The Commutants 293
6.3.6 Irreducible Representations 299
6.3.7 Commutative von Neumann Algebras 320
6.3.8 Representations of W^*–Algebras 325
6.3.9 Finite–dimensional C^*–algebras 334

6.3.10 A generalization 355

7 C^*–algebras Generated by Groups 369
7.1 Projective Representations of Groups 369
7.1.1 Schur functions 369
7.1.2 Projective Representations 404
7.1.3 Supplementary Results 431
7.1.4 Examples 466
7.2 Clifford Algebras 492
7.2.1 General Clifford Algebras 492
7.2.2 $\mathcal{C}\ell_{p,q}$ 518
7.2.3 $\mathcal{C}\ell(\mathbb{N})$ 538

Name Index 559

Subject Index 563

Symbol Index 592

Introduction

This book has evolved from the lecture course on Functional Analysis I had given several times at the ETH. The text has a strict logical order, in the style of "Definiton – Theorem – Proof – Example – Exercises". The proofs are rather thorough and there are many examples.

The first part of the book (the first three chapters, resp. the first two volumes) is devoted to the theory of Banach spaces in the most general sense of the term. The purpose of the first chapter (resp. first volume) is to introduce those results on Banach spaces which are used later or which are closely connected with the book. It therefore only contains a small part of the theory, and several results are stated (and proved) in a diluted form. The second chapter (which together with Chapter 3 makes the second volume) deals with Banach algebras (and involutive Banach algebras), which constitute the main topic of the first part of the book. The third chapter deals with compact operators on Banach spaces and linear (ordinary and partial) differential equations – applications of the theory of Banach algebras.

The second part of the book (the last four chapters, resp. the last three volumes) is devoted to the theory of Hilbert spaces, once again in the general sense of the term. It begins with a chapter (Chapter 4, resp. Volume 3) on the theory of C^*–algebras and W^*–algebras which are essentially the focus of the book. Chapter 5 (resp. Volume 4) treats Hilbert spaces for which we had no need earlier. It contains the representation theorems, i.e. the theorems on isometries between abstract C^*–algebras and the concrete C^*–algebras of operators on Hilbert spaces. Chapter 6 (which together with Chapter 7 makes Volume 5) presents the theory of $\mathcal{L}^p$–spaces of operators, its application to the self–adjoint linear (ordinary and partial) differential equations, and the von Neumann algebras. Finally, Chapter 7 presents examples of C^*–algebras defined with the aid of groups, in particular the Clifford algebras. Many important domains of C^*–algebras are ignored in the present book. It should be emphasized that the whole theory is constructed in parallel for the real and for the complex numbers, i.e. the C^*–algebras are real or complex.

In addition to the above (vertical) structure of the book, there is also a second (horizontal) division. It consists of a main strand, eight branches, and additional material. The results belonging to the main strand are marked with (0). Logically speaking, a reader could restrict himself/herself to these and ignore the rest. Results on the eight subsidiary branches are marked with $(1), (2), (3), (4), (5), (6), (7)$, and (8). The key is

1. Infinite Matrices
2. Banach Categories
3. Nuclear Maps
4. Locally Compact Groups
5. Differential Equations
6. Laurent Series
7. Clifford Algebras
8. Hilbert C^*–Modules

These are (logically) independent of each other, but all depend on the main strand. Finally, the results which belong to the additional material have no marking and – from a logical perspective – may be ignored. So the reader can shorten for himself/herself this very long book using the above marks. Also, since the proofs are given with almost all references, it is possible to get into the book at any level and not to read it linearly.

We assume that the reader is familiar with classical analysis and has rudimentary knowledge of set theory, linear algebra, point–set topology, and integration theory. The book addresses itself mainly to mathematicians, or to physicists interested in C^*–algebras.

I would like to apologize for any omissions in citations occasioned by the fact that my acquaintance with the history of functional analysis is, unfortunately, very restricted. For this history we recommend the following texts.

1. BIRKHOFF, G. and KREYSZIG, E., The Establishment of Functional Analysis, Historia Mathematica 11 (1984), 258–321.

2. BOURBAKI, N., Elements of the History of Mathematics, (21. Topological Vector Spaces), Springer–Verlag (1994).

3. DIEUDONNÉ, J., History of Functional Analysis, North–Holland (1981).

4. DIEUDONNÉ, J., A Panorama of Pure Mathematics (Chapter C III: Spectral Theory of Operators), Academic Press (1982).

5. HEUSER, H., Funktionalanalysis, 2. Auflage (Kapitel XIX: Ein Blick auf die werdende Functionalanalysis), Teubner (1986), 3. Auflage (1992).

6. KADISON, R.V., Operator Algebras, the First Forty Years, in: Proceedings of Symposia in Pure Mathematics 38 I (1982), 1–18.

7. MONNA, A.F., Functional Analysis in Historical Perspective, John Whiley & Sons (1973).

8. STEEN, L.A., Highlights in the History of Spectral Theory, Amer. Math. Monthly 80 (1973), 359–382.

There is no shortage of excellent books on C^*–algebras. Nevertheless, we hope that this book will be also of some utility to the mathematics community.

VOLUME 5:
SELECTED TOPICS

6. Selected Chapters of C^*–Algebras

6.1 $\mathcal{L}^p$–Spaces

The theory of compact operators may be taken substantially further in the case of Hilbert spaces. The $\mathcal{L}^p$–spaces $(p \in \{0\} \cup [1, \infty[\,)$ consist of classes of such operators. These spaces bear a particular relationship to the Banach spaces ℓ^p, and their theory is a special case of what is sometimes called "non–commutative integration theory". The spaces $\mathcal{L}^0$, $\mathcal{L}^1$, and $\mathcal{L}^2$ are particularly important. They comprise the sets of compact, nuclear, and Hilbert–Schmidt operators, respectively. $\mathcal{L}^2$ is itself a Hilbert space and may be identified with the Hilbert space of square summable kernels in the case of measure spaces. This allows us to view such operators as integral operators. $\mathcal{L}^1$ can be identified with the dual of $\mathcal{L}^0$ and the dual of $\mathcal{L}^1$ can be identified with the Banach space $\mathcal{L}$ of all operators. In particular, the algebra of operators on a Hilbert space is a W^*–algebra. This provides information on the dual of $\mathcal{L}$. We present some examples of differential equations as an application of this theory.

E, F, G, H denote Hilbert spaces throughout this section.

6.1.1 Characteristic Families of Eigenvalues

Definition 6.1.1.1 (0) *Let u be a compact operator on E. A **characteristic family of eigenvalues of** u is a family $(\alpha_\iota)_{\iota \in I}$ in $\mathbb{K}\backslash\{0\}$ such that for every $\alpha \in \mathbb{K}\backslash\{0\}$*

$$\mathrm{Card}\{\iota \in I | \alpha_\iota = \alpha\}$$

agrees with the algebraic multiplicity of α with respect to u.

If $(\alpha_\iota)_{\iota \in I}$ is a characteristic family of eigenvalues of u, then

$$\{\alpha_\iota | \iota \in I\} = \sigma_p(u) \backslash \{0\}$$

and

$$(\alpha_\iota)_{\iota \in I} \in c_0(I)$$

(Theorem 3.1.5.1 a),b),c)).

Proposition 6.1.1.2 (0) *Let u be a self-normal compact operator on E. Take $(\alpha_\iota)_{\iota \in I}$ in $\mathbb{K} \backslash \{0\}$. Then the following are equivalent:*

a) *$(\alpha_\iota)_{\iota \in I}$ is a characteristic family of eigenvalues of u.*

b) *There is an ortonormal family $(x_\iota)_{\iota \in I}$ in E such that*

$$u = \sum_{\iota \in I} \alpha_\iota \langle \cdot | x_\iota \rangle x_\iota .$$

a $\Rightarrow$ b. Given $\alpha \in \mathbb{K}$, define

$$I_\alpha := \{\iota \in I | \alpha_\iota = \alpha\}$$

and let π_α denote the orthogonal projection of E onto $\operatorname{Ker}(\alpha 1 - u)$.

Take $\alpha \in \mathbb{K}$. By Proposition 5.3.3.21, $\operatorname{Ker}(\alpha 1 - u)$ is the algebraic eigenspace of u corresponding to α, so that

$$\operatorname{Dim} \operatorname{Ker}(\alpha 1 - u) = \operatorname{Card}\{\iota \in I \mid \alpha_\iota = \alpha\} = \operatorname{Card} I_\alpha .$$

Let $(x_\iota)_{\iota \in I_\alpha}$ be an orthonormal basis of $\operatorname{Ker}(\alpha 1 - u)$.

Since

$$\operatorname{Ker}(\alpha 1 - u) \perp \operatorname{Ker}(\beta 1 - u)$$

for distinct $\alpha, \beta \in \mathbb{K}$ (Corollary 5.3.4.5), the family $(x_\iota)_{\iota \in I}$ is orthonormal. By Proposition 5.3.4.7 a) (and Theorem 3.1.5.1 b)),

$$u = \sum_{\alpha \in \mathbb{K}} \alpha \pi_\alpha .$$

Hence u vanishes on $\{x_\iota \mid \iota \in I\}^\perp$ and $ux_\iota = \alpha_\iota x_\iota$ for every $\iota \in I$. By Theorem 5.5.5.4 (a & j_1) $\Rightarrow j_3$,

$$u = \sum_{\iota \in I} \alpha_\iota \langle \cdot \mid x_\iota \rangle x_\iota .$$

b $\Rightarrow$ a. Take $\alpha \in \mathbb{K} \backslash \{0\}$. Since

$$ux_\iota = \alpha_\iota x_\iota$$

for every $\iota \in I$, we have that

$$\mathrm{Ker}\,(\alpha 1 - u) = \{x_\iota | \alpha_\iota = \alpha\}^{\perp\perp}$$

by Proposition 5.5.5.7 b). By Corollary 5.2.3.9, $\{x_\iota \mid \alpha_\iota = \alpha\}^{\perp\perp}$ is the closed vector subspace of E generated by $\{x_\iota \mid \alpha_\iota = \alpha\}$. Thus

$$\mathrm{Dim}\,\mathrm{Ker}(\alpha 1 - u) = \mathrm{Card}\{\iota \in I \mid \alpha_\iota = \alpha\}\,.$$

By Proposition 5.3.3.21, $\mathrm{Ker}\,(\alpha 1 - u)$ is the algebraic eigenspace of u corresponding to α, so that $\mathrm{Card}\{\iota \in I \mid \alpha_\iota = \alpha\}$ is the algebraic multiplicity of α with respect to u. ■

Proposition 6.1.1.3 (0) *Let u be a self-normal compact operator on E, $(\alpha_\iota)_{\iota\in I}$ a characteristic family of eigenvalues of u, $f \in \mathcal{C}(\sigma(u))$ such that $f(0) = 0$ whenever $0 \in \sigma(u)$, and put*

$$J := \{\iota \in I | f(\iota) \neq 0\}\,.$$

Then $(f(\alpha_\iota))_{\iota\in J}$ is a characteristic family of eigenvalues of $f(u)$.

By Corollary 5.5.6.5, $f(u)$ is compact. By Proposition 6.1.1.2 a $\Rightarrow$ b, there is an orthonormal family $(x_\iota)_{\iota\in I}$ in E such that

$$u = \sum_{\iota\in I} \alpha_\iota \langle\cdot|x_\iota\rangle x_\iota\,.$$

By Theorem 5.5.6.1 a $\Rightarrow$ f,

$$f(u) = \sum_{\iota\in I} f(\alpha_\iota)\langle\cdot|x_\iota\rangle x_\iota\,.$$

Hence, by Proposition 6.1.1.2 b $\Rightarrow$a, $(f(\alpha_\iota))_{\iota\in I}$ is a characteristic family of eigenvalues of $f(u)$. ■

Corollary 6.1.1.4 *Let u be a self-normal compact operator on a Hilbert space, $(\alpha_\iota)_{\iota\in I}$ a characteristic family of eigenvalues of u, and take $n \in \mathbb{N}$. Then $(\alpha_\iota^n)_{\iota\in I}$ is a characteristic family of eigenvalues of u^n.* ■

Corollary 6.1.1.5 (0) *Let u be a selfadjoint compact operator on a Hilbert space and $(\alpha_\iota)_{\iota\in I}$ a characteristic family of eigenvalues of u. Then $(\sup\{\alpha_\iota, 0\})_{\iota\in I}$ (resp. $(\sup\{-\alpha_\iota, 0\})_{\iota\in I}$) is a characteristic family of eigenvalues of u^+ (resp. u^-).*

This follows from Proposition 6.1.1.3 and Theorem 4.2.2.9 b) (and Theorem 5.3.1.13). ■

Corollary 6.1.1.6 *Let u be a self-normal compact operator on a Hilbert space and $(\alpha_\iota)_{\iota\in I}$ a characteristic family of eigenvalues of u. Then $(|\alpha_\iota|)_{\iota\in I}$ is a characteristic family of eigenvalues of $|u|$.*

This follows from Proposition 6.1.1.3 and Proposition 4.2.5.4 (and Theorem 5.3.1.13)). ■

Corollary 6.1.1.7 *Let u be a normal compact operator on a complex Hilbert space and $(\alpha_\iota)_{\iota\in I}$ a characteristic family of eigenvalues of u. Then $(\overline{\alpha}_\iota)_{\iota\in I}$ (resp. $(\mathrm{re}\,\alpha_\iota)_{\iota\in I}$, resp. $(\mathrm{im}\,\alpha_\iota)_{\iota\in I}$) is a characteristic family of u^* (resp. $\mathrm{re}\,u$, resp. $\mathrm{im}\,u$).*

The assertion follows from Proposition 6.1.1.3 and Corollary 4.1.3.8. ■

Proposition 6.1.1.8 *Let $\mathcal{F}$ be a set of self-normal compact operators on E. Then the following are equivalent:*

a) *$\mathcal{F}$ is commutative.*

b) *There is an orthonormal basis A of E and an $f \in \mathbb{K}^{\mathcal{F}\times A}$ with*

$$u = \sum_{x\in A} f(u,x)\langle\cdot|x\rangle x$$

for every $u \in \mathcal{F}$.

If these conditions are fulfilled, then

$$uv = \sum_{x\in A} f(u,x)f(v,x)\langle\cdot|x\rangle x$$

for all $u, v \in \mathcal{F}$.

a $\Rightarrow$ b. Given $(u,\alpha) \in \mathcal{F}\times\mathbb{K}$, let $\pi_{u\alpha}$ denote the orthogonal projection of E onto $\mathrm{Ker}\,(\alpha 1 - u)$. By Proposition 5.3.4.7 g),

$$\{\pi_{u\alpha}|\alpha \in \mathbb{K}\} \subset \{u\}^{cc}$$

for every $u \in \mathcal{F}$, and so

$$\{\pi_{u\alpha}|(u,\alpha) \in \mathcal{F}\times\mathbb{K}\} \subset \mathcal{F}^{cc}.$$

Take $u, v \in \mathcal{F}$ and $\alpha \in \mathbb{K}$. Since $\mathcal{F}^{cc}$ is commutative (Proposition 2.1.1.17 e)),

$$v\pi_{u\alpha} = \pi_{u\alpha}v\,,$$

i.e. $\mathrm{Ker}(\alpha 1 - u)$ reduces v (Corollary 5.2.3.12 b)).

Given $g \in \mathbb{K}^{\mathcal{F}}$, define

$$F_g := \bigcap_{u\in\mathcal{F}} \mathrm{Ker}\,(g(u)1 - u)$$

and let A_g be an orthonormal basis of F_g. Since

$$F_g \perp F_h$$

for distinct $g, h \in \mathbb{K}^{\mathcal{F}}$ (Corollary 5.3.4.5), it follows that

$$A := \bigcup_{g\in\mathbb{K}^{\mathcal{F}}} A_g$$

is an orthonormal set of E.

Assume there is a $u \in \mathcal{F}$ and an $\alpha \in \mathbb{K}\backslash\{0\}$ such that

$$F := A^{\perp} \cap \mathrm{Ker}\,(\alpha 1 - u) \neq \{0\}\,.$$

Order the set

$$\left\{ F \cap \left(\bigcap_{\iota\in I} \mathrm{Ker}\,(\alpha_\iota 1 - u_\iota) \right) \neq \{0\} \;\middle|\; ((u_\iota, \alpha_\iota))_{\iota\in I} \text{ family in } \mathcal{F}\times\mathbb{K} \right\}$$

by inclusion. Since F is finite–dimensional (Theorem 3.1.5.1 a)), the above set has a least element

$$G := F \cap \left(\bigcap_{\iota\in I} \mathrm{Ker}\,(\alpha_\iota 1 - u_\iota) \right)\,.$$

Take $v \in \mathcal{F}$. Since for every $g \in \mathbb{K}^{\mathcal{F}}$, F_g reduces v (Corollary 5.3.2.10), F and G also reduce v. Hence there is a $\beta \in \mathbb{K}$ with

$$G \cap \mathrm{Ker}\,(\beta 1 - v) \neq \{0\}\,.$$

Since G is a least element we get

$$G \subset \mathrm{Ker}\,(\beta 1 - v)\,.$$

Thus there is a $g \in \mathbb{K}^{\mathcal{F}}$ with

$$G \subset \operatorname{Ker}(g(v)1 - v)$$

for every $v \in \mathcal{F}$. We deduce that

$$G \subset F_g = A_g^{\perp\perp} \subset A^{\perp\perp},$$

which is a contradiction. Hence for every $u \in \mathcal{F}$ and $\alpha \in \mathbb{K}\backslash\{0\}$,

$$A^{\perp} \cap \operatorname{Ker}(\alpha 1 - u) = \{0\}.$$

Hence

$$A^{\perp} \subset \bigcap_{u \in \mathcal{F}} \operatorname{Ker} u = F_0 = A_0^{\perp\perp} \subset A^{\perp\perp},$$

$$A^{\perp} = \{0\},$$

i.e. A is an orthonormal basis of E.

Take $u \in \mathcal{F}$. Given $\alpha \in \mathbb{K}$, put

$$B_\alpha := A \cap \operatorname{Ker}(\alpha 1 - u).$$

B_α is an orthonormal basis of $\operatorname{Ker}(\alpha 1 - u)$, and so

$$\pi_{u\alpha} = \sum_{x \in B_\alpha} \langle \cdot | x \rangle x$$

if $\alpha \neq 0$ (Proposition 5.5.1.7). For each $x \in A$, let $f(u,x)$ denote the $\alpha \in \mathbb{K}$ with $x \in B_\alpha$. Then

$$u = \sum_{\alpha \in \sigma(u)} \alpha \pi_{u\alpha} = \sum_{\alpha \in \mathbb{K}} \alpha \pi_{u\alpha} = \sum_{\alpha \in \mathbb{K}\backslash\{0\}} \alpha \pi_{u\alpha} = \sum_{\alpha \in \mathbb{K}\backslash\{0\}} \sum_{x \in B_\alpha} \alpha \langle \cdot | x \rangle x =$$

$$= \sum_{x \in A} f(u,x) \langle \cdot | x \rangle x$$

(Theorem 5.5.6.1 a $\Rightarrow$ c).

b $\Rightarrow$ a and the final assertion are easy to see. ∎

Corollary 6.1.1.9 *Let u, v be self-normal compact operators on E, and $(\alpha_\iota)_{\iota \in I}$, $(\beta_\lambda)_{\lambda \in L}$ characteristic families of eigenvalues of u and v, respectively. If*

$$uv = vu = 0,$$

then the family obtained by joining the above two families is a characteristic family of eigenvalues of $u + v$.

By Proposition 6.1.1.8 a $\Rightarrow$ b, there is an orthonormal basis A of E and $f, g \in \mathbb{K}^A$ with

$$u = \sum_{x \in A} f(x)\langle \cdot | x \rangle x\,, \quad v = \sum_{x \in A} g(x)\langle \cdot | x \rangle x\,,$$

$$fg = 0\,.$$

Now

$$u + v = \sum_{x \in A} (f(x) + g(x))\langle \cdot | x \rangle x\,,$$

and the assertion follows from Proposition 6.1.1.2 b $\Rightarrow$ a. ■

6.1.2 Characteristic Sequences

Definition 6.1.2.1 (0) (von Neumann, Schatten, 1948) *Take* $u \in \mathcal{L}(E,F)$. *Given* $n \in \mathbb{N}$, *define*

$$\mathfrak{E}_n := \{G \text{ vector subspace of } E | \operatorname{Dim} G < n\}$$

$$\theta_n(u) := \inf\{\|u|G^\perp\| | G \in \mathfrak{E}_n\}$$

*(**the** n**-th characteristic number of** u*), and*

$$\theta(u) := (\theta_n(u))_{n\in\mathbb{N}} \text{ (**the characteristic sequence of** } u)$$

$$N(u) := \{n \in \mathbb{N} | \theta_n(u) \neq 0\}.$$

$(\theta_n(u))_{n\in N(u)}$ *is called* ***the characteristic family of*** u.

For $p \in [1,\infty]$, *define*

$$\mathcal{L}^p(E,F) := \{u \in \mathcal{L}(E,F) | \theta(u) \in \ell^p\},$$

$$\mathcal{L}^p(E) := \mathcal{L}^p(E,E).$$

Given $u \in \mathcal{L}^p(E,F)$, *define*

$$\|u\|_p := \|\theta(u)\|_p \quad (\textbf{the } p\textbf{-norm of } u).$$

Define further

$$\mathcal{L}^0(E,F) := \mathcal{K}(E,F), \quad \mathcal{L}^0(E) := \mathcal{K}(E),$$

and

$$\|u\|_0 := \|u\|$$

for every $u \in \mathcal{L}^0(E,F)$.

An ***initial segment of*** $\mathbb{N}$ *is a subset* N *of* $\mathbb{N}$ *such that given* $m \in \mathbb{N}$ *and* $n \in N$, $m < n \Rightarrow m \in N$.

$\theta(u)$ is decreasing and $N(u)$ is an initial segment of $\mathbb{N}$. We have

$$\mathcal{L}^\infty(E,F) = \mathcal{L}(E,F)$$

and

$$\theta_1(u) = \|u\| = \|u\|_\infty$$

for every $u \in \mathcal{L}(E,F)$. We shall show that the p-norm is a norm (Theorem 6.1.3.12 a)) and that for $p = 1$ the symbols $\mathcal{L}^1$ and $\|\cdot\|_1$ introduced above agree with the ones introduced in Definition 1.6.1.1.

Proposition 6.1.2.2 a) $n \in \mathbb{N} \Rightarrow \theta_n(1_E) = \begin{cases} 1 & if \quad n \leq \operatorname{Dim} E \\ 0 & if \quad n > \operatorname{Dim} E \,. \end{cases}$

b) *For* $p \in [1, \infty[$,

$$1_E \in \mathcal{L}^p(E) \Longleftrightarrow E \text{ finite-dimensional} \Longrightarrow \|1_E\|_p = (\operatorname{Dim} E)^{\frac{1}{p}} \,.$$

a) Assume that $n \leq \operatorname{Dim} E$ and take $G \in \mathfrak{E}_n$. Then $G^\perp \neq \{0\}$, so that

$$\|1_E | G^\perp\| = 1 \,, \quad \theta_n(E) = 1 \,.$$

If $n > \operatorname{Dim} E$, then $E \in \mathfrak{E}_n$, $E^\perp = \{0\}$,

$$\|1_E | E^\perp\| = 0 \,, \quad \theta_n(E) = 0 \,.$$

b) follows from a). ■

Proposition 6.1.2.3 (0) *If* $(x, y) \in E \times F$ *and* $p \in [1, \infty]$ *then*

$$\theta_n(\langle \cdot | x \rangle y) = \begin{cases} \|x\| \, \|y\| & if \quad n = 1 \\ 0 & if \quad n \in \mathbb{N} \backslash \{1\} \,, \end{cases}$$

$$\langle \cdot | x \rangle y \in \mathcal{L}^p(E, F) \,, \quad \|\langle \cdot | x \rangle y\|_p = \|x\| \, \|y\| \,.$$

We have

$$\theta_1(\langle \cdot | x \rangle y) = \|\langle \cdot | x \rangle y\| = \|x\| \, \|y\| \,.$$

Since the restriction of $\langle \cdot | x \rangle y$ to $\{x\}^\perp$ vanishes identically, we see that

$$\theta_n(\langle \cdot | x \rangle x) = 0$$

for every $n \in \mathbb{N} \backslash \{1\}$. ■

Proposition 6.1.2.4 (0) *Take* $p, q \in [1, \infty]$ *with* $p \leq q$, *and* $u \in \mathcal{L}^p(E, F)$. *Then* $u \in \mathcal{L}^q(E, F)$ *and*

$$\|u\| \leq \|u\|_q \leq \|u\|_p \,.$$

This follows immediately from

$$x \in \ell^p \Rightarrow x \in \ell^q \,, \; \|x\|_q \leq \|x\|_p$$

(Proposition 1.1.2.6 b),e)). ■

Proposition 6.1.2.5 *If* $u, v \in \mathcal{L}(E,F)$ *then*

$$\theta_{n+p-1}(u+v) \leq \theta_n(u) + \theta_p(v)$$

for all $n, p \in \mathbb{N}$.

Take $G_1 \in \mathfrak{E}_n$ and $G_2 \in \mathfrak{E}_p$ and let G be the vector subspace of E generated by $G_1 + G_2$. Then $G \in \mathfrak{E}_{n+p-1}$. Thus

$$\theta_{n+p-1}(u+v) \leq \|(u+v)|G^\perp\| \leq \|u|G^\perp\| + \|v|G^\perp\| \leq \|u|G_1^\perp\| + \|v|G_2^\perp\|.$$

Since G_1 and G_1 are arbitrary,

$$\theta_{n+p-1}(u+v) \leq \theta_n(u) + \theta_p(v). \qquad \blacksquare$$

Proposition 6.1.2.6 (0) *Take* $n \in \mathbb{N}$ *and put*

$$\mathcal{F} := \{v \in \mathcal{L}(E,F) \mid \operatorname{Dim} \operatorname{Im} v < n\}.$$

Then

$$\theta_n(u) = d_{\mathcal{F}}(u)$$

for every $u \in \mathcal{L}(E,F)$.

Take $v \in \mathcal{F}$. Since the map

$$(\operatorname{Ker} v)^\perp \longrightarrow \operatorname{Im} v, \quad x \longmapsto vx$$

is bijective, we get successively,

$$\operatorname{Dim} (\operatorname{Ker} v)^\perp = \operatorname{Dim} \operatorname{Im} v < n,$$

$$(\operatorname{Ker} v)^\perp \in \mathfrak{E}_n,$$

$$\theta_n(u) \leq \|u|(\operatorname{Ker} v)^{\perp\perp}\| = \|u|\operatorname{Ker} v\| = \|(u-v)|\operatorname{Ker} v\| \leq \|u-v\|$$

$$\theta_n(u) \leq d_{\mathcal{F}}(u).$$

Now take $G \in \mathfrak{E}_n$ and put

$$v := u \circ \pi_G.$$

Then $v \in \mathcal{F}$, so that

$$d_{\mathcal{F}}(u) \leq \|u - v\| = \|u \circ (1_E - \pi_G)\| = \|u \circ \pi_G^\perp\| = \|u|G^\perp\|.$$

Since G is arbitrary,

$$d_{\mathcal{F}}(u) \leq \theta_n(u),$$

$$\theta_n(u) = d_{\mathcal{F}}(u). \qquad \blacksquare$$

Corollary 6.1.2.7 *Given* $u \in \mathcal{L}(E,F)$, $v \in \mathcal{L}(F,G)$, *and* $m,n \in \mathbb{N}$,

$$\theta_{m+n-1}(v \circ u) \leq \theta_m(u)\theta_n(v).$$

Take $u_0 \in \mathcal{L}(E,F)$, $v_0 \in \mathcal{L}(F,G)$ with

$$\operatorname{Dim}\operatorname{Im} u_0 < m, \quad \operatorname{Dim}\operatorname{Im} v_0 < n.$$

Then

$$(v - v_0) \circ (u - u_0) = v \circ u - (v \circ u_0 + v_0 \circ u - v_0 \circ u_0)$$

and

$$\operatorname{Im}(v \circ u_0 + v_0 \circ u - v_0 \circ u_0) \subset v(\operatorname{Im} u_0) + \operatorname{Im} v_0,$$

$$\operatorname{Dim}\operatorname{Im}(v \circ u_0 + v_0 \circ u - v_0 \circ u_0) \leq \operatorname{Dim} v(\operatorname{Im} u_0) + \operatorname{Dim}\operatorname{Im} v_0 < m + n - 1.$$

We deduce that

$$\theta_{m+n-1}(v \circ u) \leq \|(v - v_0) \circ (u - u_0)\| \leq \|u - u_0\|\,\|v - v_0\|$$

(Proposition 6.1.2.6). Since u_0 and v_0 are arbitrary,

$$\theta_{m+n-1}(v \circ u) \leq \theta_m(u)\theta_n(v)$$

(Proposition 6.1.2.6). ■

Corollary 6.1.2.8 *If* $u,v \in \mathcal{L}(E,F)$ *and* $n \in \mathbb{N}$, *then*

$$|\theta_n(u) - \theta_n(v)| \leq \|u - v\|.$$

If $w \in \mathcal{L}(E,F)$ and

$$\operatorname{Dim}\operatorname{Im} w < n,$$

then, by Proposition 6.1.2.6,

$$\theta_n(u) \leq \|u - w\| = \|(u - v) + (v - w)\| \leq \|u - v\| + \|v - w\|.$$

By Proposition 6.1.2.6 again,

$$\theta_n(u) \leq \|u - v\| + \theta_n(v)$$

so that

$$|\theta_n(u) - \theta_n(v)| \leq \|u - v\|.$$ ■

Corollary 6.1.2.9 (0) *Take $u \in \mathcal{L}(E,F)$. Then the following are equivalent:*

a) *u is compact.*

b) $\theta(u) \in c_0$.

a $\Rightarrow$ b. Choose $\varepsilon > 0$. By Corollary 5.2.3.4, there is a $v \in \mathcal{L}_f(E,F)$ with

$$\|u - v\| < \varepsilon .$$

Put

$$n := 1 + \operatorname{Dim} \operatorname{Im} v \in \mathbb{N} .$$

Then

$$\theta_n(u) \leq \|u - v\| < \varepsilon ,$$

by Proposition 6.1.2.6. Hence $\theta(u) \in c_0$.

b $\Rightarrow$ a. Take $\varepsilon > 0$. There is an $n \in \mathbb{N}$ with $\theta_n(u) < \varepsilon$. By Proposition 6.1.2.6, there is a $v \in \mathcal{L}(E,F)$ with

$$\operatorname{Dim} \operatorname{Im} v < n , \quad \|u - v\| < \varepsilon .$$

Thus

$$u \in \overline{\mathcal{L}_f(E,F)} = \mathcal{K}(E,F)$$

by Corollary 5.2.3.4. ■

Corollary 6.1.2.10 (0) *For every $u \in \mathcal{L}_f(E,F)$, $N(u)$ is finite and so*

$$\mathcal{L}_f(E,F) \subset \mathcal{L}^p(E,F) \subset \mathcal{K}(E,F)$$

for every $p \in [1, \infty[$.

By Proposition 6.1.2.6, $N(u)$ is finite. Thus

$$\mathcal{L}_f(E,F) \subset \mathcal{L}^p(E,F) .$$

The inclusion

$$\mathcal{L}^p(E,F) \subset \mathcal{K}(E,F)$$

follows from Corollary 6.1.2.9 b $\Rightarrow$ a. ■

Proposition 6.1.2.11 (0) *Let N be an initial segment of $\mathbb{N}$, $(x_n)_{n\in N}$ and $(y_n)_{n\in N}$ orthonormal families in E and F, respectively, and take a $(\theta_n)_{n\in N} \in c_0(N)$, such that $(|\theta_n|)_{n\in N}$ decreases. Put*

$$u := \sum_{n\in N} \theta_n \langle \cdot | x_n \rangle y_n$$

(Corollary 5.5.5.5). Then $N(u) \subset N$ and

$$\theta_n(u) = |\theta_n|$$

for every $n \in N$.

Take $n \in N$. Let G be the vector subspace of E generated by $(x_m)_{m\in\mathbb{N}_n}$ and take $H \in \mathfrak{E}_n$. Then $(\pi_H x_m)_{m\in\mathbb{N}_n}$ is linearly dependent. Thus there is a family $(\alpha_m)_{m\in\mathbb{N}_n}$ in $\mathbb{K}$, such that not every α_m vanishes and

$$\sum_{m=1}^{n} \alpha_m \pi_H x_m = 0\,.$$

We deduce that

$$0 \neq x := \sum_{m=1}^{n} \alpha_m x_m \in G \cap H^\perp\,,$$

$$\|u|H^\perp\|^2 \|x\|^2 \geq \|ux\|^2 = \sum_{m\in N} |\theta_m \langle x | x_m \rangle|^2 \geq$$

$$\geq |\theta_n|^2 \sum_{m\in\mathbb{N}_n} |\langle x | x_m \rangle|^2 = |\theta_n|^2 \|x\|^2\,,$$

$$\|u|H^\perp\| \geq |\theta_n|\,,$$

$$\theta_n(u) \geq |\theta_n|\,.$$

Now take $n \in N$. Let G be the vector subspace of E generated by $(x_m)_{m\in\mathbb{N}_{n-1}}$ and take $x \in G^\perp$. Then

$$\|ux\|^2 = \sum_{\substack{m\in N\\ m\geq n}} |\theta_m \langle x | x_m \rangle|^2 \leq |\theta_n|^2 \sum_{\substack{m\in N\\ m\geq n}} |\langle x | x_m \rangle|^2 \leq |\theta_n|^2 \|x\|^2$$

by Bessel's Inequality, and so

$$\theta_n(u) \leq \|u|G^\perp\| \leq |\theta_n|\,,$$

$$\theta_n(u) = |\theta_n|\,.$$

Take $n \in \mathbb{N}\backslash N$. Since

$$\operatorname{Dim}\operatorname{Im} u < n\,,$$

it follows from Proposition 6.1.2.6 that

$$\theta_n(u) = 0\,. \qquad \blacksquare$$

Corollary 6.1.2.12 (0) *Let $(x_\iota)_{\iota\in I}$ and $(y_\iota)_{\iota\in I}$ be orthonormal families in E and F, respectively. Take $f \in c_0(I)$ define*

$$u := \sum_{\iota\in I} f(\iota)\langle\cdot|x_\iota\rangle y_\iota$$

(Corollary 5.5.5.5), and take $p \in [1,\infty]$. Then u belongs to $\mathcal{L}^p(E,F)$ iff $f \in \ell^p(I)$, and in this case

$$\|u\|_p = \|f\|_p\,. \qquad \blacksquare$$

Corollary 6.1.2.13 (0) *Let u be a compact operator on E and N an initial segment of $\mathbb{N}$. If $(\alpha_n)_{n\in N}$ is a characteristic family of eigenvalues of u such that $(|\alpha_n|)_{n\in N}$ is decreasing, then the following are equivalent:*

a) *u is self-normal.*

b) *there is an orthonormal family $(x_n)_{n\in N}$ in E with*

$$u = \sum_{n\in N} \alpha_n\langle\cdot|x_n\rangle x_n\,.$$

If these conditions are fulfilled, then:

c) *$N \supset N(u)$ and $\theta_n(u) = |\alpha_n|$ for every $n \in N$.*

d) *If $p \in [1,\infty[\,\cup\{0\}$, then $u \in \mathcal{L}^p(E)$ iff $(\alpha_n)_{n\in N} \in \ell^p(N)$, and in this case*

$$\|u\|_p = \|(\alpha_n)_{n\in N}\|_p\,.$$

a $\Rightarrow$ b follows from Proposition 6.1.1.2 a $\Rightarrow$ b.

b $\Rightarrow$ a. Let $(x_\iota)_{\iota\in I}$ be an orthonormal basis of E with $N \subset I$. Then

$$ux_n = \begin{cases} \alpha_n x_n & if\ n \in N \\ 0 & if\ n \in I\backslash N. \end{cases}$$

By Proposition 5.5.5.7 e), u is self-normal.

c) follows from Proposition 6.1.2.11.

d) follows from c). ■

Corollary 6.1.2.14 (0) *Let u be a selfadjoint operator on E. Take $p \in [1,\infty[$. Then the following are equivalent:*

a) $u \in \mathcal{L}^p(E)$.

b) $u^+, u^- \in \mathcal{L}^p(E)$.

If these conditions are fulfilled, then

$$\|u\|_p^p = \|u^+\|_p^p + \|u^-\|_p^p.$$

a) or b) implies that u is compact (Corollary 6.1.2.10). Let $(\alpha_\iota)_{\iota\in I}$ be a characteristic family of eigenvalues of u. By Corollary 6.1.1.5, $(\sup\{\alpha_\iota, 0\})_{\iota\in I}$ and $(\sup\{-\alpha_\iota, 0\})_{\iota\in I}$ are characteristic families of eigenvalues of u^+ and u^-, respectively, and the assertions now follow from Corollary 6.1.2.13 d). ■

Corollary 6.1.2.15 *Let u be a self-normal operator on E, and take $n \in \mathbb{N}$ and $p \in [1,\infty]$. Then $u^n \in \mathcal{L}^p(E)$ iff $u \in \mathcal{L}^{pn}(E)$ and in this case*

$$\|u^n\|_p = \|u\|_{np}^n.$$

By Corollary 5.3.3.11, u is compact iff u^n is compact. Hence, by Corollary 6.1.2.10 (and Proposition 4.2.4.2 b)), we may assume that u is compact. The assertion now follows from Corollary 6.1.1.4 and Corollary 6.1.2.13 d). ■

Corollary 6.1.2.16 (0) *Let u be a positive compact operator on E and f a continuous increasing real function on $\sigma(u)$ with $f(0) = 0$ whenever $0 \in \sigma(u)$. Then $(\theta_n(u))_{n\in N(u)}$ is a characteristic family of eigenvalues of u and*

$$\theta(f(u)) = (f(\theta_n(u)))_{n\in\mathbb{N}}.$$

In particular,

$$\theta(u^\alpha) = (\theta_n(u)^\alpha)_{n\in\mathbb{N}}$$

for every $\alpha \in]0,\infty[$.

By Corollary 6.1.2.13 a $\Rightarrow$ b & c, $(\theta_n(u))_{n\in N(u)}$ is a characteristic family of eigenvalues of u and there is an orthonormal family $(x_n)_{n\in N(u)}$ in E such that

$$u = \sum_{n\in N(u)} \theta_n(u)\langle\cdot|x_n\rangle x_n\,.$$

By Theorem 5.5.6.1 e $\Rightarrow$ f,

$$f(u) = \sum_{n\in N(u)} f(\theta_n(u))\langle\cdot|x_n\rangle x_n\,.$$

Hence, by Corollary 6.1.2.13 b $\Rightarrow$ c,

$$\theta(f(u)) = (f(\theta_n(u)))_{n\in\mathbb{N}}\,. \qquad \blacksquare$$

Proposition 6.1.2.17 (0) *Let u be a positive operator on E and for each $n \in \mathbb{N}$ put*

$$\alpha_n := \inf_{F\in\mathfrak{E}_n} \sup_{x\in(F^\perp)^\#} \langle ux|x\rangle\,.$$

Then the following are equivalent:

a) *u is compact.*

b) *$(\|ux_\iota\|)_{\iota\in I} \in c_0(I)$ for any orthonormal family $(x_\iota)_{\iota\in I}$ in E.*

c) *$(\langle ux|x\rangle)_{x\in A} \in c_0(A)$ for any orthonormal basis A of E.*

d) *$(\langle ux_n|x_n\rangle)_{n\in\mathbb{N}} \in c_0$ for any orthonormal sequence $(x_n)_{n\in\mathbb{N}}$ in E.*

e) *$(\alpha_n)_{n\in\mathbb{N}} \in c_0$.*

If these conditions are fulfilled, then

$$\theta_n(u) = \alpha_n$$

for every $n \in \mathbb{N}$ and there is an orthonormal family $(x_n)_{n\in N(u)}$ in E such that

$$u = \sum_{n\in N(u)} \theta_n(u)\langle\cdot|x_n\rangle x_n\,.$$

a $\Rightarrow$ b follows from Corollary 5.5.1.9.

b $\Rightarrow$ c $\Rightarrow$ d is trivial.

d $\Rightarrow$ e. Assume that

$$\alpha := \lim_{n\to\infty} \alpha_n > 0 .$$

By recursion, we construct an orthonormal sequence $(x_n)_{n\in\mathbb{N}}$ in E such that

$$\langle ux_n | x_n \rangle > \frac{\alpha}{2}$$

for every $n \in \mathbb{N}$. Take $n \in \mathbb{N}$, and suppose the sequence has been constructed up to $n-1$. Let F denote the vector subspace of E generated by $\{x_m | m \in \mathbb{N}_{n-1}\}$. Then $F \in \mathfrak{E}_n$, so that

$$\sup_{x\in(F^\perp)^\#} \langle ux|x\rangle \geq \alpha_n > \frac{\alpha}{2} .$$

Hence there is an $x_n \in F^\perp$ such that

$$\|x_n\| = 1 , \quad \langle ux_n | x_n \rangle > \frac{\alpha}{2} .$$

This completes the inductive construction. The existence of the orthonormal sequence $(x_n)_{n\in\mathbb{N}}$ in E contradicts d).

e $\Rightarrow$ a. Let F be a vector subspace of E. Then

$$\sup_{x\in(F^\perp)^\#} \langle ux|x\rangle = \sup_{x\in(F^\perp)^\#} \langle u^{\frac{1}{2}}x | u^{\frac{1}{2}}x\rangle = \sup_{x\in(F^\perp)^\#} \|u^{\frac{1}{2}}x\|^2 = \|u^{\frac{1}{2}}|F^\perp\|^2 .$$

Hence

$$\alpha_n = \inf_{F\in\mathfrak{E}_n} \|u^{\frac{1}{2}}|F^\perp\|^2 = \theta_n(u^{\frac{1}{2}})^2 .$$

By Corollary 6.1.2.9 b $\Rightarrow$ a, $u^{\frac{1}{2}}$ is compact. Hence u is compact.

Now we prove the final assertion. By Corollary 6.1.2.13 a $\Rightarrow$ b & c, there is an orthonormal family $(x_n)_{n\in\mathbb{N}(u)}$ in E such that

$$u = \sum_{n\in N(u)} \theta_n(u) \langle \cdot | x_n \rangle x_n .$$

Take a vector subspace F of E. Then

$$\sup_{x\in(F^\perp)^\#} \langle ux|x\rangle \leq \sup_{x\in(F^\perp)^\#} \|ux\|\,\|x\| \leq \|u|F^\perp\| ,$$

and so

$$\alpha_n = \inf_{F\in\mathfrak{E}_n} \sup_{x\in(F^\perp)^\#} \langle ux|x\rangle \leq \inf_{F\in\mathfrak{E}_n} \|u|F^\perp\| = \theta_n(u) .$$

To prove the reverse inequality, we may assume that $n \in N(u)$. Let G be the vector subspace of E generated by $(x_m)_{m\in\mathbb{N}_n}$ and take $F \in \mathfrak{E}_n$. There is a $y \in G \cap F^\perp$ with $\|y\| = 1$. Now

$$\sup_{x\in(F^\perp)^\#}\langle ux|x\rangle \geq \langle uy|y\rangle = \sum_{m=1}^{n}\theta_m(u)\langle y|x_m\rangle\langle x_m|y\rangle =$$

$$= \sum_{m=1}^{n}\theta_m(u)|\langle x_m|y\rangle|^2 \geq \theta_n(u)\sum_{m=1}^{n}|\langle y|x_m\rangle|^2 = \theta_n(u)\,,$$

and so

$$\alpha_n = \inf_{F\in\mathfrak{E}_n}\sup_{x\in(F^\perp)^\#}\langle ux|x\rangle \geq \theta_n(u)\,. \qquad \blacksquare$$

Corollary 6.1.2.18 (0) *Let u be a positive operator on E and take $p,q\in[1,\infty[$. Then $u\in\mathcal{L}^p(E)$ iff $u^{\frac{p}{q}}\in\mathcal{L}^q(E)$, and in this case*

$$\|u\|_p^p = \|u^{\frac{p}{q}}\|_q^q\,.$$

We may assume that u is compact (Proposition 5.3.3.10 b), Corollary 6.1.2.10). By Corollary 6.1.2.16,

$$\theta(u^{\frac{p}{q}}) = (\theta_n(u)^{\frac{p}{q}})_{n\in\mathbb{N}}\,,$$

and the assertion now follows (Corollary 6.1.2.13 a $\Rightarrow$ d). $\blacksquare$

6.1.3 Properties of the $\mathcal{L}^p$–spaces

Theorem 6.1.3.1 (0) *Take* $u \in \mathcal{K}(E,F)$ *and put* $v := (u^* \circ u)^{\frac{1}{2}}$. *Take* $w \in \mathcal{L}(E,F)$ *such that*

$$\operatorname{Ker} u = \operatorname{Ker} w\,, \quad u = w \circ v\,,$$

(Proposition 5.5.5.10).

a) *There are orthonormal families* $(x_n)_{n\in N(u)}$ *and* $(y_n)_{n\in N(u)}$ *in* E *and* F, *respectively, such that*

$$u = \sum_{n\in N(u)} \theta_n(u)\langle\cdot|x_n\rangle y_n\,.$$

b) $\theta(u) = \theta(u^*) = \theta(v)$.

c) $v = \sum_{n\in N(u)} \theta_n(u)\langle\cdot|x_n\rangle x_n$.

d) $n \in N(u) \Rightarrow y_n = wx_n$.

v is compact (Proposition 5.3.3.10 b)). By Proposition 6.1.2.17, there is an orthonormal family $(x_n)_{n\in N(v)}$ such that

$$v = \sum_{n\in N(v)} \theta_n(v)\langle\cdot|x_n\rangle x_n\,.$$

Given $n \in N(v)$, put

$$y_n := wx_n\,.$$

By Proposition 5.5.5.10 f), $(y_n)_{n\in N(v)}$ is an orthonormal family in F and

$$u = \sum_{n\in N(v)} \theta_n(v)\langle\cdot|x_n\rangle y_n\,.$$

It follows that

$$u^* = \sum_{n\in N(v)} \theta_n(v)\langle\cdot|y_n\rangle x_n$$

(Proposition 5.3.2.13 b)), and so

$$\theta(u) = \theta(u^*) = \theta(v)$$

by Proposition 6.1.2.11. ■

Corollary 6.1.3.2 (0) *Take* $p \in \{0\} \cup [1, \infty]$ *and* $u \in \mathcal{L}(E,F)$. *Then the following are equivalent:*

a) $u \in \mathcal{L}^p(E,F)$.

b) $u^* \in \mathcal{L}^p(F,E)$.

c) $(u^* \circ u)^{\frac{1}{2}} \in \mathcal{L}^p(E)$.

If these conditions are fulfilled, then

$$\|u\|_p = \|u^*\|_p = \|(u^* \circ u)^{\frac{1}{2}}\|_p .$$

By Corollary 6.1.2.10, Proposition 5.3.3.10 b), and Theorem 5.3.1.4, we may assume that u is compact, and so the assertion follows from Theorem 6.1.3.1 b). ■

Corollary 6.1.3.3 (0) *Take* $u \in \mathcal{L}(E,F)$ *and* $v \in \mathcal{K}(E,G)$. *If there is an* $\alpha \in]0, \infty[$ *with*

$$(u^* \circ u)^\alpha \leq (v^* \circ v)^\alpha ,$$

then:

a) $\theta_n(u) \leq \theta_n(v)$ *for every* $n \in \mathbb{N}$.

b) *If* $p \in \{0\} \cup [1, \infty]$ *and* $v \in \mathcal{L}^p(E,G)$, *then* $u \in \mathcal{L}^p(E,G)$ *and* $\|u\|_p \leq \|v\|_p$.

a) Given $x \in E$,

$$\langle (u^* \circ u)^\alpha x | x \rangle \leq \langle (v^* \circ v)^\alpha x | x \rangle$$

(Corollary 5.3.3.7), and so

$$\theta_n(u) = \theta_n((u^* \circ u)^{\frac{1}{2}}) = \theta_n((u^* \circ u)^\alpha)^{\frac{1}{2\alpha}} \leq \theta_n((v^* \circ v)^\alpha)^{\frac{1}{2\alpha}} = \theta_n(v)$$

(Corollary 6.1.2.16, Theorem 6.1.3.1 b), Proposition 6.1.2.17).

b) follows from a). ■

Definition 6.1.3.4 (0) *Take* $u \in \mathcal{K}(E,F)$. *A* ***Schatten decomposition of*** u *is an expression of the form*

$$\sum_{n \in N} \theta_n \langle \cdot | x_n \rangle y_n$$

such that:

1) N *is an initial segment of* $\mathbb{N}$.

2) $(\theta_n)_{n\in N}$ *is a decreasing family in* $\mathbb{R}_+\backslash\{0\}$.

3) $(x_n)_{n\in N}$ *and* $(y_n)_{n\in N}$ *are orthonormal families in* E *and* F, *respectively.*

4) $u = \sum_{n\in N} \theta_n\langle\cdot|x_n\rangle y_n$.

By Proposition 6.1.2.11, $N = N(u)$ and

$$\theta_n(u) = \theta_n$$

for every $n \in N$. By Theorem 6.1.3.1 a), every compact operator admits a Schatten decomposition.

Corollary 6.1.3.5 (0) *If* $u \in \mathcal{K}(E,F)$ *and* $v \in \mathcal{L}(F,E)$ *then there are orthonormal bases* $(x_\iota)_{\iota\in I}$, $(y_\lambda)_{\lambda\in L}$ *of* E *and* F, *respectively, such that*

$$n \in I \cap L \Longrightarrow \langle vux_n|x_n\rangle = \langle uvy_n|y_n\rangle\,,$$

$$\iota \in I\backslash L \Longrightarrow \langle vux_\iota|x_\iota\rangle = 0\,,$$

$$\lambda \in L\backslash I \Longrightarrow \langle uvy_\lambda|y_\lambda\rangle = 0\,.$$

In particular, $(\langle vux_\iota|x_\iota\rangle)_{\iota\in I}$ *is summable iff* $(\langle uvy_\lambda|y_\lambda\rangle)_{\lambda\in L}$ *is summable and in this case*

$$\sum_{\iota\in I}\langle vux_\iota|x_\iota\rangle = \sum_{\lambda\in L}\langle uvy_\lambda|y_\lambda\rangle\,.$$

Let

$$\sum_{n\in N}\theta_n\langle\cdot|x_n\rangle y_n$$

be a Schatten decomposition of u and let $(x_\iota)_{\iota\in I}$, $(y_\lambda)_{\lambda\in L}$ be orthonormal bases of E and F, respectively, such that

$$N = I \cap L\,.$$

By Theorem 5.5.5.4 b $\Rightarrow$ e & k,

$$n \in N \Longrightarrow \langle vux_n|x_n\rangle = \langle uvy_n|y_n\rangle$$

$$\iota \in I\backslash N \Longrightarrow \langle vux_\iota|x_\iota\rangle = 0\,,$$

$$\lambda \in L\backslash N \Longrightarrow \langle uvy_\lambda|y_\lambda\rangle = \langle vy_\lambda|u^*y_\lambda\rangle = 0\,. \qquad \blacksquare$$

Proposition 6.1.3.6 (3) *The definitions of* $\mathcal{L}^1(E,F)$ *and* $\|\cdot\|_1$ *given in Definition 1.6.1.1 and Definition 6.1.2.1 coincide in the case of Hilbert spaces.*

We use $\mathcal{L}^1(E,F)_{\text{old}}$, $\|\cdot\|_{1\text{old}}$ and $\mathcal{L}^1(E,F)_{\text{new}}$, $\|\cdot\|_{1\text{new}}$ to denote the corresponding symbols introduced in Definitions 1.6.1.1 and 6.1.2.1, respectively.

Take $u \in \mathcal{L}(E,F)$. By Corollary 6.1.2.10 and Corollary 3.1.1.5, we may assume that u is compact. Let

$$\sum_{n\in N} \theta_n \langle \cdot | x_n \rangle y_n$$

be a Schatten decomposition of u (Theorem 6.1.3.1 a)).

First suppose that $u \in \mathcal{L}^1(E,F)_{\text{new}}$. Then $u \in \mathcal{L}^1(E,F)_{\text{old}}$ and

$$\|u\|_{1\text{old}} \leq \sum_{n\in N} \|\theta_n x_n\| \, \|y_n\| = \sum_{n\in N} \theta_n = \|u\|_{1\text{new}} .$$

Now suppose that $u \in \mathcal{L}^1(E,F)_{\text{old}}$ and let $((a_\iota, b_\iota))_{\iota\in I}$ be a family in $E\times F$, such that

$$\sum_{\iota\in I} \|a_\iota\| \, \|b_\iota\| < \infty , \quad u = \sum_{\iota\in I} \langle \cdot | a_\iota \rangle b_\iota .$$

Then

$$\theta_n = \langle u x_n | y_n \rangle = \sum_{\iota\in I} \langle x_n | a_\iota \rangle \langle b_\iota | y_n \rangle \leq \sum_{\iota\in I} |\langle x_n | a_\iota \rangle \langle b_\iota | y_n \rangle| ,$$

$$\sum_{n\in N} \theta_n \leq \sum_{n\in N} \sum_{\iota\in I} |\langle x_n | \alpha_\iota \rangle \langle b_\iota | y_n \rangle| \leq \sum_{\iota\in I} \|a_\iota\| \, \|b_\iota\| < \infty$$

by Proposition 5.5.1.19. It follows that $u \in \mathcal{L}^1(E,F)_{1\text{new}}$ and

$$\|u\|_{1\text{new}} \leq \|u\|_{1\text{old}} . \qquad \blacksquare$$

Lemma 6.1.3.7 (0) *If* $n \in \mathbb{N}$ *and* $(\alpha_k)_{k\in\mathbb{N}_n}$, $(\beta_k)_{k\in\mathbb{N}_n}$ *are families in* $\mathbb{R}$, *then*

$$\sum_{k=1}^{n} \alpha_k \beta_k = \sum_{k=1}^{n-1} (\alpha_k - \alpha_{k+1}) \sum_{i=1}^{k} \beta_i + \alpha_n \sum_{k=1}^{n} \beta_k .$$

Now

$$\sum_{k=1}^{n-1} (\alpha_k - \alpha_{k+1}) \sum_{i=1}^{k} \beta_i + \alpha_n \sum_{k=1}^{n} \beta_k =$$

$$= \sum_{k=1}^{n-1} \alpha_k \sum_{i=1}^{k} \beta_i - \sum_{k=1}^{n-1} \alpha_{k+1} \sum_{i=1}^{k} \beta_i + \alpha_n \sum_{k=1}^{n} \beta_k =$$

$$= \sum_{k=1}^{n-1} \alpha_k \sum_{i=1}^{k} \beta_i - \sum_{k=2}^{n} \alpha_k \sum_{i=1}^{k-1} \beta_i + \alpha_n \sum_{k=1}^{n} \beta_k =$$

$$= \sum_{k=1}^{n-1} \alpha_k \beta_k - \alpha_n \sum_{i=1}^{n-1} \beta_i + \alpha_n \sum_{k=1}^{n} \beta_k = \sum_{k=1}^{n} \alpha_k \beta_k . \qquad \blacksquare$$

Lemma 6.1.3.8 (0) *Take* $n \in \mathbb{N}$, *and let* $(\alpha_i)_{i \in \mathbb{N}_n}$, $(\beta_i)_{i \in \mathbb{N}_n}$ *be decreasing families in* $\mathbb{R}_+$ *such that*

$$\sum_{i=1}^{k} \alpha_i \leq \sum_{k=1}^{k} \beta_i$$

for every $k \in \mathbb{N}_n$. *Then*

$$\sum_{i=1}^{n} \alpha_i^p \leq \sum_{i=1}^{n} \beta_i^p$$

for every $p \in [1, \infty[$.

For every $x \in \mathbb{R}^n$ we need in the proof the family $(\widetilde{x}_i)_{i \in \mathbb{N}_n}$ obtained by writing the coordinates of x as a decreasing family. Formally this can be described in the following way. Given $i \in \mathbb{N}_n$ and $x \in \mathbb{R}^n$, define

$$\mathfrak{A}_i := \{A \subset \mathbb{N}_n | \mathrm{Card}\,(\mathbb{N}_n \backslash A) < i\} ,$$

$$\widetilde{x}_i := \inf_{A \in \mathfrak{A}_i} \sup_{k \in A} |x_k| , \quad \widetilde{x} := (\widetilde{x}_i)_{i \in \mathbb{N}_n} .$$

Furthermore, put

$$\alpha := (\alpha_i)_{i \in \mathbb{N}_n} , \quad \beta := (\beta_i)_{i \in \mathbb{N}_n} ,$$

$$K := \{x \in \mathbb{R}^n | \widetilde{x} = \beta\} ,$$

and let L denote the absolutely convex hull of K in $\mathbb{R}^n$. Since K is finite, L is compact.

Let x' be a linear form on $\mathbb{R}^n$. Put

$$\gamma := (x'(e_k))_{k \in \mathbb{N}_n} .$$

Then, by Lemma 6.1.3.7,

$$|x'(\alpha)| \leq \sum_{k=1}^{n} |\gamma_k| \alpha_k = \sum_{k=1}^{n-1} (\alpha_k - \alpha_{k+1}) \sum_{i=1}^{k} |\gamma_i| + \alpha_n \sum_{i=1}^{n} |\gamma_i| \leq$$

$$\leq \sum_{k=1}^{n-1} (\alpha_k - \alpha_{k+1}) \sum_{i=1}^{k} \widetilde{\gamma}_i + \alpha_n \sum_{i=1}^{n} \widetilde{\gamma}_i = \sum_{k=1}^{n} \widetilde{\gamma}_k \alpha_k =$$

$$= \sum_{k=1}^{n-1} (\widetilde{\gamma}_k - \widetilde{\gamma}_{k+1}) \sum_{i=1}^{k} \alpha_i + \widetilde{\gamma}_n \sum_{i=1}^{n} \alpha_i \leq$$

$$\leq \sum_{k=1}^{n-1} (\widetilde{\gamma}_k - \widetilde{\gamma}_{k+1}) \sum_{i=1}^{k} \beta_i + \widetilde{\gamma}_n \sum_{i=1}^{n} \beta_i = \sum_{k=1}^{n} \widetilde{\gamma}_k \beta_k .$$

There is an $x \in K$ with

$$x'(x) = \sum_{k=1}^{n} \widetilde{\gamma}_k \beta_k .$$

Thus

$$|x'(\alpha)| \leq \sup_{x \in K} x'(x) \leq \sup_{x \in L} x'(x) .$$

By Corollary 1.3.1.7, $\alpha \in L$. Hence there is a family $(\delta_x)_{x \in K}$ in $\mathbb{R}$ such that

$$\sum_{x \in K} |\delta_x| \leq 1 , \quad \alpha = \sum_{x \in K} \delta_x x .$$

We deduce that

$$\|\alpha\|_p \leq \sum_{x \in K} |\delta_x| \, \|x\|_p = \sum_{x \in K} |\delta_x| \, \|\beta\|_p \leq \|\beta\|_p ,$$

$$\sum_{i=1}^{n} \alpha_i^p \leq \sum_{i=1}^{n} \beta_i^p . \qquad \blacksquare$$

Lemma 6.1.3.9 (0) *Let* $(\alpha_{\iota\lambda})_{(\iota,\lambda) \in I \times L}$ *be a family in* $\mathbb{K}$ *such that*

$$\iota \in I \Longrightarrow \sum_{\lambda \in L} |\alpha_{\iota\lambda}| \leq 1 ,$$

$$\lambda \in L \Longrightarrow \sum_{\iota \in I} |\alpha_{\iota\lambda}| \leq 1 .$$

Take $p \in [1, \infty]$, $x \in \ell^p(L)$, *and for each* $\iota \in I$ *put*

$$y_\iota := \sum_{\lambda \in L} \alpha_{\iota\lambda} x_\lambda .$$

Then

$$y := (y_\iota)_{\iota \in I} \in \ell^p(I) , \quad \|y\|_p \leq \|x\|_p .$$

It is obviously sufficient to consider the case when I and L are finite. We may also assume that each $\alpha_{\iota\lambda}$ and x_λ is a positive real number. Finally, we may assume that $I = L = \mathbb{N}_n$ for some $n \in \mathbb{N}$ and that $(x_i)_{i \in \mathbb{N}_n}$, $(y_i)_{i \in \mathbb{N}_n}$ are decreasing. Then for $k \in \mathbb{N}_n$,

$$\sum_{j=1}^{k} \Big(\sum_{i=1}^{k} \alpha_{ij} \Big) x_j = \sum_{j=1}^{k} x_j + \sum_{j=1}^{k} \Big(\sum_{i=1}^{k} \alpha_{ij} - 1 \Big) x_j \leq$$

$$\leq \sum_{j=1}^{k} x_j + \sum_{j=1}^{k} \Big(\sum_{i=1}^{k} \alpha_{ij} - 1 \Big) x_k = \sum_{j=1}^{k} x_j + \sum_{j=1}^{k} \Big(\sum_{i=1}^{k} \alpha_{ij} \Big) x_k - k x_k ,$$

so that

$$\sum_{i=1}^{k} y_i = \sum_{i=1}^{k} \Big(\sum_{j=1}^{n} \alpha_{ij} x_j \Big) = \sum_{j=1}^{n} \Big(\sum_{i=1}^{k} \alpha_{ij} \Big) x_j =$$

$$= \sum_{j=1}^{k} \Big(\sum_{i=1}^{k} \alpha_{ij} \Big) x_j + \sum_{j=k+1}^{n} \Big(\sum_{i=1}^{k} \alpha_{ij} \Big) x_j \leq$$

$$\leq \sum_{j=1}^{k} x_j + \sum_{j=1}^{k} \Big(\sum_{i=1}^{k} \alpha_{ij} \Big) x_k - k x_k + \sum_{j=k+1}^{n} \Big(\sum_{i=1}^{k} \alpha_{ij} \Big) x_k =$$

$$= \sum_{j=1}^{k} x_j + \sum_{j=1}^{n} \Big(\sum_{i=1}^{k} \alpha_{ij} \Big) x_k - k x_k =$$

$$= \sum_{j=1}^{k} x_j + \sum_{i=1}^{k} \Big(\sum_{j=1}^{n} \alpha_{ij} - 1 \Big) x_k \leq \sum_{j=1}^{k} x_j .$$

By Lemma 6.1.3.8,

$$\|y\|_p \leq \|x\|_p . \qquad \blacksquare$$

Proposition 6.1.3.10 (0) *Take $u \in \mathcal{L}(E,F)$ and $p \in [1,\infty[$. If Ω denotes the class of families $((a_\iota,b_\iota))_{\iota\in I}$ such that $(a_\iota)_{\iota\in I}$ and $(b_\iota)_{\iota\in I}$ are orthonormal families in E and F, respectively, then*

$$\sum_{n\in\mathbb{N}} \theta_n(u)^p = \sup\Big\{ \sum_{\iota\in I} |\langle ua_\iota|b_\iota\rangle|^p \,\Big|\, ((a_\iota,b_\iota))_{\iota\in I} \in \Omega\Big\}.$$

First we prove the inequality $\leq$. By Theorem 5.5.5.1 d $\Rightarrow$ a, we may assume u to be compact. Let

$$\sum_{n\in N} \theta_n\langle\cdot|x_n\rangle y_n$$

be a Schatten decomposition of u. Then

$$\langle ux_n|y_n\rangle = \theta_n$$

for every $n \in N$, and so, by Proposition 6.1.2.11,

$$\sum_{n\in N} \theta_n(u)^p = \sum_{n\in N} |\langle ux_n|y_n\rangle|^p \leq$$

$$\leq \sup\Big\{ \sum_{\iota\in I} \Big|\langle ua_\iota|b_\iota\rangle|^p \,\Big|\, ((a_\iota,b_\iota))_{\iota\in I} \in \Omega\Big\}.$$

Now we prove the reverse inequality. Take $((a_\iota,b_\iota))_{\iota\in I} \in \Omega$. We may assume that $u \in \mathcal{L}^p(E,F)$. Let

$$\sum_{n\in N} \theta_n\langle\cdot|x_n\rangle y_n$$

be a Schatten decomposition of u (Corollary 6.1.2.10). Then

$$\langle ua_\iota|b_\iota\rangle = \sum_{n\in N} \theta_n\langle a_\iota|x_n\rangle\langle y_n|b_\iota\rangle$$

for every $\iota \in I$. By Proposition 5.5.1.19,

$$\sum_{n\in N} |\langle a_\iota|x_n\rangle\langle y_n|b_\iota\rangle| \leq \|a_\iota\|\,\|b_\iota\| \leq 1$$

for every $\iota \in I$ and

$$\sum_{\iota\in I} |\langle a_\iota|x_n\rangle\langle y_n|b_\iota\rangle| \leq \|x_n\|\,\|y_n\| \leq 1.$$

Thus

$$\sum_{\iota\in I} |\langle ua_\iota|b_\iota\rangle|^p \leq \sum_{n\in N} \theta_n^p = \sum_{n\in\mathbb{N}} \theta_n(u)^p$$

(Proposition 6.1.2.11, Lemma 6.1.3.9). Hence

$$\sup\Big\{ \sum_{\iota\in I} |\langle ua_\iota|b_\iota\rangle|^p \,\Big|\, ((a_\iota,b_\iota))_{\iota\in I} \in \Omega\Big\} \leq \sum_{n\in\mathbb{N}} \theta_n(u)^p. \qquad \blacksquare$$

Corollary 6.1.3.11 (0) *If u is a self-normal compact opertor on E, then*

$$\sum_{n\in\mathbb{N}} \theta_n(u)^p = \sup\Big\{ \sum_{x\in A} |\langle ux|x\rangle|^p \,\Big|\, A \text{ orthonormal basis of } E\Big\},$$

for every $p \in [1,\infty[$.

By Theorem 5.5.6.1 a $\Rightarrow$ e, there is an orthonormal basis A of E and an $f \in c_0(A)$ with

$$u = \sum_{x\in A} f(x)\langle\cdot|x\rangle x\,.$$

Then

$$f(x) = \langle ux|x\rangle$$

for every $x \in A$. By Corollary 6.1.2.12,

$$\sum_{n\in\mathbb{N}} \theta_n(u)^p = \sum_{x\in A} |f(x)|^p = \sum_{x\in A} |\langle ux|x\rangle|^p\,.$$

The assertion now follows from Proposition 6.1.3.10. ■

Theorem 6.1.3.12 (0) *Let $p \in [1,\infty[$.*

a) *$\mathcal{L}^p(E,F)$ is a vector subspace of $\mathcal{K}(E,F)$ and the map*

$$\mathcal{L}^p(E,F) \longrightarrow \mathbb{R}_+\,, \quad u \longmapsto \|u\|_p$$

is a norm. We consider $\mathcal{L}^p(E,F)$ with this norm.

b) *$\mathcal{L}^p(E,F)$ is complete.*

c) *If $(x_\iota)_{\iota\in I}$, $(y_\iota)_{\iota\in I}$ are orthonormal families in E and F, respectively, and $(\alpha_\iota)_{\iota\in I} \in \ell^p(I)$, then the family $(\alpha_\iota\langle\cdot|x_\iota\rangle y_\iota)_{\iota\in I}$ is summable in $\mathcal{L}^p(E,F)$.*

d) *$\mathcal{L}_f(E,F)$ is a dense vector subspace of $\mathcal{L}^p(E,F)$.*

a) Take $u,v \in \mathcal{L}^p(E,F)$ and let $(x_\iota)_{\iota\in I}$, $(y_\iota)_{\iota\in I}$ be orthonormal families in E and F, respectively. Then

$$\Big(\sum_{\iota\in I} |\langle (u+v)x_\iota|y_\iota\rangle|^p\Big)^{\frac{1}{p}} = \Big(\sum_{\iota\in I} |\langle ux_\iota|y_\iota\rangle + \langle vx_\iota|y_\iota\rangle|^p\Big)^{\frac{1}{p}} \leq$$

$$\leq \Big(\sum_{\iota\in I} |\langle ux_\iota|y_\iota\rangle|^p\Big)^{\frac{1}{p}} + \Big(\sum_{\iota\in I} |\langle vx_\iota|y_\iota\rangle|^p\Big)^{\frac{1}{p}} \leq \|u\|_p + \|v\|_p$$

(Proposition 6.1.3.10). Hence $u+v \in \mathcal{L}^p(E,F)$ and

$$\|u+v\|_p \leq \|u\|_p + \|v\|_p$$

(Proposition 6.1.3.10). It is easy to see that $\alpha u \in \mathcal{L}(E,F)$ and that

$$\|\alpha u\|_p = |\alpha|\,\|u\|_p$$

for every $\alpha \in \mathbb{K}$. The assertion now follows with the help of Proposition 6.1.2.4.

b) Let $(u_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in $\mathcal{L}^p(E,F)$. By Proposition 6.1.2.4 (and Corollary 6.1.2.10, Theorem 3.1.1.3), $(u_n)_{n\in\mathbb{N}}$ converges in $\mathcal{K}(E,F)$ to a u. Let $\varepsilon > 0$. There is an $n_0 \in \mathbb{N}$ such that

$$\|u_m - u_n\|_p < \varepsilon$$

for all $m,n \in \mathbb{N}\backslash\mathbb{N}_{n_0}$. Let $(x_\iota)_{\iota\in I}$, $(y_\iota)_{\iota\in I}$ be orthonormal families in E and F, respectively. Then

$$\sum_{\iota\in I} |\langle (u_m-u_n)x_\iota|y_\iota\rangle|^p \leq \|u_m-u_n\|_p^p < \varepsilon^p$$

for all $m,n \in \mathbb{N}\backslash\mathbb{N}_{n_0}$ (Proposition 6.1.3.10). Hence

$$\sum_{\iota\in I} |\langle (u-u_n)x_\iota|y_\iota\rangle|^p \leq \varepsilon^p\,,$$

$$u-u_n \in \mathcal{L}^p(E,F)\,, \quad \|u-u_n\|_p < \varepsilon$$

for every $n \in \mathbb{N}\backslash\mathbb{N}_{n_0}$ (Proposition 6.1.3.10). By a), $u \in \mathcal{L}^p(E,F)$ and $(u_n)_{n\in\mathbb{N}}$ converges to u in $\mathcal{L}^p(E,F)$. Thus $\mathcal{L}^p(E,F)$ is complete.

c) Take $\varepsilon > 0$. Choose $J \in \mathfrak{P}_f(I)$, such that

$$\sum_{\iota\in I\backslash J} |\alpha_\iota|^p < \varepsilon^p\,.$$

Then

$$\Big\|\sum_{\iota\in K} \alpha_\iota\langle\cdot|x_\iota\rangle y_\iota\Big\|_p \leq \Big(\sum_{\iota\in K} |\alpha_\iota|^p\Big)^{\frac{1}{2}} < \varepsilon$$

for every $K \in \mathfrak{P}_f(I\backslash J)$ (Proposition 6.1.3.10). By b) and Proposition 1.1.6.6, the family $(\alpha_\iota\langle\cdot|x_\iota\rangle y_\iota)_{\iota\in I}$ is summable in $\mathcal{L}^p(E,F)$.

d) follows from Corollaries 6.1.2.10, 6.1.2.12, and Theorem 6.1.3.1 a). ■

Corollary 6.1.3.13 (0) *If $u \in \mathcal{L}(E,F)$, $v \in \mathcal{L}(F,G)$, $w \in \mathcal{L}(G,H)$ and $p \in \{0\} \cup [1,\infty]$, then:*

a) $n \in \mathbb{N} \Rightarrow \theta_n(w \circ v \circ u) \leq \|w\| \theta_n(v) \|u\|$.

b) $v \in \mathcal{L}^p(F,G) \Rightarrow \begin{cases} w \circ v \circ u \in \mathcal{L}^p(E,H), \\ \|w \circ v \circ u\|_p \leq \|w\| \, \|v\|_p \|u\|. \end{cases}$

c) *The map*

$$\mathcal{L}^p(F,G) \longrightarrow \mathcal{L}^p(E,H), \quad v \longmapsto w \circ v \circ u$$

is continuous and linear with norm at most $\|w\| \, \|u\|$.

d) *$\mathcal{L}^p(E)$ is an involutive Banach algebra. If $p \in [1,\infty[$ then this Banach algebra is quasiunital iff E is one–dimensional.*

a) The inequality

$$\theta_n(w \circ v) \leq \|w\| \theta_n(v)$$

follows immediately from the definition. By Theorem 6.1.3.1 b), we deduce that

$$\theta_n(v \circ u) = \theta_n((v \circ u)^*) = \theta_n(u^* \circ v^*) \leq \|u^*\| \theta_n(v^*) = \theta_n(v) \|u\|.$$

Hence

$$\theta_n(w \circ v \circ u) \leq \|w\| \theta_n(v \circ u) \leq \|w\| \theta_n(v) \|u\|.$$

b) follows from a) and Proposition 3.1.1.11.

c) follows from b) and Theorem 6.1.3.12 a).

d) By Theorem 6.1.3.12 b) and Corollary 6.1.3.2, $\mathcal{L}^p(E)$ is an involutive Banach space. By b) and Proposition 6.1.2.4, $\mathcal{L}^p(E)$ is an involutive Banach algebra.

Let $\mathfrak{F}$ be an approximate unit of E. Assume that E is not one–dimensional. Then there are $x, y \in E$ such that

$$\|x\| = \|y\| = 1, \quad x \perp y.$$

Then, by b),

$$\|(\langle \cdot | x \rangle x + \langle \cdot | y \rangle y) \circ u\|_p \leq \|u\|_p \leq 1$$

for every $u \in \mathcal{L}^p(E)^\#$. We get by Proposition 6.1.2.11,

$$2^{\frac{1}{p}} = \|\langle \cdot | x \rangle x + \langle \cdot | y \rangle y\|_p = \lim_{u,\mathfrak{F}} \|(\langle \cdot | x \rangle x + \langle \cdot | y \rangle y) \circ u\|_p \leq 1,$$

which is the expected contradiction. ■

Corollary 6.1.3.14 (0) *Take* $p \in \{0\} \cup [1, \infty]$. *Then* $\mathcal{L}^p(E)$ *is an involutive ideal of* $\mathcal{L}(E)$ *and an involutive unital* $\mathcal{L}(E)$*-algebra.*

The assertion follows from Theorem 6.1.3.12 a),b) and Corollary 6.1.3.13 b),d). ■

Corollary 6.1.3.15 (Fatou's Lemma) *Take* $p \in [1, \infty[$, $u \in \mathcal{L}(E, F)$, *and let* $(u_n)_{n \in \mathbb{N}}$ *be a bounded sequence in* $\mathcal{L}^p(E, F)$ *converging pointwise to* u. *Then* $u \in \mathcal{L}^p(E, F)$ *and*

$$\|u\|_p \leq \liminf_{n \to \infty} \|u_n\|_p .$$

Let $(x_\iota)_{\iota \in I}$, $(y_\iota)_{\iota \in I}$ be orthonormal families in E and F, respectively. By Proposition 6.1.3.10,

$$\sum_{\iota \in I} |\langle u x_\iota | y_\iota \rangle|^p \leq \liminf_{n \to \infty} \sum_{\iota \in I} |\langle u_n x_\iota | y_\iota \rangle|^p \leq \liminf_{n \to \infty} \|u_n\|_p^p < \infty ,$$

$u \in \mathcal{L}^p(E, F)$, and

$$\|u\|_p \leq \liminf_{n \to \infty} \|u_n\|_p . \qquad \blacksquare$$

Remark. The hypothesis that $(u_n)_{n \in \mathbb{N}}$ is bounded in $\mathcal{L}^p(E, F)$ cannot be relinquished, as the following example shows: Take $p := 1$, $E := F := \ell^2$. Put

$$u := \sum_{k = \mathbb{N}} \frac{1}{k} \langle \cdot | e_k \rangle e_k ,$$

and

$$u_n := \sum_{k=1}^{n} \frac{1}{k} \langle \cdot | e_k \rangle e_k ,$$

for every $n \in \mathbb{N}$.

Corollary 6.1.3.16 *Let* $\mathbb{K} = \mathbb{C}$. *Take* $p \in \{0\} \cup [1, \infty[$ *and* $u \in \mathcal{L}(E)$. *Then* $u \in \mathcal{L}^p(E)$ *iff* $\operatorname{re} u$, $\operatorname{im} u \in \mathcal{L}^p(E)$ *and in this case*

$$\sup\{\|\operatorname{re} u\|_p , \|\operatorname{im} u\|_p\} \leq \|u\|_p \leq \|\operatorname{re} u\|_p + \|\operatorname{im} u\|_p .$$

If $u \in \mathcal{L}^p(E)$, then, by Corollary 6.1.3.2, $u^* \in \mathcal{L}^p(E)$ and

$$\|u^*\|_p = \|u\|_p .$$

By Theorem 6.1.3.12 a), $\operatorname{re} u$, $\operatorname{im} u \in \mathcal{L}^p(E)$ and

$$\|\mathrm{re}\, u\|_p \leq \|u\|_p\,, \quad \|\mathrm{im}\, u\|_p \leq \|u\|\,.$$

If $\mathrm{re}\, u\,,\ \mathrm{im}\, u \in \mathcal{L}^p(E)$, then, by Theorem 6.1.3.12 a), $u \in \mathcal{L}^p(E)$ and

$$\|u\|_p \leq \|\mathrm{re}\, u\|_p + \|\mathrm{im}\, u\|_p\,. \qquad \blacksquare$$

Corollary 6.1.3.17 *Let $u \in \mathcal{L}(E)$ and $p \in [1, \infty[$. If $\mathbb{K} = \mathbb{C}$ and*

$$\sup\left\{\sum_{x\in A} |\langle ux|x\rangle|^p \,\middle|\, A \text{ orthonormal basis of } E\right\} < \infty$$

then $u \in \mathcal{L}^p(E)$.

Since

$$\langle (\mathrm{re}\, u)x|x\rangle = \mathrm{re}\,\langle ux|x\rangle$$

$$\langle (\mathrm{im}\, u)x|x\rangle = \mathrm{im}\,\langle ux|x\rangle$$

for every $x \in A$, it follows

$$\sup\left\{\sum_{x\in A} |\langle (\mathrm{re}\, u)x|x\rangle|^p \,\middle|\, A \text{ orthonormal basis of } E\right\} < \infty\,,$$

$$\sup\left\{\sum_{x\in A} |\langle (\mathrm{im}\, u)x|x\rangle|^p \,\middle|\, A \text{ orthonormal basis of } E\right\} < \infty\,.$$

By Corollary 6.1.3.11, $\mathrm{re}\, u\,,\ \mathrm{im}\, u \in \mathcal{L}^p(E)$. Hence $u \in \mathcal{L}^p(E)$ by Corollary 6.1.3.16. $\blacksquare$

Corollary 6.1.3.18 (0) *Let u be a compact self-normal operator on E. Then there are an orthonormal basis A of E and an $f \in \mathbb{K}^A$ such that*

$$ux = f(x)x$$

for every $x \in A$. If A and f are as above, then:

a) *Given $p \in [1, \infty[\cup \{0\}$, $u \in \mathcal{L}^p(E)$ iff $f \in \ell^p(A)$ and in this case*

$$\|u\|_p = \|f\|_p$$

and

$$u = \sum_{x\in A} f(x)\langle\cdot|x\rangle x \qquad (\text{in } \mathcal{L}^p(E))\,.$$

b) $\sigma_p(u) = f(A)$.

c) $y \in E \Rightarrow uy = \sum_{x \in A} f(x)\langle y|x\rangle x$.

d) *If* $\alpha \in \mathbb{K}\backslash\{0\}$, *then*

$$\mathrm{Im}\,(\alpha 1 - u) = \overset{-1}{f}(\alpha)^{\perp}$$

and

$$\frac{1}{\alpha}y + \frac{1}{\alpha}\sum_{y \in A\backslash \overset{-1}{f}(\alpha)} \frac{f(x)\langle y|x\rangle}{\alpha - f(x)}x \in \overbrace{\alpha 1 - u}^{-1}(y)$$

for every $y \in \mathrm{Im}\,(\alpha 1 - u)$.

By Corollary 6.1.2.13 a $\Rightarrow$ b, there are an orthonormal basis A of E and an $f \in \mathbb{K}^A$ such that

$$ux = f(x)x$$

for every $x \in A$.

a) By Theorem 5.5.5.4 a & $j_1 \Rightarrow j_2$ & j_3, $f \in c_0(A)$ and

$$u = \sum_{x \in A} f(x)\langle \cdot|x\rangle x\,.$$

By Corollary 6.1.2.12, $u \in \mathcal{L}^p(E)$ iff $f \in \ell^p(A)$ and in this case

$$\|u\|_p = \|f\|_p\,.$$

Moreover, by Theorem 6.1.3.12 c), if $f \in \ell^p(A)$ then

$$u = \sum_{x \in A} f(x)\langle \cdot|x\rangle x \qquad (\text{in } \mathcal{L}^p(E))\,.$$

b) follows from Proposition 5.5.5.7 c).

c) follows from a).

d) follows from a) and Theorem 5.5.6.1 e $\Rightarrow$ h. ■

Proposition 6.1.3.19 *If* $E \neq \{0\}$, $F \neq \{0\}$, *and* $p \in [1, \infty[$, *then* $\mathcal{L}^p(E,F)$ *is separable iff* E *and* F *are separable.*

First suppose that E and F are separable. Let A and B be countable dense subsets of E and F, respectively. Take $(x,y) \in E \times F$. There are sequences $(x_n)_{n\in\mathbb{N}}$, $(y_n)_{n\in\mathbb{N}}$ in A and B, respectively, which converge to x and y, respectively. Take $n \in \mathbb{N}$. Then

$$\|\langle\cdot|x_n\rangle y_n - \langle\cdot|x\rangle y\|_p \leq \|\langle\cdot|x_n - x\rangle y_n\|_p + \|\langle\cdot|x\rangle(y_n - y)\|_p =$$

$$= \|x_n - x\| \, \|y_n\| + \|x\| \, \|y_n - y\|$$

(Proposition 6.1.2.3), so

$$\lim_{n\to\infty} \langle\cdot|x_n\rangle y_n = \langle\cdot|x\rangle y \,.$$

Let C be a countable dense subset of $\mathbb{K}$ and let $\mathcal{F}$ be the set of linear combinations of the elements of $\{(\langle\cdot|x\rangle y|(x,y) \in A \times B\}$ with coefficients from C. By the above considerations

$$\mathcal{L}_f(E,F) \subset \overline{\mathcal{F}} \,,$$

where $\overline{\mathcal{F}}$ denotes the closure of $\mathcal{F}$ in $\mathcal{L}^p(E,F)$. Since $\mathcal{F}$ is countable and since $\mathcal{L}_f(E,F)$ is dense in $\mathcal{L}^p(E,F)$ (Theorem 6.1.3.12 d)), it follows that $\mathcal{L}^p(E,F)$ is separable.

Now suppose that F is not separable. Since $\mathcal{L}^p(\mathbb{K},F)$ is isomorphic to F, it follows that $\mathcal{L}^p(\mathbb{K},F)$ is not separable. Hence $\mathcal{L}^p(E,F)$ is not separable.

Finally, suppose that E is not separable. By Corollary 6.1.3.2, the map

$$\mathcal{L}^p(E,F) \longrightarrow \mathcal{L}^p(F,E) \,, \quad u \longmapsto u^*$$

is an isometry of real Banach spaces. By the above considerations, $\mathcal{L}^p(E,F)$ is not separable. ■

Proposition 6.1.3.20 (Lebesgue's Dominated Convergence Theorem) *Take* $u \in \mathcal{L}(E,F)$, $p \in [1,\infty[$, $v \in \mathcal{L}^p(E,F)$, *and let* $(u_n)_{n\in\mathbb{N}}$ *be a sequence in* $\mathcal{L}^p(E,F)$ *such that*

$$\theta_m(u_n) \leq \theta_m(v)$$

for all $m,n \in \mathbb{N}$ *and*

$$\lim_{n\to\infty} \|u_n - u\| = 0 \,.$$

Then $u \in \mathcal{L}^p(E,F)$ *and*

$$\lim_{n\to\infty} \|u_n - u\|_p = 0 \,.$$

Choose $m \in \mathbb{N}$. Then

$$\theta_m(u_n - u) \leq \|u_n - u\|$$

for every $n \in \mathbb{N}$, so that

$$\lim_{n\to\infty} \theta_m(u_n - u) = 0 .$$

By Proposition 6.1.2.5,

$$\theta_m(u) \leq \theta_1(u_n - u) + \theta_m(u_n) \leq \|u_n - u\| + \theta_m(v) ,$$

$$\theta_m(u) \leq \theta_m(v) , \quad u \in \mathcal{L}^p(E,F) ,$$

$$\theta_{2m-1}(u_n - u) \leq \theta_m(u_n) + \theta_m(u) \leq 2\theta_m(v) ,$$

$$\theta_{2m}(u_n - u) \leq \theta_m(u_n) + \theta_{m+1}(u) \leq 2\theta_m(v) .$$

Take $\varepsilon > 0$. There is an $m_0 \in \mathbb{N}$ with

$$\sum_{m=m_0}^{\infty} \theta_m(v)^p < \frac{\varepsilon^p}{8} .$$

There is, furthermore, an $n_0 \in \mathbb{N}$ such that

$$\sum_{m=1}^{2m_0} \theta_m(u_n - u)^p < \frac{\varepsilon^p}{2} ,$$

for every $n \in \mathbb{N}$, with $n \geq n_0$. It follows that

$$\|u_n - u\|_p^p = \sum_{m=1}^{\infty} \theta_m(u_n - u)^p =$$

$$= \sum_{m=1}^{2m_0} \theta_m(u_n - u)^p + \sum_{m=1}^{\infty} \theta_{2(m_0+m)}(u_n - u)^p + \sum_{m=1}^{\infty} \theta_{2(m_0+m)-1}(u_n - u)^p <$$

$$< \frac{\varepsilon^p}{2} + 2\sum_{m=1}^{\infty} \theta_{m_0+m}(v) + 2\sum_{m=1}^{\infty} \theta_{m_0+m}(v) <$$

$$< \frac{\varepsilon^p}{2} + \frac{\varepsilon^p}{4} + \frac{\varepsilon^p}{4} = \varepsilon^p$$

for every $n \in \mathbb{N}$, with $n \geq n_0$, i.e.

$$\lim_{n\to\infty} \|u_n - u\|_p = 0 .$$

■

Theorem 6.1.3.21 (0) *Let p, q be conjugate exponents. Take $u \in \mathcal{L}^p(E,F)$ and $v \in \mathcal{L}^q(F,G)$. Then $v \circ u \in \mathcal{L}^1(E,G)$ and*

$$\|v \circ u\|_1 \leq \|u\|_p \|v\|_q \qquad \textbf{(Hölder inequality)}.$$

By Corollary 6.1.3.13 b), we may assume that $p, q \in [1, \infty[$. Let

$$\sum_{m \in M} \alpha_m \langle \cdot | a_m \rangle x_m \,, \quad \sum_{m \in N} \beta_n \langle \cdot | y_n \rangle b_n$$

by Schatten decompositions of u and v, respectively. Put

$$w = ((v \circ u)^* \circ (v \circ u))^{\frac{1}{2}} .$$

By Proposition 5.5.5.10, there is a $t \in \mathcal{L}(E,G)$ such that

$$\operatorname{Ker}(v \circ u) = \operatorname{Ker} t \,, \quad v \circ u = t \circ w \,.$$

Let A be an orthonormal basis of E and take $B \in \mathfrak{P}_f(A)$. Then

$$w = t^* \circ v \circ u$$

(Proposition 5.5.5.10 e)), so

$$\langle wx | x \rangle = \langle (t^* \circ v \circ u)x | x \rangle = \langle ux | v^* tx \rangle =$$

$$= \Big\langle \sum_{m \in M} \alpha_m \langle x | a_m \rangle x_m \,\Big|\, \sum_{n \in N} \overline{\beta}_n \langle tx | b_n \rangle y_n \Big\rangle =$$

$$= \sum_{m \in M} \sum_{n \in N} \alpha_m \beta_n \langle x | a_m \rangle \langle t^* b_n | x \rangle \langle x_m | y_n \rangle$$

for every $x \in A$ (Proposition 5.3.2.13 b)). It follows that

$$\sum_{x \in B} \langle wx | x \rangle = \sum_{m \in M} \sum_{n \in N} \alpha_m \beta_n \langle x_m | y_n \rangle \langle \pi_{B^{\perp\perp}} t^* b_n | \pi_{B^{\perp\perp}} a_m \rangle$$

(Proposition 5.5.1.7). We have

$$\sum_{n \in N} |\langle x_m | y_n \rangle \langle \pi_{B^{\perp\perp}} t^* b_n | \pi_{B^{\perp\perp}} a_m \rangle| =$$

$$= \sum_{n \in N} |\langle x_m | y_n \rangle \langle b_n | t \pi_{B^{\perp\perp}} a_m \rangle| \leq \|x_m\| \, \|t \pi_{B^{\perp\perp}} a_m\| \leq 1$$

for every $m \in M$ and

$$\sum_{m\in M} |\langle x_m|y_n\rangle\langle \pi_{B^{\perp\perp}} t^* b_n|\pi_{B^{\perp\perp}} a_m\rangle| =$$

$$= \sum_{m\in M} |\langle x_m|y_n\rangle\langle \pi_{B^{\perp\perp}} t^* b_n|a_m\rangle| \leq \|y_n\|\,\|\pi_{B^{\perp\perp}} t^* b_n\| \leq 1$$

for every $n \in N$ (Propositions 5.2.3.1 a $\Rightarrow$ b, 5.5.1.19, 5.5.5.10 c)). Given $m \in M$, put

$$\gamma_m := \sum_{n\in N} \beta_n \langle x_m|y_n\rangle\langle \pi_{B^{\perp\perp}} t^* b_n|\pi_{B^{\perp\perp}} a_m\rangle\,.$$

By Lemma 6.1.3.9, $(\gamma_m)_{m\in M} \in \ell^q(M)$ and

$$\|(\gamma_m)_{m\in M}\|_q \leq \|(\beta_n)_{n\in\mathbb{N}}\|_q = \|v\|_q\,.$$

By the classical Hölder inequality,

$$\sum_{x\in B} |\langle wx|x\rangle| \leq \sum_{m\in M} |\alpha_m\gamma_m| \leq \|(\alpha_m)_{m\in M}\|_p \|(\gamma_m)_{m\in M}\|_q \leq \|u\|_p\|v\|_q\,.$$

Since B is arbitrary,

$$\sum_{x\in A} |\langle wx|x\rangle| \leq \|u\|_p\|v\|_q\,.$$

By Corollary 6.1.3.11, $w \in \mathcal{L}^1(E)$ and

$$\|w\|_1 \leq \|u\|_p\|v\|_q\,.$$

It follows that

$$v\circ u = t\circ w \in \mathcal{L}^1(E,G)\,,$$

$$\|v\circ u\|_1 = \|t\circ w\|_1 \leq \|t\|\,\|w\|_1 \leq \|u\|_p\|v\|_q$$

(Corollary 6.1.3.13 b), Proposition 5.5.5.10 c)). ■

Proposition 6.1.3.22 (0) *Take* $u, v \in \mathcal{L}(E,F)$ *with*

$$u\circ v^* = 0 \text{ and } v^*\circ u = 0\,.$$

a) $(\operatorname{Ker} u)^\perp \subset \operatorname{Ker} v$, $(\operatorname{Ker} u^*)^\perp \subset \operatorname{Ker} v^*$, $\operatorname{Im} u \perp \operatorname{Im} v$.

b) $\|u+v\| = \sup(\|u\|, \|v\|)$.

c) *If u, v are compact, then there are orthonormal families $(x_\iota)_{\iota\in I}$ in E and $(y_\iota)_{\iota\in I}$ in F, respectively, and functions $f, g \in \mathbb{K}^I$ such that*

$$u = \sum_{\iota\in I} f(\iota)\langle\cdot|x_\iota\rangle y_\iota\,, \quad v = \sum_{\iota\in I} g(\iota)\langle\cdot|x_\iota\rangle y_\iota\,, \quad fg = 0\,.$$

d) *If $p \in [1,\infty[$ and $u, v \in \mathcal{L}^p(E,F)$, then*

$$\|u+v\|_p^p = \|u\|_p^p + \|v\|_p^p\,.$$

a) We have

$$\operatorname{Im} v^* \subset \operatorname{Ker} u\,, \quad \operatorname{Im} u \subset \operatorname{Ker} v^*\,,$$

so

$$(\operatorname{Ker} v)^\perp = \overline{\operatorname{Im} v^*} \subset \operatorname{Ker} u\,, \quad (\operatorname{Ker} u^*)^\perp = \overline{\operatorname{Im} u} \subset \operatorname{Ker} v^* = (\operatorname{Im} v)^\perp$$

(Proposition 5.3.2.4) and

$$(\operatorname{Ker} u)^\perp \subset (\operatorname{Ker} v)^{\perp\perp} = \operatorname{Ker} v\,, \quad \operatorname{Im} u \perp \operatorname{Im} v\,.$$

b) We may assume that

$$\|u\| \leq \|v\|\,.$$

Put

$$G := \operatorname{Ker} u\,.$$

By a),

$$\|vx\| = \|v\pi_G x\| = \|(u+v)\pi_G x\| \leq \|u+v\|\,\|\pi_G x\| \leq \|u+v\|\,\|x\|$$

for every $x \in E$, so that

$$\|v\| \leq \|u+v\|\,.$$

By a),

$$\|(u+v)x\|^2 = \|ux+vx\|^2 = \|u\pi_{G^\perp}x + v\pi_G x\|^2 =$$

$$= \|u\pi_{G^\perp}x\|^2 + \|v\pi_G x\|^2 \leq \|u\|^2\|\pi_G x\|^2 + \|v\|^2\|\pi_G x\|^2 \leq$$

$$\leq \|v\|^2\Big(\|\pi_{G^\perp}x\|^2 + \|\pi_G x\|^2\Big) = \|v\|^2\|x\|^2$$

for every $x \in E$, and so

$$\|u+v\| \leq \|v\|\,.$$

c) Let

$$u = \sum_{n\in N(u)} \theta_n\langle\cdot|x_n\rangle a_n\,, \quad v = \sum_{n\in N(v)} \theta_n\langle\cdot|y_n\rangle b_n$$

be Schatten decomposition of u and v. Then

$$\{a_n|n \in N(u)\} \subset \operatorname{Im} u\,, \quad \{b_n|n \in N(v)\} \subset \operatorname{Im} v$$

and

$$\{x_n|n \in N(u)\} \subset (\operatorname{Ker} u)^\perp\,, \quad \{y_n|n \in N(v)\} \subset (\operatorname{Ker} v)^\perp$$

(Theorem 5.5.5.4 a $\Rightarrow$ g). By a),

$$\{a_n|n \in N(u)\} \perp \{b_n|n \in N(v)\}\,,$$

$$\{x_n|n \in N(u)\} \perp \{y_n|n \in N(v)\}\,,$$

and now the assertion follws.

d) follows from c) and Corollary 6.1.2.12. ■

Proposition 6.1.3.23 (0) *If E and F are infinite-dimensional then for every $u \in \mathcal{K}(E,F)^\#$ there is a pair $(x,y) \in E \times F$ with $x \neq 0\,, y \neq 0$ such that*

$$\|u \pm \langle\cdot|x\rangle y\| \leq 1\,.$$

In particular, $\mathcal{K}(E,F)^\#$ has no extreme points and $\mathcal{K}(E,F)$ is not a dual space.

Let

$$u = \sum_{n\in N} \theta_n\langle\cdot|x_n\rangle y_n$$

be a Schatten decomposition of u. First suppose that N is finite. Then there are

$$x \in \{x_n | n \in \mathbb{N}\}^\perp , \quad y \in \{y_n | n \in \mathbb{N}\}^\perp$$

with

$$\|x\| = \|y\| = 1 .$$

By Proposition 6.1.3.22 b),

$$\|u \pm \langle \cdot | x \rangle y\| \leq 1 .$$

Now suppose that N is infinite. Then there is an $n \in N$ with $\theta_n < 1$. Put

$$x := (1 - \theta_n) x_n , \quad y := y_n .$$

By Corollary 6.1.2.12,

$$\|u \pm \langle \cdot | x \rangle y\| \leq 1 .$$

u is not an extreme point of $\mathcal{K}(E,F)^\#$, since

$$u \pm \langle \cdot | x \rangle y \in \mathcal{K}(E,F)^\#$$

and

$$u = \frac{1}{2}(u + \langle \cdot | x \rangle y) + \frac{1}{2}(u - \langle \cdot | x \rangle y) .$$

By Corollary 1.3.1.12, $\mathcal{K}(E,F)$ is not a dual space. ■

Lemma 6.1.3.24 *Let $n \in \mathbb{N}$ and let u be a nilpotent operator on $\mathbb{K}^n$. Then there is an orthonormal basis $(x_m)_{m \in \mathbb{N}_n}$ of $\mathbb{K}^n$ such that ux_m belongs to the vector subspace of $\mathbb{K}^n$ generated by $(x_p)_{p \in \mathbb{N}_{m-1}}$ for every $m \in \mathbb{N}_n$.*

We prove the assertion by complete induction. u^* is nilpotent, and so it is not invertible. Hence there is an $x_n \in \mathbb{K}^n$ with

$$\|x_n\| = 1 , \quad u^* x_n = 0 .$$

Thus

$$\langle uy | x_n \rangle = \langle y | u^* x_n \rangle = 0$$

for every $y \in \mathbb{K}^n$, i.e.

$$\operatorname{Im} u \subset \{x_n\}^\perp .$$

By the hypothesis of the induction, there is an orthonormal basis $(x_m)_{m \in \mathbb{N}_{n-1}}$ of $\{x_n\}^\perp$ such that, for every $m \in \mathbb{N}_{n-1}$, ux_m belongs to the vector subspace of $\{x_n\}^\perp$ generated by $(x_p)_{p \in \mathbb{N}_{m-1}}$ The family $(x_m)_{m \in \mathbb{N}_n}$ has the required property. ■

Proposition 6.1.3.25 *For every $u \in \mathcal{K}(E)$, there is an orthonormal basis A of the closed vector subspace of E generated by*

$$\bigcup_{\alpha \in \mathbb{K}\backslash\{0\}} E_a(1 - \frac{1}{\alpha}u)$$

(Definition 3.1.3.18) such that $(\langle ux|x\rangle)_{x\in A}$ is a characteristic family of eigenvalues of u.

Take $\alpha \in \mathbb{K}\backslash\{0\}$. By Theorem 3.1.3.17 b),d), $E_a(1 - \frac{1}{\alpha}u)$ is finite-dimensional and invariant with respect to u, and the map

$$E_a(1 - \frac{1}{\alpha}u) \longrightarrow E_a(1 - \frac{1}{\alpha}u), \quad x \longmapsto (1 - \frac{1}{\alpha}u)x$$

is nilpotent. By Lemma 6.1.3.24, there is an orthonormal basis $(x_{\alpha n})_{n\in\mathbb{N}_{q(\alpha)}}$ of $E_a(1 - \frac{1}{\alpha}u)$ such that $x_{\alpha n} - \frac{1}{\alpha}ux_{\alpha n}$ belongs to the vector subspace of $E_a(1 - \frac{1}{\alpha}u)$ generated by $(x_{\alpha m})_{m\in\mathbb{N}_{n-1}}$ for any $n \in \mathbb{N}_{q(\alpha)}$.

By Proposition 3.1.4.3, the set

$$B := \{x_{\alpha n} | \alpha \in \mathbb{K}\backslash\{0\}, n \in \mathbb{N}_{q(\alpha)}\}$$

is linearly independent. By Theorem 3.1.5.1 c), B is countable. We write B as a family $(y_n)_{n\in N}$ such that N is an initial segment of $\mathbb{N}$, and that for every $n \in N$, $\alpha y_n - uy_n$ belongs to the vector subspace of E generated by $(y_m)_{m\in\mathbb{N}_{n-1}}$. Let $(x_n)_{n\in N}$ be the family constructed from $(y_n)_{n\in N}$ by the Gram–Schmidt orthonormalization. $(x_n)_{n\in N}$ is an orthonormal basis of the closed vector subspace F of E generated by

$$\bigcup_{\alpha \in \mathbb{K}\backslash\{0\}} E_a(1 - \frac{1}{\alpha}u).$$

Take $\alpha \in \sigma(u)\backslash\{0\}$ and choose $n \in N$ such that y_n is an algebraic eigenvector of u corresponding to α. Then $\alpha x_n - ux_n$ belongs to the vector subspace of E generated by $(x_m)_{m\in\mathbb{N}_{n-1}}$. It follows that

$$\alpha - \langle ux_n|x_n\rangle = \langle \alpha x_n - ux_n|x_n\rangle = 0$$

$$\langle ux_n|x_n\rangle = \alpha.$$

Hence

$$\text{Card}\,\{n \in N | \langle ux_n|x_n\rangle = \alpha\}$$

agrees with the algebraic multiplicity $q(\alpha)$ of α with respect to u and $(\langle ux_n|x_n\rangle)_{n\in N}$ is a characteristic family of eigenvalues of u. ■

Corollary 6.1.3.26 (H. Weyl) *If u is a compact operator on E and $(\alpha_\iota)_{\iota\in I}$ is a characteristic family of eigenvalues of u, then*

$$\sum_{\iota\in I}|\alpha_\iota|^p \leq \sum_{n\in N(u)}\theta_n(u)^p\,.$$

By Proposition 6.1.3.25, there is an orthonormal family $(x_\iota)_{\iota\in I}$ in E such that

$$\langle ux_\iota|x_\iota\rangle = \alpha_\iota$$

for every $\iota\in I$. It follows that

$$\sum_{\iota\in I}|\alpha_\iota|^p = \sum_{\iota\in I}|\langle ux_\iota|x_\iota\rangle|^p \leq \sum_{n\in\mathbb{N}}\theta_n(u)^p$$

(Proposition 6.1.3.10). ■

Proposition 6.1.3.27 *Take $p\in\{0\}\cup[1,\infty[$ and $u\in\mathcal{L}^p(F,G)$. Then the map*

$$\mathcal{L}(F,E)_F^\#\times\mathcal{L}(G,H)_G^\#\longrightarrow\mathcal{L}^p(E,H)\,,\quad (v,w)\longmapsto w\circ u\circ v^*$$

(Theorem 6.1.3.12 a), Corollary 6.1.3.13 b)) is continuous. (Recall that $\mathcal{L}(F,E)_F^\#$ denotes the set $\mathcal{L}(F,E)^\#$ endowed with the topology of pointwise convergence.)

Case 1 $\exists\,(x,y)\in F\times G\,,\quad u=\langle\cdot|x\rangle y$

Take $(v_0,w_0)\in\mathcal{L}(F,E)\times\mathcal{L}(G,H)$. Then

$$\|w\circ u\circ v^*-w_0\circ u\circ v_0^*\|_p = \|\langle\cdot|vx\rangle wy-\langle\cdot|v_0x\rangle w_0y\|_p \leq$$

$$\leq \|\langle\cdot|(v-v_0)x\rangle wy\|_p+\|\langle\cdot|v_0x\rangle(w-w_0)y\|_p =$$

$$= \|(v-v_0)x\|\,\|wy\|+\|v_0x\|\,\|(w-w_0)y\|$$

for every $(v,w)\in\mathcal{L}(F,E)\times\mathcal{L}(G,H)$ (Proposition 5.3.2.13 c),d), Proposition 6.1.2.3), so that

$$\lim_{(v,w)\mapsto(v_0,w_0)} w\circ u\circ v^* = w_0\circ u\circ v_0^*\,.$$

Case 2 $u\in\mathcal{L}_f(F,G)$.

This follows from Case 1 and Corollary 5.5.1.11.

Case 3 $u \in \mathcal{L}^p(F,G)$

Take $\varepsilon > 0$. By Corollary 5.2.3.4 and Theorem 6.1.3.12 d), there is a $u_0 \in \mathcal{L}_f(F,G)$ such that

$$\|u_0 - u\|_p < \varepsilon .$$

We have

$$\|w \circ u_0 \circ v^* - w \circ u \circ v^*\|_p = \|w \circ (u_0 - u) \circ v^*\|_p \leq$$

$$\leq \|w\| \, \|u_0 - u\|_p \|v\| < \varepsilon$$

for every $(v,w) \in \mathcal{L}(E,F)^\# \times \mathcal{L}(G,H)^\#$ (Corollary 6.1.3.13 b)). By Case 2, the map associated to u_0 is continuous, so our map, being the uniform limit of continuous maps, is continuous. ■

Proposition 6.1.3.28 *Let A be an orthonormal basis of E and take $p \in \{0\} \cup [1,\infty[$. If $\mathbb{K} = \mathbb{C}$ then the vector subspace of $\mathcal{L}^p(E)$ generated by*

$$\{\langle \cdot | x + i^k y \rangle (x + i^k y) | x, y \in A \, , \, k \in \mathbb{N}_4\}$$

(Proposition 6.1.2.3) is dense in $\mathcal{L}^p(E)$.

Let $\mathcal{F}$ be the vector subspace of $\mathcal{L}^p(E)$ generated by

$$\{\langle \cdot | x + i^k y \rangle (x + i^k y) | x, y \in A \, , \, k \in \mathbb{N}_4\}$$

and $\overline{\mathcal{F}}$ its closure in $\mathcal{L}^p(E)$.

Step 1 $\langle \cdot | y \rangle x \in \mathcal{F}$ for all $x, y \in A$

Since the map

$$E \times E \longrightarrow \mathcal{L}^p(E) \, , \quad (\xi, \eta) \longmapsto \langle \cdot | \eta \rangle \xi$$

is sesquilinear (Proposition 5.3.2.13 a)),

$$\langle \cdot | y \rangle x = \frac{1}{4} \sum_{k=1}^{4} i^k \langle \cdot | x + i^k y \rangle (x + i^k y) \in \mathcal{F}$$

by the polarization identity (Proposition 2.3.3.2 d)).

Step 2 $\langle\cdot|x\rangle\xi \in \overline{\mathcal{F}}$ for every $(x,\xi) \in A \times E$

Take $\varepsilon > 0$. There is a $B \in \mathfrak{P}_f(A)$ such that

$$\left\|\xi - \sum_{y\in B}\langle\xi|y\rangle y\right\| < \varepsilon .$$

Thus

$$\left\|\langle\cdot|x\rangle\xi - \sum_{y\in B}\langle\xi|y\rangle\langle\cdot|x\rangle y\right\|_p =$$

$$= \left\|\langle\cdot|x\rangle(\xi - \sum_{y\in B}\langle\xi|y\rangle y)\right\|_p = \|x\| \left\|\xi - \sum_{y\in B}\langle\xi|y\rangle y\right\| < \varepsilon$$

(Proposition 6.1.2.3). Since ε is arbitrary, it follows from Step 1 that

$$\langle\cdot|x\rangle\xi \in \overline{\mathcal{F}} .$$

Step 3 $\langle\cdot|\xi\rangle\eta \in \overline{\mathcal{F}}$ for all $\xi,\eta \in E$

Take $\varepsilon > 0$. There is a $B \in \mathfrak{P}_f(A)$ such that

$$\|\xi - \sum_{x\in B}\langle\xi|x\rangle x\| < \varepsilon .$$

Thus

$$\|\langle\cdot|\xi\rangle\eta - \sum_{x\in B}\overline{\langle\xi|x\rangle}\ \langle\cdot|x\rangle\eta\|_p =$$

$$= \|\langle\cdot|\xi - \sum_{x\in B}\langle\xi|x\rangle x\rangle\eta\|_p = \|\xi - \sum_{x\in B}\langle\xi|x\rangle x\| \ \|\eta\| < \varepsilon$$

(Proposition 6.1.2.3). Since ε is arbitrary, Step 2 implies that $\langle\cdot|\xi\rangle\eta \in \overline{\mathcal{F}}$.

Step 4 The assertion

By Step 3 and Corollary 5.5.1.11,

$$\mathcal{L}_f(E) \subset \overline{\mathcal{F}} .$$

By Theorem 6.1.3.12 d) and Corollary 5.2.3.4,

$$\overline{\mathcal{F}} = \mathcal{L}^p(E) .$$

■

6.1.4 Hilbert–Schmidt Operators

Definition 6.1.4.1 (0) *The operators of* $\mathcal{L}^2$ *are called* ***Hilbert–Schmidt operators*** (Carleman, 1921)

Proposition 6.1.4.2 (0) *If* A *and* B *are orthonormal bases of* E *and* F, *respectively, then*

$$\sum_{x\in A}\|ux\|^2 = \sum_{(x,y)\in A\times B}|\langle ux|y\rangle|^2 = \sum_{y\in B}\|u^*y\|^2$$

for every $u \in \mathcal{L}(E,F)$. *If the above sums are finite, then* u *is compact.*

By Parseval's equation,

$$\sum_{x\in A}\|ux\|^2 = \sum_{x\in A}\sum_{y\in B}|\langle ux|y\rangle|^2 = \sum_{(x,y)\in A\times B}|\langle ux|y\rangle|^2 =$$

$$= \sum_{(x,y)\in A\times B}|\langle x|u^*y\rangle|^2 = \sum_{y\in B}\sum_{x\in A}|\langle x|u^*y\rangle|^2 = \sum_{y\in B}\|u^*y\|^2 .$$

The final assertion follows from Theorem 5.5.5.1 d $\Rightarrow$ a. ■

Proposition 6.1.4.3 (0) *If* A *is an orthonormal basis of* E *and* $u \in \mathcal{L}(E,F)$. *Then the following are equivalent:*

a) $u \in \mathcal{L}^2(E,F)$.

b) $\sum_{x\in A}\|ux\|^2 < \infty$.

If these conditions are fulfilled. then

$$\|u\|_2 = \Big(\sum_{x\in A}\|ux\|^2\Big)^{\frac{1}{2}} .$$

By Corollary 6.1.2.10 and Proposition 6.1.4.2, we may assume that u is compact. Let

$$\sum_{n\in N}\theta_n\langle\cdot|x_n\rangle y_n$$

be a Schatten decomposition of u. Then

$$ux = \sum_{n\in N}\theta_n\langle x|x_n\rangle y_n ,$$

$$\|ux\|^2 = \sum_{n\in N} \theta_n^2 |\langle x|x_n\rangle|^2$$

for every $x \in A$ (Parseval's equation), so

$$\sum_{x\in A} \|ux\|^2 = \sum_{x\in A}\sum_{n\in N} \theta_n^2 |\langle x|x_n\rangle|^2 =$$

$$= \sum_{n\in N} \theta_n^2 \sum_{x\in A} |\langle x|x_n\rangle|^2 = \sum_{n\in N} \theta_n^2$$

(Parseval's equation). ■

Corollary 6.1.4.4 (0) *Let u be a positive operator on E and A an orthonormal basis of E. u is nuclear iff*

$$\sum_{x\in A} \langle ux|x\rangle < \infty ,$$

and in this case

$$\|u\|_1 = \sum_{x\in A} \langle ux|x\rangle .$$

By Corollary 6.1.2.18, $u \in \mathcal{L}^1(E)$ iff $u^{\frac{1}{2}} \in \mathcal{L}^2(E)$, and in this case

$$\|u\|_1 = \|u^{\frac{1}{2}}\|_2^2 .$$

By Proposition 6.1.4.3, $u^{\frac{1}{2}} \in \mathcal{L}^2(E)$ iff

$$\sum_{x\in A} \|u^{\frac{1}{2}}x\|^2 < \infty ,$$

and in this case

$$\|u^{\frac{1}{2}}\|_2^2 = \sum_{x\in A} \|u^{\frac{1}{2}}x\|^2 .$$

The assertion now follows from

$$\sum_{x\in A} \|u^{\frac{1}{2}}x\|^2 = \sum_{x\in A} \langle u^{\frac{1}{2}}x|u^{\frac{1}{2}}x\rangle = \sum_{x\in A} \langle ux|x\rangle .$$ ■

Corollary 6.1.4.5 (0) *Let u, v be positive operators on E such that $u \leq v$. If $v \in \mathcal{L}^1(E)$, then $u \in \mathcal{L}^1(E)$.*

If A is an orthonormal basis of E, then

$$\sum_{x\in A}\langle ux|x\rangle \leq \sum_{x\in A}\langle vx|x\rangle$$

(Corollary 5.3.3.7) and the assertion follows from Corollary 6.1.4.4. ■

Corollary 6.1.4.6 (0) *Let $u \in \mathcal{L}(E,F)$. $u \in \mathcal{L}^2(E,F)$ iff $u^*\circ u \in \mathcal{L}^1(E)$, and in this case*

$$\|u^* \circ u\|_1 = \|u\|_2^2.$$

Let A be an orthonormal basis of E. Then

$$\sum_{x\in A}\langle u^*ux|x\rangle = \sum_{x\in A}\langle ux|ux\rangle = \sum_{x\in A}\|ux\|^2$$

and the assertion follows from Proposition 6.1.4.3 and Corollary 6.1.4.4. ■

Theorem 6.1.4.7 (0) *Let A, B be orthonormal bases of E and F, respectively.*

a) *The map*

$$\mathcal{L}^2(E,F)\times\mathcal{L}^2(E,F) \longrightarrow \mathbb{K}, \quad (u,v) \longmapsto \sum_{x\in A}\langle ux|vx\rangle$$

is a scalar product which generates the norm of $\mathcal{L}^2(E,F)$.

b) $u \in \mathcal{L}^2(E,F)$, $(x,y) \in E\times F \Rightarrow \langle u|\langle\cdot|x\rangle y\rangle = \langle ux|y\rangle$.

c) $\{\langle\cdot|x\rangle y | (x,y) \in A\times B\}$ *is an orthonormal basis of $\mathcal{L}^2(E,F)$.*

d) *If $u \in \mathcal{L}^2(E,F)$, then*

$$u = \sum_{(x,y)\in A\times B}\langle ux|y\rangle\langle\cdot|x\rangle y,$$

where the sum is taken in $\mathcal{L}^2(E,F)$.

e) *If $u, v \in \mathcal{L}^2(E,F)$, then $\langle u|v\rangle = \langle v^*|u^*\rangle$.*

a) Now

$$\sum_{x\in A}|\langle ux|vx\rangle| \leq \sum_{x\in A}\|ux\|\,\|vx\| \leq$$

$$\leq \Big(\sum_{x\in A}\|ux\|^2\Big)^{\frac{1}{2}}\Big(\sum_{x\in A}\|vx\|^2\Big)^{\frac{1}{2}} = \|u\|_2\|v\|_2\,,$$

(Proposition 6.1.4.3), so the sum in a) is well–defined. The assertion now follows from

$$\sum_{x\in A}\langle ux|ux\rangle = \sum_{x\in A}\|ux\|^2 = \|u\|_2^2$$

(Proposition 6.1.4.3).

b) By Theorem 5.5.1.15 c),

$$\langle u|\langle\cdot|x\rangle y\rangle = \sum_{z\in A}\langle uz|\langle z|x\rangle y\rangle = \sum_{z\in A}\overline{\langle z|x\rangle}\,\langle uz|y\rangle =$$

$$= \sum_{z\in A}\langle x|z\rangle\langle z|u^*y\rangle = \langle x|u^*y\rangle = \langle ux|y\rangle\,.$$

c) By b), given (x_1,y_1), $(x_2,y_2)\in A\times B$,

$$\langle\langle\cdot|x_1\rangle y_1|\langle\cdot|x_2\rangle y_2\rangle = \langle\langle x_2|x_1\rangle y_1|y_2\rangle =$$

$$= \langle x_2|x_1\rangle\langle y_1|y_2\rangle = \begin{cases} 1 & \text{if } (x_1,y_1)=(x_2,y_2) \\ 0 & \text{if } (x_1,y_1)\neq(x_2,y_2) \end{cases},$$

so $\{\langle\cdot|x\rangle y\,|\,(x,y)\in A\times B\}$ is an orthonormal set of $\mathcal{L}^2(E,F)$. By b), Proposition 6.1.4.3, and Proposition 6.1.4.2

$$\|u\|_2^2 = \sum_{(x,y)\in A\times B}|\langle ux|y\rangle|^2 = \sum_{(x,y)\in A\times B}|\langle u|\langle\cdot|x\rangle y\rangle|^2$$

for every $u\in\mathcal{L}^2(E,F)$. Hence by Theorem 5.5.1.15 d $\Rightarrow$ a,

$$\{\langle\cdot|x\rangle y|(x,y)\in A\times B\}$$

is an orthonormal basis of $\mathcal{L}^2(E,F)$.

d) follows from b), c), and the Fourier expansion.

e) By Theorem 5.5.1.15 a $\Rightarrow$ c,

$$\langle u|v\rangle = \sum_{x\in A}\langle ux|vx\rangle = \sum_{x\in A}\sum_{y\in B}\langle ux|y\rangle\langle y|vx\rangle = \sum_{y\in B}\sum_{x\in A}\langle x|u^*y\rangle\langle v^*y|x\rangle =$$

$$= \sum_{y\in B}\langle v^*y|u^*y\rangle = \langle v^*|u^*\rangle\,.$$

■

Corollary 6.1.4.8 *If u, v are positive Hilbert–Schmidt operators on a Hilbert space, then $\langle u|v\rangle \in \mathbb{R}_+$.*

Let $(\alpha_\iota)_{\iota\in I}$ be a characteristic family of eigenvalues of u. By Proposition 6.1.1.2 a $\Rightarrow$ b, there is an orthonormal family $(x_\iota)_{\iota\in I}$ in E such that

$$u = \sum_{\iota\in I} \alpha_\iota \langle\cdot|x_\iota\rangle x_\iota .$$

Thus

$$\langle u|v\rangle = \sum_{\iota\in I} \langle ux_\iota|vx_\iota\rangle =$$

$$= \sum_{\iota\in I} \alpha_\iota \langle x_\iota|vx_\iota\rangle \in \mathbb{R}_+$$

by Corollary 5.3.3.7. ∎

Proposition 6.1.4.9 (0) *Let $(S, \mathfrak{S}, \mu)$, $(T, \mathfrak{T}, \nu)$ be σ– finite measure spaces. We set*

$$\widetilde{k} : L^2(\nu) \longrightarrow L^2(\mu), \quad x \longmapsto kx$$

for every $k \in L^2(\mu\otimes\nu)$ (Proposition 3.1.6.17 a)).

a) *$\widetilde{k} \in \mathcal{L}^2(L^2(\nu), L^2(\mu))$ for every $k \in L^2(\mu\otimes\nu)$ and the map*

$$L^2(\mu\otimes\nu) \longrightarrow \mathcal{L}^2(L^2(\nu), L^2(\mu)), \quad k \longmapsto \widetilde{k}$$

is an isometry.

b) *If $\mathcal{F}, \mathcal{G}$ are orthonormal bases of $L^2(\nu)$ and $L^2(\mu)$, respectively, and if $u \in \mathcal{L}^2(L^2(\nu), L^2(\mu))$, then $(\langle ux|y\rangle y\otimes\overline{x})_{(x,y)\in\mathcal{F}\times\mathcal{G}}$ is summable in $L^2(\mu\otimes\nu)$ and if we put*

$$k := \sum_{(x,y)\in\mathcal{F}\times\mathcal{G}} \langle ux|y\rangle y\otimes\overline{x},$$

then

$$\widetilde{k} = u .$$

c) *If* $(x_\iota)_{\iota\in I}$, $(y_\iota)_{\iota\in I}$ *are orthonormal families in* $L^2(\nu)$ *and* $L^2(\mu)$, *respectively, and if* $f \in \ell^2(I)$, *then* $(f(\iota)y_\iota \otimes \overline{x}_\iota)_{\iota\in I}$ *is summable in* $L^2(\mu\otimes\nu)$. *If we put*

$$k := \sum_{\iota\in I} f(\iota)y_\iota \otimes \overline{x}_\iota \quad (\text{in } L^2(\mu\otimes\nu)),$$

then

$$\widetilde{k} = \sum_{\iota\in I} f(\iota)\langle\cdot|x_\iota\rangle y_\iota \quad (\text{in } \mathcal{L}^2(L^2(\nu), L^2(\mu))).$$

a & b. $(\overline{x})_{x\in\mathcal{F}}$ is an orthonormal basis of $L^2(\nu)$ and $(y\otimes\overline{x})_{(x,y)\in\mathcal{F}\times\mathcal{G}}$ is an orthonormal basis of $L^2(\mu\otimes\nu)$ (Proposition 5.5.3.8).

Take $k \in L^2(\mu\otimes\nu)$. Then

$$\langle\widetilde{k}x|y\rangle = \int(\int k(s,t)x(t)d\nu(t))\overline{y}(s)d\mu(s) =$$

$$= \int k(\overline{y}\otimes x)d(\mu\otimes\nu) = \langle k|y\otimes\overline{x}\rangle$$

for every $(x,y) \in \mathcal{F}\times\mathcal{G}$. We get

$$\sum_{(x,y)\in\mathcal{F}\times\mathcal{G}} |\langle\widetilde{k}x|y\rangle|^2 = \sum_{(x,y)\in\mathcal{F}\times\mathcal{G}} |\langle k|y\otimes\overline{x}\rangle|^2 = \|k\|^2_{2,\mu\otimes\nu}$$

by Parseval's equation. By Proposition 6.1.4.2 and Proposition 6.1.4.3,

$$\widetilde{k} \in \mathcal{L}^2(L^2(\nu), L^2(\mu)), \quad \|\widetilde{k}\|_2 = \|k\|_{2,\mu\otimes\nu}.$$

Now take $u \in \mathcal{L}^2(L^2(\nu), L^2(\mu))$. By Proposition 6.1.4.3 and Proposition 6.1.4.2,

$$\sum_{(x,y)\in\mathcal{F}\times\mathcal{G}} |\langle ux|y\rangle|^2 < \infty.$$

Hence the family $(\langle ux|y\rangle y\otimes\overline{x})_{(x,y)\in\mathcal{F}\times\mathcal{G}}$ is summable in $L^2(\mu\otimes\nu)$ (Corollary 5.2.2.5). Put

$$k := \sum_{(x,y)\in\mathcal{F}\times\mathcal{G}} \langle ux|y\rangle y\otimes\overline{x}.$$

By the above considerations,

$$\langle\widetilde{k}x|y\rangle = \langle k|y\otimes\overline{x}\rangle = \langle ux|y\rangle$$

for every $(x, y) \in \mathcal{F} \times \mathcal{G}$, so that

$$\widetilde{k} = u .$$

c) Put

$$u := \sum_{\iota \in I} f(\iota) \langle \cdot | x_\iota \rangle y_\iota \quad (\text{in } \mathcal{L}^2(L^2(\nu), L^2(\mu)))$$

(Theorem 6.1.3.12 c)). Then

$$\langle u x_\iota | y_\lambda \rangle = f(\iota) \delta_{\iota\lambda}$$

for all $\iota, \lambda \in I$. By b), $(f(\iota) y_\iota \otimes \overline{x}_\iota)_{\iota \in I}$ is summable in $L^2(\mu \otimes \nu)$ and

$$\widetilde{k} = u . \qquad \blacksquare$$

Remark. The relation

$$\|k\| \leq \|\widetilde{k}\|_2 = \|k\|_{2,\mu\otimes\nu}$$

(Proposition 6.1.2.4, Proposition 6.1.4.9 a)) clarifies Remark a) of Proposition 3.1.6.17.

Proposition 6.1.4.10 *Take $p \in \{0\} \cup [1, \infty]$. For $u \in \mathcal{L}^p(F)$, define*

$$\widetilde{u} \,.\, \mathcal{L}^2(E, F) \longrightarrow \mathcal{L}^2(E, F) , \quad v \longmapsto u \circ v$$

(Corollary 6.1.3.13 b)).

a) *$\widetilde{u} \in \mathcal{L}(\mathcal{L}^2(E, F))$ for every $u \in \mathcal{L}^p(E)$.*

b) *The map*

$$\mathcal{L}^p(F) \longrightarrow \mathcal{L}(\mathcal{L}^2(E, F)) , \quad u \longmapsto \widetilde{u}$$

is an injective involutive algebra homomorphism (Corollary 6.1.3.13 d), Theorem 6.1.4.7 a)).

c) *The representation in b) is non-degenerate.*

d) *If G is a closed vector subspace of E, then*

$$\{v \in \mathcal{L}^2(E, F) | G \subset \operatorname{Ker} v\}$$

is a closed vector subspace of $\mathcal{L}^2(E, F)$ which is invariant under $\{\widetilde{u} | u \in \mathcal{L}^p(F)\}$. In particular, if $\operatorname{Dim} E \neq 0$, then the representation in b) is not irreducible.

e) *$v \in \mathcal{L}^2(E,F)$ is cyclic for $\{\widetilde{u} \mid u \in \mathcal{L}^p(F)\}$ iff it is injective.*

f) *$\{\widetilde{u} \mid u \in \mathcal{L}^p(F)\}$ has a cyclic vector iff E is separable and*

$$\textit{Hilbert dimension of } E \leq \textit{ Hilbert dimension of } F\,.$$

a) follows from Corollary 6.1.3.13 b).

b) It is obvious that the map is an algebra homomorphism. Take $u \in \mathcal{L}^p(F)$ and $v, w \in \mathcal{L}^2(E,F)$. If A is an orthonormal basis for E, then

$$\langle \widetilde{u}v|w\rangle = \langle u \circ v|w\rangle =$$

$$= \sum_{x\in A}\langle uvx|wx\rangle = \sum_{x\in A}\langle vx|u^*wx\rangle = \langle v|u^* \circ w\rangle = \langle v|\widetilde{u^*}w\rangle\,,$$

so that

$$\widetilde{u}^* = \widetilde{u^*}\,,$$

whence the map is involutive.

Take $u \in \mathcal{L}^p(F)$ with

$$\widetilde{u} = 0\,.$$

Then

$$0 = \widetilde{u}(\langle\cdot|x\rangle y) = u \circ (\langle\cdot|x\rangle y) = \langle\cdot|x\rangle uy$$

for every $(x,y) \in E \times F$ (Proposition 5.3.2.13 c)), so that

$$u = 0\,,$$

whence the map is injective.

c) Take $(x,y) \in E \times F$. Then $\langle\cdot|y\rangle y \in \mathcal{L}^p(F)$ and

$$\widetilde{\langle\cdot|y\rangle y}(\langle\cdot|x\rangle y) = (\langle\cdot|y\rangle y) \circ (\langle\cdot|x\rangle y) = \langle\cdot|x\rangle y$$

(Proposition 5.3.2.13 e)), so that

$$\langle\cdot|x\rangle y \in \{\widetilde{u}v|(u,v) \in \mathcal{L}^p(F) \times \mathcal{L}^2(E,F)\}\,.$$

Thus

$$\mathcal{L}_f(E,F) \subset \{\widetilde{u}v|(u,v) \in \mathcal{L}^p(F) \times \mathcal{L}^2(E,F)\}^{\perp\perp}$$

(Corollary 5.5.1.11) and

$$\mathcal{L}^2(E,F) = \{\widetilde{u}v | (u,v) \in \mathcal{L}^p(F) \times \mathcal{L}^2(E,F)\}^{\perp\perp}$$

(Theorem 6.1.3.12 d)). Hence the representation in b) is non-degenerate.

d) is easy to see.

e) If v is not injective, then, by d), it is not cyclic for $\{\widetilde{u} | u \in \mathcal{L}^p(F)\}$. Suppose that v is injective. Let

$$\sum_{n\in N} \theta_n \langle \cdot | x_n \rangle y_n$$

be a Schatten decomposition of v. Since v is injective, $(x_n)_{n\in N}$ is an orthonormal basis of E. Take $(x,y) \in E \times F$ and $\varepsilon > 0$. There is an $m \in N$ such that

$$\sum_{\substack{n\in N \\ n>m}} |\langle x_n | x \rangle|^2 < \frac{\varepsilon^2}{1 + \|y\|^2}.$$

Define

$$u := \sum_{n\in \mathbb{N}_m} \frac{\langle x_n | x \rangle}{\theta_n} \langle \cdot | y_n \rangle y.$$

Then $u \in \mathcal{L}^p(F)$ (Proposition 6.1.2.3) and

$$\widetilde{u}v = u \circ v = \sum_{n\in \mathbb{N}_m} \frac{\langle x_n | x \rangle}{\theta_n} \theta_n \langle \cdot | x_n \rangle y =$$

$$= \Big\langle \cdot \Big| \sum_{n\in \mathbb{N}_m} \langle x | x_n \rangle x_n \Big\rangle y$$

(Proposition 5.3.2.13 e)). Hence

$$\|\widetilde{u}v - \langle \cdot | x \rangle y\|_2^2 = \Big\| \langle \cdot | \sum_{\substack{n\in N \\ n>m}} \langle x | x_n \rangle x_n \rangle y \Big\|_2^2 =$$

$$= \Big\| \sum_{\substack{n\in N \\ n>m}} \langle x | x_n \rangle x_n \Big\| \, \|y\| = \Big(\sum_{\substack{n\in N \\ n>m}} |\langle x | x_n \rangle|^2 \Big)^{\frac{1}{2}} \|y\| < \varepsilon$$

(Proposition 6.1.2.3). Since ε is arbitrary,

$$\langle \cdot | x \rangle y \in \{\widetilde{u}v | u \in \mathcal{L}^p(F)\}^{\perp\perp}.$$

It follows that

$$\mathcal{L}_f(E,F) \subset \{\widetilde{u}v | u \in \mathcal{L}^p(F)\}^{\perp\perp}$$

(Corollary 5.5.1.11) and that

$$\mathcal{L}^2(E,F) = \{\widetilde{u}v | u \in \mathcal{L}^p(F)\}^{\perp\perp}$$

(Theorem 6.1.3.12 d)).

f) First suppose that v is a cyclic vector for $\{\widetilde{u} | u \in \mathcal{L}^p(F)\}$. By e), v is injective. By Corollary 5.5.2.3, the Hilbert dimension of E is less than that of F. Let

$$\sum_{n \in N} \theta_n \langle \cdot | x_n \rangle y_n$$

be a Schatten decomposition of v. Then $(x_n)_{n \in N}$ is an orthonormal basis of E and E is separable (Corollary 5.5.2.7 b $\Rightarrow$ a).

Now suppose that E is separable and that the Hilbert dimension of E is less than that of F. Let $(x_n)_{n \in N}$ be an orthonormal basis of E and $(y_n)_{n \in N}$ be an orthonormal family in F. By Corollary 5.5.2.7 a $\Rightarrow$ b, we may assume that $N \subset \mathbb{N}$. Put

$$v := \sum_{n \in N} \frac{1}{2^n} \langle \cdot | x_n \rangle y_n .$$

Then v is injective and belongs to $\mathcal{L}^2(E,F)$. By e), v is cyclic for

$$\{\widetilde{u} | u \in \mathcal{L}^p(F)\}. \qquad \blacksquare$$

6.1.5 The Trace

Theorem 6.1.5.1 (0) *Take $u \in \mathcal{L}^1(E)$ and let $(x_\iota)_{\iota\in I}$, $(y_\iota)_{\iota\in I}$ be families in E such that*

$$\sum_{\iota\in I} \|x_\iota\| \, \|y_\iota\| < \infty\,, \quad u = \sum_{\iota\in I} \langle\cdot|x_\iota\rangle y_\iota\,.$$

Then

$$\sum_{x\in A} \langle ux|x\rangle = \sum_{\iota\in I} \langle y_\iota|x_\iota\rangle$$

for every orthonormal basis A of E. This number is called ***the trace of*** *u and is denoted by $tr\, u$. The operators of $\mathcal{L}^1(E)$ sometimes are called* ***trace operators on*** *E.*

We have

$$\sum_{x\in A}\sum_{\iota\in I} |\langle x|x_\iota\rangle\langle y_\iota|x\rangle| \leq \sum_{\iota\in I} \|x_\iota\| \, \|y_\iota\| < \infty$$

(Proposition 5.5.1.19), so that

$$\sum_{x\in A} \langle ux|x\rangle = \sum_{x\in A}\sum_{\iota\in I} \langle x|x_\iota\rangle\langle y_\iota|x\rangle =$$

$$= \sum_{\iota\in I}\sum_{x\in A} \langle y_\iota|x\rangle\langle x|x_\iota\rangle = \sum_{\iota\in I} \langle y_\iota|x_\iota\rangle$$

(Theorem 5.5.1.15 a $\Rightarrow$ c)). ■

Corollary 6.1.5.2 (0) *Given $u \in \mathcal{L}(E,F)$ and (x,y), $(a,b) \in E \times F$,*

$$tr\,(u \circ (\langle\cdot|y\rangle x)) = \langle ux|y\rangle = tr\,((\langle\cdot|y\rangle x) \circ u)\,,$$

$$tr\,((\langle\cdot|a\rangle b) \circ (\langle\cdot|y\rangle x)) = \langle x|a\rangle\langle b|y\rangle\,.$$

Now

$$tr\,(u \circ (\langle\cdot|y\rangle x)) = tr\,(\langle\cdot|y\rangle ux) = \langle ux|y\rangle\,,$$

$$tr\,((\langle\cdot|y\rangle x) \circ u) = tr\,(\langle\cdot|u^*y\rangle x) = \langle x|u^*y\rangle = \langle ux|y\rangle\,,$$

(Proposition 5.3.2.13 c),d), Theorem 6.1.5.1), and so

$$tr\,((\langle\cdot|a\rangle b) \circ (\langle\cdot|y\rangle x)) = \langle\langle x|a\rangle b|y\rangle = \langle x|a\rangle\langle b|y\rangle\,.$$ ■

Corollary 6.1.5.3 (0) *The map*

$$\mathcal{L}^1(E) \longrightarrow \mathbb{K}\,, \quad u \longmapsto tr\, u$$

is a continuous involutive linear form with norm 1 *(Corollary 6.1.3.14) and*

$$tr\, u = \|u\|_1$$

for any positive element u *of* $\mathcal{L}^1(E)$.

By Theorem 6.1.5.1, the map is continuous with norm 1. By Proposition 5.3.2.13 b) and Theorem 6.1.3.12 d), it is involutive. Finally, by Corollary 6.1.2.13 a $\Rightarrow$ b, $tr\, u = \|u\|_1$ for every positive element u of $\mathcal{L}^1(E)$. ■

Corollary 6.1.5.4 (0) *Let* u *be a positive operator on* E *and take* $p \in [1, \infty[$. *Then* $u \in \mathcal{L}^p(E)$ *iff* $u^p \in \mathcal{L}^1(E)$ *and in this case*

$$\|u\|_p = (tr\, u^p)^{\frac{1}{p}}\,.$$

By Corollary 6.1.2.18, $u \in \mathcal{L}^p(E)$ iff $u^p \in \mathcal{L}^1(E)$, and in this case

$$\|u\|_p^p = \|u^p\|_1\,.$$

By Corollary 6.1.5.3,

$$\|u\|_p = (tr\, u^p)^{\frac{1}{p}}\,.$$ ■

Corollary 6.1.5.5 (0) *If* u *is a self-normal operator in* $\mathcal{L}^1(E)$, *then*

$$tr\, u = \sum_{\iota \in I} \alpha_\iota$$

for any characteristic family $(\alpha_\iota)_{\iota \in I}$ *of eigenvalues of* u.

By Proposition 6.1.1.2 a $\Rightarrow$ b, there is an orthonormal family $(x_\iota)_{\iota \in I}$ in E such that

$$u = \sum_{\iota \in I} \alpha_\iota \langle \cdot | x_\iota \rangle x_\iota\,.$$ ■

Corollary 6.1.5.6 (0) *If* $u, v \in \mathcal{L}^2(E, F)$ *then* $v^* \circ u \in \mathcal{L}^1(E)$ *and*

$$tr\,(v^* \circ u) = \langle u | v \rangle$$

(Theorem 6.1.4.7 a)).

That $v^* \circ u \in \mathcal{L}^1(E)$ follows from Theorem 6.1.3.1 b) and Theorem 6.1.3.21. If A is an orthonormal basis of E, then

$$tr\,(v^* \circ u) = \sum_{x \in A} \langle v^* u x | x \rangle = \sum_{x \in A} \langle ux | vx \rangle = \langle u | v \rangle$$

(Theorem 6.1.4.7 a)). ■

Proposition 6.1.5.7 (0) *If $u \in \mathcal{K}(E,F)$ and $v \in \mathcal{L}(F,E)$, then*

$$u \circ v \in \mathcal{L}^1(F),\ v \circ u \geq 0 \Longrightarrow v \circ u \in \mathcal{L}^1(E),$$

$$v \circ u \in \mathcal{L}^1(F),\ u \circ v \geq 0 \Longrightarrow u \circ v \in \mathcal{L}^1(F),$$

$$u \circ v \in \mathcal{L}^1(F),\ v \circ u \in \mathcal{L}^1(E) \Longrightarrow tr\,(u \circ v) = tr\,(v \circ u).$$

By Corollary 6.1.3.5, there are orthonormal bases $(x_\iota)_{\iota \in I}$ and $(y_\lambda)_{\lambda \in L}$ of E and F, respectively, such that $(\langle vux_\iota | x_\iota \rangle)_{\iota \in I}$ is summable iff $(\langle uvy_\lambda | y_\lambda \rangle)_{\lambda \in L}$ is summable, and in this case

$$\sum_{\iota \in I} \langle vux_\iota | x_\iota \rangle = \sum_{\lambda \in L} \langle uvy_\lambda | y_\lambda \rangle .$$

The final implication now follows. The first two implications follow by Corollary 6.1.4.4. ■

Corollary 6.1.5.8 (0) *If $u \in \mathcal{L}^1(E)$ and $v : F \to E$ is an isomorphism, then*

$$tr\,(v^{-1} \circ u \circ v) = tr\,u .$$

By Proposition 6.1.5.7 (and Corollary 6.1.3.13 b)),

$$tr\,(v^{-1} \circ u \circ v) = tr\,(u \circ v \circ v^{-1}) = tr\,u . \qquad ■$$

Corollary 6.1.5.9 *Take $u, v \in \mathrm{Re}\,\mathcal{L}(E)$ such that one of the operators is compact and $uv \in \mathcal{L}^1(E)$.*

a) $vu \in \mathcal{L}^1(E)$, $tr\,(uv) = tr\,(vu) \in \mathbb{R}$.

b) $u \geq 0$, $v \geq 0 \Rightarrow tr\,(uv) = \|v^{\frac{1}{2}} u v^{\frac{1}{2}}\|_1 \geq \|u^{\frac{1}{2}} v^{\frac{1}{2}}\|^2$.

c) $u \geq 0$, $v \geq 0$, $tr\,(uv) = 0 \Rightarrow uv = vu = 0$.

a) We have

$$vu = v^*u^* = (uv)^* \in \mathcal{L}^1(E)$$

(Theorem 6.1.3.1 b)), and

$$\overline{tr\,(uv)} = tr\,(uv)^* = tr\,(v^*u^*) = tr\,(vu) = tr\,(uv)$$

(Corollary 6.1.5.3, Proposition 6.1.5.7), so

$$tr\,(uv) \in \mathbb{R}\,.$$

b) By a), we may assume u to be compact. Then $uv^{\frac{1}{2}}$ is compact. By Corollary 5.3.3.9, $v^{\frac{1}{2}}uv^{\frac{1}{2}}$ is positive. Hence by Proposition 6.1.5.7, $v^{\frac{1}{2}}uv^{\frac{1}{2}} \in \mathcal{L}^1(E)$ and

$$tr\,(uv) = tr\,(v^{\frac{1}{2}}uv^{\frac{1}{2}}) = \|v^{\frac{1}{2}}uv^{\frac{1}{2}}\|_1 \geq \|v^{\frac{1}{2}}uv^{\frac{1}{2}}\| =$$

$$= \|v^{\frac{1}{2}}u^{\frac{1}{2}}u^{\frac{1}{2}}v^{\frac{1}{2}}\| = \|u^{\frac{1}{2}}v^{\frac{1}{2}}\|^2$$

(Corollary 6.1.5.3, Proposition 6.1.2.4, Theorem 5.3.1.4).

c) By b),

$$u^{\frac{1}{2}}v^{\frac{1}{2}} = 0\,,$$

so

$$uv = u^{\frac{1}{2}}(u^{\frac{1}{2}}v^{\frac{1}{2}})v^{\frac{1}{2}} = 0\,,$$

$$vu = (uv)^* = 0\,. \qquad \blacksquare$$

Proposition 6.1.5.10 (0) *Let p, q be conjugate exponents and let $\mathcal{L}_f(E,F)$ be endowed with the $\|\cdot\|_p$-norm. If u' is a continuous linear form on $\mathcal{L}_f(E,F)$, then there is a unique $u \in \mathcal{L}^q(F,E)$ such that*

$$u'(v) = tr\,(u \circ v)$$

for every $v \in \mathcal{L}_f(E,F)$. Furthermore,

$$\|u\|_q = \|u'\|\,.$$

Given $y \in F$, define

$$u_y : E \longrightarrow \mathbb{K}, \quad x \longmapsto \overline{u'(\langle \cdot | x \rangle y)}$$

(Proposition 6.1.2.3).

Take $y \in F$. u_y is linear and

$$|u_y(x)| = |u'(\langle \cdot | x \rangle y)| \leq \|u'\| \, \|\langle \cdot | x \rangle y\|_p = \|u'\| \, \|x\| \, \|y\|$$

for every $x \in E$ (Proposition 6.1.2.3), i.e. u_y is continuous and

$$\|u_y\| \leq \|u'\| \, \|y\| .$$

By the Fréchet–Riesz Theorem, there is a unique $\widetilde{y} \in E$ such that

$$u_y = \langle \cdot | \widetilde{y} \rangle .$$

Then

$$\|\widetilde{y}\| = \|u_y\| \leq \|u'\| \, \|y\| .$$

Define

$$u : F \longrightarrow E, \quad y \longmapsto \widetilde{y} .$$

Choose $y, z \in F$ and $\alpha, \beta \in \mathbb{K}$. Then

$$\langle x | \widetilde{\alpha y + \beta z} \rangle = u_{\alpha y + \beta z}(x) = \overline{u'(\langle \cdot | x \rangle (\alpha y + \beta z))} =$$

$$= \overline{\alpha} \, \overline{u'(\langle \cdot | x \rangle y)} + \overline{\beta} \, \overline{u'(\langle \cdot | x \rangle z)} =$$

$$= \overline{\alpha} u_y(x) + \overline{\beta} u_z(x) = \overline{\alpha} \langle x | \widetilde{y} \rangle + \overline{\beta} \langle x | \widetilde{z} \rangle = \langle x | \alpha \widetilde{y} + \beta \widetilde{z} \rangle$$

for every $x \in E$, so that

$$u(\alpha y + \beta z) = \widetilde{\alpha y + \beta z} = \alpha \widetilde{y} + \beta \widetilde{z} = \alpha u y + \beta u z .$$

Thus u is linear. By the above considerations, u is continuous and $\|u\| \leq \|u'\|$.

We have

$$u'(\langle \cdot | x \rangle y) = \overline{u_y(x)} = \overline{\langle x | \widetilde{y} \rangle} = \langle u y | x \rangle = tr\,(u \circ (\langle \cdot | x \rangle y))$$

for every $(x, y) \in E \times F$ (Corollary 6.1.5.2), so that

$$u'(v) = tr\,(u \circ v)$$

for every $v \in \mathcal{L}_f(E,F)$ (Corollary 5.5.1.11).

Let $(x_\iota)_{\iota\in I}$, $(y_\iota)_{\iota\in I}$ be orthonormal families in E and F, respectively. If we endow $\mathbb{K}^{(I)}$ with the norm induced by $\ell^p(I)$, then

$$\Big|\sum_{\iota\in I} f(\iota)\langle uy_\iota|x_\iota\rangle\Big| = \Big|\sum_{\iota\in I} f(\iota)tr\,(u\circ(\langle\cdot|x_\iota\rangle y_\iota))\Big| =$$

$$= \Big|\sum_{\iota\in I} f(\iota)u'(\langle\cdot|x_\iota\rangle y_\iota)\Big| = \Big|u'(\sum_{\iota\in I} f(\iota)\langle\cdot|x_\iota\rangle y_\iota)\Big| \leq$$

$$\leq \|u'\|\,\Big\|\sum_{\iota\in I} f(\iota)\langle\cdot|x_\iota\rangle y_\iota\Big\|_p \leq \|u'\|\,\|f\|_p$$

for every $f \in \mathbb{K}^{(I)}$ (Corollary 6.1.2.12). We deduce that

$$(\langle uy_\iota|x_\iota\rangle)_{\iota\in I} \in \ell^q(I)\,,$$

$$\|(\langle uy_\iota|x_\iota\rangle)_{\iota\in I}\|_q \leq \|u'\|$$

(Proposition 1.2.2.2). Since $(x_\iota)_{\iota\in I}$, $(y_\iota)_{\iota\in I}$ are also arbitrary, it follows that

$$u \in \mathcal{L}^q(F,E)\,, \quad \|u\|_q \leq \|u'\|$$

(Proposition 6.1.3.10 and $\|u\| \leq \|u'\|$).

Given $v \in \mathcal{L}_f(E,F)$

$$|u'(v)| = |tr\,(u\circ v)| \leq \|u\circ v\|_1 \leq \|u\|_q\|v\|$$

(Hölder's Inequality), so that

$$\|u'\| \leq \|u\|_q\,,$$

$$\|u'\| = \|u\|_q\,,$$

The uniqueness of u follows from Theorem 6.1.3.12 d). ■

Theorem 6.1.5.11 (0) *Take* $p,q,r \in [1,\infty]$ *with*

$$\frac{1}{r} \leq \frac{1}{p} + \frac{1}{q}\,,$$

$u \in \mathcal{L}^p(E,F)$ *and* $v \in \mathcal{L}^q(F,G)$. *Then* $v\circ u \in \mathcal{L}^r(E,G)$ *and*

$$\|v\circ u\|_r \leq \|u\|_p\|v\|_q \qquad \textbf{\textit{(Hölder's Inequality)}}.$$

By Corollary 6.1.3.13 b) and Proposition 6.1.2.4, we may assume that $p, q, r \in [1, \infty]$ and

$$\frac{1}{p} + \frac{1}{q} = \frac{1}{r}.$$

First suppose that $p, q \in [2, \infty[$. By Corollary 6.1.3.2 and Corollary 6.1.2.18,

$$u^* \circ u \in \mathcal{L}^{\frac{p}{2}}(E), \quad v \circ v^* \in \mathcal{L}^{\frac{q}{2}}(G),$$

$$\|u^* \circ u\|_{\frac{p}{2}} = \|(u^* \circ u)^{\frac{1}{2}}\|_p^2 = \|u\|_p^2,$$

$$\|v \circ v^*\|_{\frac{q}{2}} = \|v\|_q^2.$$

Let $(x_\iota)_{\iota \in I}$, $(y_\iota)_{\iota \in I}$ be orthonormal families in E and G, respectively. Then

$$\sum_{\iota \in I} |\langle (v \circ u) x_\iota | y_\iota \rangle|^r = \sum_{\iota \in I} |\langle u x_\iota | v^* y_\iota \rangle|^r \leq \sum_{\iota \in I} \|u x_\iota\|^r \|v^* y_\iota\|^r \leq$$

$$\leq \Big(\sum_{\iota \in I} \|u x_\iota\|^p \Big)^{\frac{r}{p}} \Big(\sum_{\iota \in I} \|v^* y_\iota\|^q \Big)^{\frac{r}{q}} =$$

$$= \Big(\sum_{\iota \in I} \langle u^* u x_\iota | x_\iota \rangle^{\frac{p}{2}} \Big)^{\frac{r}{p}} \Big(\sum_{\iota \in I} \langle v v^* y_\iota | y_\iota \rangle^{\frac{q}{2}} \Big)^{\frac{r}{q}} \leq$$

$$\leq \Big\| u^* \circ u \Big\|_{\frac{p}{2}}^{\frac{r}{2}} \Big\| v \circ v^* \Big\|_{\frac{q}{2}}^{\frac{r}{2}} = \|u\|_p^r \|v\|_q^r$$

(Proposition 6.1.3.10). We deduce that $v \circ u \in \mathcal{L}^r(E, G)$ and

$$\|v \circ u\|_r \leq \|u\|_p \|v\|_q.$$

Now suppose that $p < 2$. Take $p', r' \in [1, \infty]$ with

$$\frac{1}{p} + \frac{1}{p'} = 1 \text{ and } \frac{1}{r} + \frac{1}{r'} = 1.$$

Then $q > 2$, $r < 2$, and $r' > 2$. Choose $w \in \mathcal{L}_f(G, E)$. Since

$$\frac{1}{q} + \frac{1}{r'} = \frac{1}{r} - \frac{1}{p} + 1 - \frac{1}{r} = \frac{p-1}{p} = \frac{1}{\frac{p}{p-1}},$$

it follows by the above considerations that $w \circ v \in \mathcal{L}^{\frac{p}{p-1}}(F, E)$ and

$$\|w \circ v\|_{\frac{p}{p-1}} \leq \|v\|_q \|w\|_{r'} .$$

By Theorem 6.1.3.21, $w \circ v \circ u \in \mathcal{L}^1(E)$ and

$$\|w \circ v \circ u\|_1 \leq \|u\|_p \|w \circ v\|_{\frac{p}{p-1}} \leq \|u\|_p \|v\|_q \|w\|_{r'} .$$

By Proposition 6.1.5.10, $v \circ u \in \mathcal{L}^r(E,G)$ and

$$\|v \circ u\|_r \leq \|u\|_p \|v\|_q . \qquad \blacksquare$$

Remark. By the above theorem and Proposition 6.1.2.4, $\mathcal{L}^p$ is a Banach category for every $p \in \{0\} \cup [1,\infty]$. Hence, by Corollary 6.1.3.13 b), it is even a Banach $\mathcal{L}$–category.

Corollary 6.1.5.12 *If $p \in [1,\infty[$, $u \in \mathcal{L}^p(E)$, and $n \in \mathbb{N}$ then*

$$u^n \in \mathcal{L}^{\frac{p}{n}}(E) ,$$

and

$$\|u^n\|_{\frac{p}{n}} \leq \|u\|_p^n .$$

The assertion follows from Theorem 6.1.5.11 by complete induction. $\blacksquare$

Corollary 6.1.5.13 (0) *Take $p,q,r \in [1,\infty[$ with*

$$\frac{1}{p} + \frac{1}{q} = \frac{1}{r} ,$$

and $u \in \mathcal{L}(E,F)$. Then the following are equivalent:

a) $u \in \mathcal{L}^r(E,F)$.

b) *There are $v \in \mathcal{L}^p(E)_+$ and $w \in \mathcal{L}^q(E,F)$ such that*

$$u = w \circ v , \quad \|v\|_p = \left\|u\right\|_r^{\frac{r}{p}} , \quad \|w\|_q = \left\|u\right\|_r^{\frac{r}{q}} .$$

c) *There are $v \in \mathcal{L}^p(E,F)$ and $w \in \mathcal{L}^q(F)_+$ such that*

$$u = w \circ v , \quad \|v\|_p = \left\|u\right\|_r^{\frac{r}{p}} , \quad \|w\|_q = \left\|u\right\|_r^{\frac{r}{q}} .$$

d) *There are a Hilbert space G and operators $v \in \mathcal{L}^p(E,G)$, $w \in \mathcal{L}^q(G,F)$ with*

$$u = w \circ v .$$

a $\Rightarrow$ b. Put

$$s := (u^* \circ u)^{\frac{1}{2}} .$$

Then s is positive and, by Corollary 6.1.3.2,

$$s \in \mathcal{L}^r(E) , \quad \|s\|_r = \|u\|_r .$$

By Corollary 6.1.2.18,

$$s^{\frac{r}{p}} \in \mathcal{L}^p(E) , \quad \|s^{\frac{r}{p}}\|_p = \left\|s\right\|_r^{\frac{r}{p}} = \left\|u\right\|_r^{\frac{r}{p}} ,$$

$$s^{\frac{r}{q}} \in \mathcal{L}^q(E) , \quad \|s^{\frac{r}{q}}\|_q = \left\|s\right\|_r^{\frac{r}{q}} = \left\|u\right\|_r^{\frac{r}{q}} .$$

By Proposition 5.5.5.10, there is a $t \in \mathcal{L}(E, F)$ such that

$$u = t \circ s , \quad \|t\| \leq 1 .$$

Define

$$v := s^{\frac{r}{p}} , \quad w := t \circ s^{\frac{r}{q}} .$$

Then

$$w \circ v = u ,$$

$$w \in \mathcal{L}^q(E) , \quad \|w\|_q \leq \|t\| \, \|s^{\frac{r}{q}}\|_q = \left\|u\right\|_r^{\frac{r}{q}}$$

(Corollary 6.1.3.13 b)). By Hölder's Inequality,

$$\|u\|_r \leq \|v\|_p \|w\|_q \leq \left\|u\right\|_r^{\frac{r}{p}} \left\|u\right\|_r^{\frac{r}{q}} = \|u\|_r ,$$

so

$$\|w\|_q = \left\|u\right\|_r^{\frac{r}{q}} .$$

a $\Rightarrow$ c. By Corollary 6.1.3.2,

$$u^* \in \mathcal{L}^r(F, E) , \quad \|u^*\|_r = \|u\|_r .$$

By a $\Rightarrow$ b, there are $w \in \mathcal{L}^q(F)_+$ and $t \in \mathcal{L}^p(F, E)$, such that

$$u^* = t \circ w , \quad \|w\|_q = \left\|u^*\right\|_r^{\frac{r}{q}} = \left\|u\right\|_r^{\frac{r}{q}} ,$$

$$\|t\|_p = \left\|u^*\right\|_r^{\frac{r}{p}} = \left\|u\right\|_r^{\frac{r}{p}} .$$

Put

$$v := t^* .$$

Then

$$u = w \circ v .$$

By Corollary 6.1.3.2,

$$v \in \mathcal{L}^p(E,F) , \quad \|v\|_p = \|t\|_p = \left\|u\right\|_r^{\frac{r}{p}} .$$

b $\Rightarrow$ d and c $\Rightarrow$ d are trivial.

d $\Rightarrow$ a follows from Theorem 6.1.5.11. ■

Remark. For $p = q = 2$ and $r = 1$, we get characterizations of nuclear operators by means of Hilbert–Schmidt operators.

Proposition 6.1.5.14 *If* $\mathbb{K} = \mathbb{C}$, *then* $u \in \mathcal{L}^1(E)$ *is positive iff*

$$\operatorname{re} tr\, u \geq \operatorname{re} tr\,(vu)$$

for every unitary operator v *on* E.

If u is positive, then

$$\operatorname{re} tr\,(vu) \leq |tr\, vu| \leq \|vu\|_1 \leq \|v\|\, \|u\|_1 = tr\, u = \operatorname{re} tr\, u$$

for every unitary operator v on E (Corollary 6.1.5.3, Corollary 6.1.3.13 b)).

Now suppose that

$$\operatorname{re} tr\, u \geq \operatorname{re} tr\,(vu)$$

for every unitary operator v on E. Take $v \in \operatorname{Re} \mathcal{L}(E)$ and $\alpha \in \mathbb{R}$ with

$$2|\alpha|\, \|v\| < 1 .$$

Define

$$w := i\alpha v ,$$

$$u_1 := (1+w)(1-w)^{-1},$$

$$u_2 := (1+w)^{-1}(1-w),$$

$$u_3 := u_1 - 1 - 2w.$$

Then

$$w^* = -w, \quad u_1^* = u_2,$$

$$u_1 u_1^* = u_1 u_2 = 1, \quad u_1^* u_1 = u_2 u_1 = 1,$$

i.e. u_1 is unitary. We have

$$u_3 = (1+w)\sum_{n=0}^{\infty} w^n - 1 - 2w = \sum_{n=0}^{\infty} w^n + \sum_{n=1}^{\infty} w^n - 1 - 2w =$$

$$= 2\sum_{n=2}^{\infty} w^n,$$

so that

$$\|u_3\| \leq 2\sum_{n=2}^{\infty} \|w\|^n = 2\frac{\alpha^2\|v\|^2}{1-|\alpha|\,\|v\|} \leq 4\alpha^2\|v\|^2.$$

Hence we see, successively, that

$$\operatorname{re} tr\, u \geq \operatorname{re} tr\,(u_1 u) = \operatorname{re} tr\,((1+2w+u_3)u) =$$

$$= \operatorname{re} tr\, u + 2\alpha \operatorname{re} tr\,(ivu) + \operatorname{re} tr\,(u_3 u),$$

$$2\alpha \operatorname{re} tr\,(ivu) + \operatorname{re} tr\,(u_3 u) \leq 0,$$

$$2\alpha \operatorname{re} tr\,(ivu) - 4\alpha^2\|v\|^2\|u\|_1 \leq 2\alpha \operatorname{re} tr\,(ivu) - \|u_3\|\,\|u\|_1 \leq$$

$$\leq 2\alpha \operatorname{re} tr\,(ivu) - \|u_3 u\|_1 \leq 2\alpha \operatorname{re} tr\,(ivu) + \operatorname{re} tr\,(u_3 u) \leq 0$$

$$2\alpha \operatorname{re} tr\,(ivu) \leq 4\alpha^2\|v\|^2\|u\|_1.$$

(Corollary 6.1.3.13 b)). Since α is arbitrary,

$$\operatorname{re} tr\,(ivu) = 0\,,$$

$$\operatorname{im} tr\,(vu) = \operatorname{re}\,(-itr\,(vu)) = -\operatorname{re} tr\,(ivu) = 0$$

$$tr\,(vu) \in \mathbb{R}\,,$$

$$x \in E \Longrightarrow \langle ux|x\rangle = tr\,(u \circ (\langle\cdot|x\rangle x)) =$$

$$= tr\,((\langle\cdot|x\rangle x) \circ u) \in \mathbb{R}$$

(Corollary 6.1.5.2, Proposition 6.1.5.7, Proposition 5.3.2.13 b)). By Corollary 5.3.3.3, u is selfadjoint.

By Corollary 6.1.2.13 a $\Rightarrow$ b & d, there is an orthonormal family $(x_n)_{n\in N}$ in E and a characteristic family $(\alpha_n)_{n\in N}$ of eigenvalues of u such that

$$(\alpha_n)_{n\in N} \in \ell^1(N)\,,$$

$$u = \sum_{n\in N} \alpha_n \langle\cdot|x_n\rangle x_n\,.$$

$\alpha_n \in \mathbb{R}$ for every $n \in N$. Take $v \in \mathcal{L}(E)$ such that

$$vx_n = \begin{cases} x_n & if\ \ \alpha_n > 0 \\ -x_n & if\ \ \alpha_n < 0 \end{cases}$$

for every $n \in N$ and

$$vx = x$$

for every $x \in \{x_n | n \in N\}^\perp$. Then v is unitary and

$$vu = \sum_{n\in N} \alpha_n \langle\cdot|x_n\rangle v x_n = \sum_{n\in N} |\alpha_n| \langle\cdot|x_n\rangle x_n\,.$$

It follows that

$$\sum_{n\in N} \alpha_n = tr\,u = \operatorname{re} tr\,u \geq \operatorname{re} tr\,vu = \sum_{n\in N} |\alpha_n|$$

(Corollary 6.1.5.5). Hence $\alpha_n \in \mathbb{R}_+$ for every $n \in N$, and so u is positive. ■

Corollary 6.1.5.15 *Let* $\mathbb{K} = \mathbb{C}$. *Take* $u \in \mathcal{L}^1(E)$ *and* $v \in \operatorname{Un} E$. *Then*

$$vu = |u|$$

iff

$$\operatorname{re} tr(vu) \geq \operatorname{re} tr(wu)$$

for every $w \in \operatorname{Un} E$.

We have

$$\operatorname{re} tr(vu) \geq \operatorname{re} tr(wu)$$

for every $w \in \operatorname{Un} E$ iff

$$\operatorname{re} tr(vu) \geq \operatorname{re} tr(wvu)$$

for every $w \in \operatorname{Un} E$. By Proposition 6.12.5.14, the last relation is equivalent to $vu \geq 0$. If vu is positive, then

$$(vu)^2 = (vu)^*(vu) = u^*v^*vu = u^*u = |u|^2,$$

so

$$vu = |u|.$$

Conversely, it follows from

$$vu = |u|$$

that vu is positive. ■

Corollary 6.1.5.16 *Let* $\mathbb{K} = \mathbb{C}$. *Take* $u \in \mathcal{L}^1(E)$, *and let* v *be a unitary operator on* E. *Then*

$$\operatorname{re} tr(vu) = \|u\|_1$$

iff

$$vu = |u|.$$

Given any unitary operator w on E,

$$\operatorname{re} tr(wu) \leq |tr(wu)| \leq \|wu\|_1 \leq \|w\| \, \|u\|_1 = \|u\|_1$$

(Corollary 6.1.5.3, Corollary 6.1.3.13 b)). It follows from

$$\operatorname{re} tr(vu) = \|u\|_1$$

that

$$vu = |u|$$

(Corollary 6.1.5.15). Conversely, it follows from

$$vu = |u|$$

that

$$\|u\|_1 = \| \, |u| \, \|_1 = tr|u| = tr(vu) = \operatorname{re} tr(vu)$$

(Corollary 6.1.3.2, Corollary 6.1.5.3). ■

Corollary 6.1.5.17 *Let $\mathbb{K} = \mathbb{C}$, and take $u, v \in \mathcal{L}(E)$ such that at least one of the operators is nuclear. If*

$$\operatorname{re} tr(uv) \geq \operatorname{re} tr(wuv), \quad \operatorname{re} tr(uv) \geq \operatorname{re} tr(uwv)$$

for every unitary operator w on E, then uv and vu are positive.

uv and vu are nuclear (Corollary 6.1.3.13 b)). Given a unitary operator w on E,

$$\operatorname{re} tr(uv) \geq \operatorname{re} tr(wuv)$$

and

$$\operatorname{re} tr(vu) = \operatorname{re} tr(uv) \geq \operatorname{re} tr(uwv) = \operatorname{re} tr(wvu)$$

(Corollary 6.1.3.13 b), Proposition 6.1.5.7). By Proposition 6.1.5.14, uv and vu are positive. ■

Proposition 6.1.5.18 *The representation defined in Proposition 6.1.4.10 b) for $F = E$ and $p = 1$ is the representation of $\mathcal{L}^1(E)$ associated to the trace (Corollary 6.1.5.3).*

Take $u, v \in L^1(E)$. Then $u, v \in \mathcal{L}^2(E)$ (Proposition 6.1.2.4) and for any orthonormal basis A of E,

$$\langle u|v\rangle = \sum_{x\in A}\langle ux|vx\rangle = \sum_{x\in A}\langle v^*ux|x\rangle = tr(v^*u)\,.$$

The assertion now follows. ■

Remark. By Proposition 6.1.4.10 f), this representation does not always admit a cyclic vector. This does not contradict Theorem 5.4.1.2 k), since $\mathcal{L}^1(E)$ is not quasiunital (Proposition 6.1.3.13 d)).

Proposition 6.1.5.19 (8) *Let $n \in \mathbb{N}$ and $A \in \{\mathbb{R},\mathbb{C},\mathbb{H}\}$ and denote by A^n the Hilbert right A-module $\ell^2(\mathbb{N}_n, A)$ (Example 5.6.4.2 a)). Put*

$$\varphi : A_{n,n} \longrightarrow \mathbb{K}\,, \quad a \longmapsto \begin{cases} \sum\limits_{r=1}^{n} a_{rr} & \text{if } A = \mathbb{K} \\ 2\mathrm{re}\sum\limits_{r=1}^{n} a_{rr} & \text{if } \mathbb{K} = \mathbb{R} \text{ and } A = \mathbb{C} \\ 4\mathrm{re}\sum\limits_{r=1}^{n} a_{rr} & \text{if } \mathbb{K} = \mathbb{R} \text{ and } A = \mathbb{H} \end{cases}$$

and denote by E the Hilbert space obtained by endowing the vector space A^n with the scalar product

$$A^n \times A^n \longrightarrow \mathbb{K}\,, \quad (\xi,\eta) \longmapsto \varphi[\eta_s^*\xi_t]_{s,t\in\mathbb{N}_n}$$

(Proposition 5.6.2.5 a),e), Proposition 5.6.6.17 a)). For every $a \in A_{n,n}$,

$$tr\,\widetilde{a} = \varphi(a)\,,$$

where

$$\widetilde{a} : E \longrightarrow E\,, \quad \xi \longmapsto a\xi\,.$$

In particular,

$$\varphi(ab) = \varphi(ba)$$

for all $a, b \in A_{n,n}$.

Denote

$$B := \begin{cases} \{1\} & \text{if } A = \mathbb{K} \\ \{1, i\} & \text{if } \mathbb{K} = \mathbb{R} \text{ and } A = \mathbb{C} \\ \{1, i, j, k\} & \text{if } \mathbb{K} = \mathbb{R} \text{ and } A = \mathbb{H}\,. \end{cases}$$

Fix $r \in \mathbb{N}_n$ and put

$$e_r := [\delta_{rs}]_{s \in \mathbb{N}_n} \in A^n ,$$

$$B_r := \{e_r y \mid y \in B\} .$$

Then

$$\langle \widetilde{a} e_r | e_r \rangle = \sum_{s=1}^{n} \delta_{rs} \left(\sum_{t=1}^{n} a_{st} \delta_{rt} \right) = a_{rr} .$$

Since $\bigcup_{r=1}^{n} B_r$ is an orthonormal basis of E (Corollary 5.6.4.11 e)),

$$tr\, \widetilde{a} = \sum_{r=1}^{n} \sum_{\eta \in B_r} \langle \widetilde{a}\eta | \eta \rangle = \sum_{r=1}^{n} \sum_{y \in B} \langle \widetilde{a} e_r y | e_r y \rangle =$$

$$= \sum_{r=1}^{n} \sum_{y \in B} y^* \langle \widetilde{a} e_r | e_r \rangle y = \sum_{r=1}^{n} \sum_{y \in B} y^* a_{rr} y = \varphi(a) .$$

The final assertion follows from Proposition 6.1.5.7. ■

6.1.6 Duals of $\mathcal{L}^p$-spaces

Throughout this subsection, $p \in \{0\} \cup [1, \infty]$ and q is the conjugate of p.

Definition 6.1.6.1 (0) *Given* $u \in \mathcal{L}^q(F, E)$, *define*

$$\widetilde{u} := \overset{p}{\widetilde{u}} : \mathcal{L}^p(E, F) \longrightarrow \mathbb{K}, \quad v \longmapsto tr(u \circ v)$$

(Theorem 6.1.3.21).

Proposition 6.1.6.2 (0) *Given any* (a, b), $(x, y) \in E \times F$, $u \in \mathcal{L}^p(E, F)$, *and* $v \in \mathcal{L}^q(F, E)$,

$$\widetilde{\langle \cdot | y \rangle x}(u) = \langle ux | y \rangle,$$

$$\widetilde{v}(\langle \cdot | x \rangle y) = \langle vy | x \rangle,$$

$$\widetilde{\langle \cdot | b \rangle a}(\langle \cdot | x \rangle y) = \langle a | x \rangle \langle y | b \rangle.$$

Now

$$\widetilde{\langle \cdot | y \rangle x}(u) = tr((\langle \cdot | y \rangle x) \circ u) = \langle ux | y \rangle,$$

$$\widetilde{v}(\langle \cdot | x \rangle y) = tr(v \circ (\langle \cdot | x \rangle y)) = \langle vy | x \rangle,$$

$$\widetilde{\langle \cdot | b \rangle a}(\langle \cdot | x \rangle y) = \langle (\langle \cdot | b \rangle a) y | x \rangle = \langle y | b \rangle \langle a | x \rangle$$

(Corollary 6.1.5.2). ■

Corollary 6.1.6.3 (0) *Let* E *be a complex (real)* C^**-algebra and* (H, φ) *the (complex) universal representation of* E. *Then for every* $x' \in E'$ *there is a* $u \in \mathcal{L}_f(H)$ *such that*

$$x' = \overset{\infty}{\widetilde{u}} \circ \varphi.$$

The assertion follows from Corollary 5.4.2.4 (Proposition 5.4.2.6 e)) and Proposition 6.1.6.2. ■

Corollary 6.1.6.4 *If $\mathfrak{E}$, $\mathfrak{F}$ are filters on E and F, respectively, which converge weakly to 0 and such that $E^\# \in \mathfrak{E}$, $F^\# \in \mathfrak{F}$, then*

$$\lim_{(x,y),\mathfrak{E}\times\mathfrak{F}} \overbrace{\langle\cdot|x\rangle y}^{0} = 0$$

in the topology of pointwise convergence.

Given $u \in \mathcal{K}(F,E)$ and $(x,y) \in E \times F$,

$$|\overbrace{\langle\cdot|x\rangle y}^{0}(u)| = |\langle uy|x\rangle| \leq \|uy\|\,\|x\|$$

(Proposition 6.1.6.2) and the assertion follows from Theorem 5.5.5.1 a $\Rightarrow$ c. ■

Proposition 6.1.6.5 *If $u \in \mathcal{L}^2(F,E)$, then*

$$\widetilde{u} = \langle\cdot|u^*\rangle.$$

Given $v \in \mathcal{L}^2(E,F)$,

$$\widetilde{u}(v) = tr(u \circ v) = \langle v|u^*\rangle$$

(Corollary 6.1.5.6). ■

Theorem 6.1.6.6 (0) *If $u \in \mathcal{L}^q(F,E)$, then $\overset{p}{\widetilde{u}} \in \mathcal{L}^p(E,F)'$ and the map*

$$\mathcal{L}^q(F,E) \longrightarrow \mathcal{L}^p(E,F)', \quad u \longmapsto \overset{p}{\widetilde{u}}$$

is a norm preserving linear map. If $p \neq \infty$, then it is an isometry.

$\widetilde{u}$ is linear and if $v \in \mathcal{L}^p(E,F)$, then

$$|\widetilde{u}(v)| = |tr(u \circ v)| \leq \|u \circ v\|_1 \leq \|u\|_q \|v\|_p$$

(Corollary 6.1.5.3, Theorem 6.1.3.21). Hence $\widetilde{u} \in \mathcal{L}^p(E,F)'$. The final assertions follow from Proposition 6.1.5.10, Theorem 6.1.3.12 d), and Corollary 5.2.3.4. ■

Remark. We will often identify $\mathcal{K}(E,F)'$ with $\mathcal{L}^1(F,E)$ using the isometry

$$\mathcal{L}^1(F,E) \longrightarrow \mathcal{K}(E,F)', \quad u \longmapsto \overset{0}{\widetilde{u}},$$

and $\mathcal{L}^1(F,E)'$ with $\mathcal{L}(E,F)$ using the isometry

$$\mathcal{L}(E,F) \longrightarrow \mathcal{L}^1(F,E)', \quad u \longmapsto \overset{1}{\widetilde{u}}.$$

$\mathcal{K}(E,F)''$ is thereby identified with $\mathcal{L}(E,F)$ and $\mathcal{L}^1(F,E)''$ with $\mathcal{L}(E,F)'$.

Corollary 6.1.6.7 (0) *$\mathcal{L}^p(E,F)$ is reflexive for every $p \in]1,\infty[$. $\mathcal{L}^1(E,F)$ and $\mathcal{L}(E,F)$ are dual spaces.* ■

Corollary 6.1.6.8 (0) *Invoking identifications of the Remark of Theorem 6.1.6.6, the inclusion*

$$\mathcal{K}(E,F) \longrightarrow \mathcal{L}(E,F)$$

is given by the evaluation map.

Let

$$j : \mathcal{K}(E,F) \longrightarrow \mathcal{L}(E,F)$$

be the inclusion map and take $u \in \mathcal{K}(E,F)$. Then, given any $v \in \mathcal{L}^1(F,E)$,

$$\langle u, v\rangle = \widetilde{v}(u) = tr(v \circ u) = tr(u \circ v) = tr((ju) \circ v) = \widetilde{ju}(v) = \langle v, ju\rangle$$

(Proposition 6.1.5.7). Hence, the inclusion is the evaluation map. ■

Remark. Assume that E and F are infinite-dimensional. By Corollary 5.5.2.8, $\mathcal{K}(E,F)$ is not a complemented subspace of $\mathcal{L}(E,F)$. By the above corollary and Corollary 1.4.2.16 c $\Rightarrow$ a (and the Murray's Theorem) $\mathcal{K}(E,F)$ is not a dual space. If fact, this was already proved in Proposition 6.1.3.23.

Corollary 6.1.6.9 *The following are equivalent:*

a) *One of E and F is finite-dimensional.*

b) $\mathcal{K}(E,F) = \mathcal{L}^1(E,F) = \mathcal{L}(E,F)$.

c) *One of $\mathcal{K}(E,F)$, $\mathcal{L}^1(E,F)$, and $\mathcal{L}(E,F)$ is reflexive.*

d) *$\mathcal{K}(E,F)$, $\mathcal{L}^1(E,F)$, and $\mathcal{L}(E,F)$ are reflexive.*

e) *$\mathcal{K}(E,F)$ is a dual space.*

a $\Rightarrow$ b. Take $u \in \mathcal{L}(E,F)$. From

$$u = u \circ 1_E = 1_F \circ u$$

we get that $u \in \mathcal{K}(E,F)$ (Proposition 3.1.1.11), so that

$$\mathcal{K}(E,F) = \mathcal{L}(E,F).$$

Take $u \in \mathcal{K}(E,F)$ and let

$$\sum_{n\in N} \theta_n\langle\cdot|x_n\rangle y_n$$

be a Schatten decomposition of u (Theorem 6.1.3.1 a)). By a), N is finite, so that $u \in \mathcal{L}^1(E,F)$ (Corollary 6.1.2.12). Hence

$$\mathcal{K}(E,F) = \mathcal{L}^1(E,F)$$

(Corollary 6.1.2.10).

b $\Rightarrow$ c. By Corollary 6.1.6.8, $\mathcal{K}(E,F)$ is reflexive.

c $\Rightarrow$ d follows from Theorem 6.1.6.6 and Proposition 1.3.8.4.

d $\Rightarrow$ e is trivial.

e $\Rightarrow$ a. Let I be a predual of $\mathcal{K}(E,F)$. By Theorem 6.1.6.6, $\mathcal{L}(E,F)$ is isometric to the tridual of I. Hence, by Proposition 1.3.6.19 b) (and by Murray's Theorem), $\mathcal{K}(E,F)$ is a complemented subspace of $\mathcal{L}(E,F)$. By Corollary 5.5.2.8, one of the spaces E and F must be finite–dimensional. ■

Corollary 6.1.6.10 (0) *If we identify $\mathcal{L}^1(E,F)''$ with $\mathcal{L}(F,E)'$ (Remark of Theorem 6.1.6.6), then the mapping*

$$\mathcal{L}^1(E,F) \longrightarrow \mathcal{L}^1(E,F)'', \quad u \longmapsto \overset{\infty}{\widetilde{u}}$$

is the evaluation map.

If $u \in \mathcal{L}^1(E,F)$ and $v \in \mathcal{L}(F,E)$, then

$$\langle u, \overset{1}{\widetilde{v}}\rangle = tr(v \circ u) = tr(u \circ v) = \langle \overset{\infty}{\widetilde{u}}, v\rangle$$

(Proposition 6.1.5.7). ■

Corollary 6.1.6.11 (0) *The ultraweak topology (which will be introduced and studied in Subsection 6.3.1) on $\mathcal{L}(E,F)$ is obtained by identification with $\mathcal{L}^1(F,E)'_{\mathcal{L}^1(F,E)}$ (Remark of Theorem 6.1.6.6). Let $\mathcal{F}$ be an ultraweak closed vector subspace of $\mathcal{L}(E,F)$ and*

$$q : \mathcal{L}^1(F,E) \longrightarrow \mathcal{L}^1(F,E)/^\circ\mathcal{F}$$

the quotient map. Then the map

$$(\mathcal{L}^1(F,E)/^\circ\mathcal{F})' \longrightarrow \mathcal{F}, \quad x' \longmapsto q'x'$$

is an isometry.

The assertion follows immediately from Corollary 1.3.5.5. ■

Proposition 6.1.6.12 *Given* $u \in \mathcal{L}^1(E)$, *the linear forms* $\overset{0}{\widetilde{u}}$ *and* $\overset{\infty}{\widetilde{u}}$ *are order continuous.*

Let

$$\sum_{n\in N} \theta_n \langle \cdot | x_n \rangle y_n$$

be a Schatten decomposition of u (Theorem 6.1.3.1 a)), $\mathcal{F}$ a downward directed set in $\mathcal{L}(E)$ with infimum 0 and $\mathfrak{F}$ its lower section filter. By Theorem 5.3.3.14 c),

$$\lim_{v,\mathfrak{F}} \langle v y_n | x_n \rangle = 0$$

for every $n \in N$. An easy consequence is that

$$\lim_{v,\mathfrak{F}} \overset{\infty}{\widetilde{u}}(v) = 0\,.$$

Hence $\overset{\infty}{\widetilde{u}}$ is order continuous. Since $\mathcal{K}(E)$ is a hereditary set in $\mathcal{L}(E)$ (Corollary 3.1.1.13, Proposition 4.3.4.5 e)), $\overset{0}{\widetilde{u}}$ is also order continuous. ■

Corollary 6.1.6.13 *If* $\mathbb{K} = \mathbb{C}$ $(\mathbb{K} = \mathbb{R})$, *then*

$$\mathcal{K}(E)' = \mathcal{K}(E)^\pi = \{u' | u' \text{ is an order continuous linear form on } \mathcal{K}(E)\}\,,$$

$$(\mathrm{Re}\,\mathcal{K}(E)' = \mathrm{Re}\,\mathcal{K}(E)^\pi =$$

$$= \{u' | u' \text{ is a selfadjoint order continuous linear form on } \mathcal{K}(E)\})\,.$$

The inclusion

$$\mathcal{K}(E)^\pi \subset \{u' | u' \text{ is an order continuous linear form on } \mathcal{K}(E)\}\,,$$

$$(\mathrm{Re}\,\mathcal{K}(E)^\pi \subset \{u' | u' \text{ is a selfadjoint order continuous linear form on } \mathcal{K}(E)\})$$

is trivial. The inclusion

$$\{u' | u' \text{ is an order continuous linear form on } \mathcal{K}(E)\} \subset \mathcal{K}(E)'$$

$$\{u' | u' \text{ is a selfadjoint order continuous linear form on } \mathcal{K}(E)\} \subset \mathrm{Re}\,\mathcal{K}(E)')$$

follows from Proposition 1.7.2.4 b) and Theorem 4.2.2.9 a). Take $u' \in \mathcal{K}(E)'_+$. By Proposition 6.1.6.12, u' is order continuous and so it belongs to $\mathcal{K}(E)^\pi$. Hence

$$\mathcal{K}(E)' \subset \mathcal{K}(E)^\pi \qquad (\mathrm{Re}\,\mathcal{K}(E)' \subset \mathcal{K}(E)^\pi)$$

(Corollary 4.1.2.7 d)). ■

Remark. If E is infinite–dimensional, then the supplementary hypothesis in the real case is necessary. Indeed, if we put

$$\mathcal{F} := \{u \in \mathcal{K}(E) | u^* = -u\}$$

and consider a non–continuous linear form on $\mathcal{F}$, then its linear extension to $\mathcal{K}(E)$ restricting to 0 on $\mathrm{Re}\,\mathcal{K}(E)$ is a non–continuous order continuous linear form on $\mathcal{K}(E)$.

Proposition 6.1.6.14 *Given $u, v \in \mathcal{L}(E)$, the following are equivalent:*

a) $uv = vu$.

b) *For $x, y \in E$,*

$$\langle ux | v^* y \rangle = \langle vx | u^* y \rangle .$$

In the complex case, the above assertions are equivalent to:

c) $x \in E \Rightarrow \langle ux | v^* x \rangle = \langle vx | u^* x \rangle$.

a $\Rightarrow$ b. $\langle ux | v^* y \rangle = \langle vux | y \rangle = \langle uvx | y \rangle = \langle vx | u^* y \rangle$.
b $\Rightarrow$ a. By Proposition 6.1.6.2,

$$\overbrace{uv - vu}^{1}(\langle \cdot | y \rangle x) = \langle uvx | y \rangle - \langle vux | y \rangle = \langle vy | u^* y \rangle - \langle ux | v^* y \rangle = 0$$

for all $x, y \in E$. Hence $\overbrace{uv - vu}^{1}$ vanishes on $\mathcal{L}_f(E)$ (Corollary 5.5.1.11) and therefore on $\mathcal{L}^1(E)$ (Theorem 6.1.3.12 d)). By Theorem 6.1.6.6,

$$uv - vu = 0 ,$$

i.e.

$$uv = vu\,.$$

b $\Rightarrow$ c is trivial.

c $\Rightarrow$ b. The map

$$\varphi : E \times E \longrightarrow \mathbb{C}\,, \quad (x,y) \longmapsto \langle ux|v^*y\rangle - \langle vx|u^*y\rangle$$

is sesquilinear. By c),

$$\varphi(x,x) = 0$$

for all $x \in E$. By the polarisation identity, φ vanishes identically. ■

Proposition 6.1.6.15 *Let Ω be the class of Hilbert spaces and $\mathcal{K}_\Omega$ the subcategory of $\mathcal{L}_\Omega$ defined by the map*

$$(E,F) \longmapsto \mathcal{K}(E,F)$$

on Ω^2. If we identify $\mathcal{K}(E,F)''$ with $\mathcal{L}(E,F)$ for all $E,F \in \Omega$ (Remark of Theorem 6.1.6.6), then the Arens multiplications of $\mathcal{K}''_\Omega$ coincide with the multiplication in $\mathcal{L}_\Omega$. ■

6.1.7 Exterior Multiplication and $\mathcal{L}^p$–Spaces

Throughout this subsection, $p \in \{0\} \cup [1, \infty]$ and q is the conjugate of p.

Definition 6.1.7.1 (0) *Define*

$$E \xrightarrow{u} F :\Longleftrightarrow u \in \mathcal{L}(E, F)\,,$$

$$E \underset{p}{\xrightarrow{u'}} F :\Longleftrightarrow u' \in \mathcal{L}^p(F, E)'\,.$$

Suppose that $E \underset{p}{\xrightarrow{u'}} F$, $F \xrightarrow{v} G$, $H \xrightarrow{w} E$. *We define*

$$vu' : \mathcal{L}^p(G, E) \longrightarrow \mathbb{K}\,, \quad u \longmapsto \langle u \circ v, u' \rangle\,,$$

$$u'w : \mathcal{L}^p(F, H) \longrightarrow \mathbb{K}\,, \quad u \longmapsto \langle w \circ u, u' \rangle\,,$$

$$u'^* : \mathcal{L}^p(E, F) \longrightarrow \mathbb{K}\,, \quad u \longmapsto \overline{\langle u^*, u' \rangle}$$

(Corollary 6.1.3.13 b), Corollary 6.1.3.2 a $\Rightarrow$ b)).

If $E = F = G = H$, then the above definitions coincide with the definitions given in Proposition 2.2.7.2 and Definition 2.3.1.1.

Proposition 6.1.7.2 *Suppose that* $F \underset{\infty}{\xrightarrow{u'}} E$. *If* $u' \in \mathcal{K}(E, F)^\circ$, *then*

$$uu' = 0\,, \quad u'v = 0$$

for every $u \in \mathcal{K}(E, G)$ *and* $v \in \mathcal{K}(G, F)$, *and*

$$\pi_{G^\perp} u' = u'\,, \quad u' \pi_{H^\perp} = u'$$

for all finite-dimensional vector subspaces G *of* E *and* H *of* F.

If $G \xrightarrow{w} F$ (resp. $E \xrightarrow{w} G$), then

$$\langle w, uu' \rangle = \langle w \circ u, u' \rangle = 0$$

$$(\langle w, u'v \rangle = \langle v \circ w, u' \rangle = 0)$$

(Proposition 3.1.1.11), so

$$uu' = 0 \quad (u'v = 0)\,.$$

It follows that

$$u' = (\pi_G + \pi_{G^\perp})u' = \pi_G u' + \pi_{G^\perp} u' = \pi_{G^\perp} u'\,,$$

$$u' = u'(\pi_H + \pi_{H^\perp}) = u'\pi_H + u'\pi_{H^\perp} = u'\pi_{H^\perp}\,.$$

■

Proposition 6.1.7.3 (0) *Suppose that* $E \overset{u'}{\underset{p}{\to}} F$.

a) *If* $F \overset{v}{\to} G$, *then* $E \overset{vu'}{\underset{p}{\to}} G$ *and* $\|vu'\| \leq \|v\| \, \|u'\|$.

b) *If* $G \overset{v}{\to} E$, *then* $G \overset{u'v}{\underset{p}{\to}} F$ *and* $\|u'v\| \leq \|u'\| \, \|v\|$.

c) *If* $F \overset{v}{\to} G \overset{w}{\to} H$, *then* $w(vu') = (w \circ v)u'$.

d) *If* $H \overset{w}{\to} G \overset{v}{\to} E$, *then* $(u'v)w = u'(w \circ v)$.

e) *If* $F \overset{v}{\to} G$ *and* $H \overset{w}{\to} E$, *then* $(vu')w = v(u'w)$.

f) $1_F u' = u' 1_E = u'$.

g) $F \overset{u'^*}{\underset{p}{\to}} E$, $\|u'^*\| = \|u'\|$, $u'^{**} = u'$.

h) *If* $F \overset{v}{\to} G$, *then* $(vu')^* = u'^* v^*$.

i) *If* $G \overset{v}{\to} E$, *then* $(u'v)^* = v^* u'^*$.

a) Given $u \in \mathcal{L}^p(G, E)$

$$|\langle u, vu' \rangle| = |\langle u \circ v, u' \rangle| \leq \|u \circ v\|_p \|u'\| \leq \|u\|_p \|v\| \, \|u'\|$$

(Corollary 6.1.3.13 b)).

b) Given $u \in \mathcal{L}^p(F, G)$,

$$|\langle u, u'v \rangle| = |\langle v \circ u, u' \rangle| \leq \|v \circ u\|_p \|u'\| \leq \|v\| \, \|u\|_p \|u'\|$$

(Corollary 6.1.3.13 b)).

c),d),e),f),h),i) are trivial.

g) Given $u \in \mathcal{L}^p(E, F)$

$$|\langle u, u'^* \rangle| = |\overline{\langle u^*, u' \rangle}| \leq \|u^*\|_p \|u'\| = \|u\|_p \|u'\|$$

(Corollary 6.1.3.2), so

$$F \overset{u'^*}{\underset{p}{\to}} E, \quad \|u'^*\| \leq \|u'\|.$$

The relation

$$u'^{**} = u'$$

is trivial. We deduce from the above inequality that

$$\|u'\| = \|u'^{**}\| \leq \|u'^*\| \leq \|u'\|,$$

$$\|u'^*\| = \|u'\|. \qquad \blacksquare$$

Corollary 6.1.7.4 (0) *$\mathcal{L}^p(E)'$ is an involutive unital $\mathcal{L}(E)$–module.* ■

Remark. This assertion was already proved in Corollary 6.1.3.14.

Corollary 6.1.7.5 (0) *If $u' \in \mathcal{L}(E)'_+$, then $u^*u'u \in \mathcal{L}(F)'_+$ for every $u \in \mathcal{L}(F,E)$.*

By Proposition 6.1.7.3 e),h),i),

$$(u^*u'u)^* = u^*u'u\,.$$

Thus $u^*u'u$ is selfadjoint. Given $v \in \mathcal{L}(F)$,

$$\langle v^*v, u^*u'u\rangle = \langle uv^*vu^*, u'\rangle = \langle (vu^*)^*(vu^*), u'\rangle \in \mathbb{R}_+\,,$$

and so $u^*u'u \in \mathcal{L}(F)'_+$. ■

Corollary 6.1.7.6 *Let F be a closed vector subspace of E, and take $u' \in \mathcal{L}(E)'_+$ such that*

$$u'\pi_F = \pi_F u'\,.$$

Then $\pi_F u'$, $\pi_{F^\perp} u' \in \mathcal{L}(E)'_+$.

By Corollary 6.1.7.5,

$$\pi_F u' = \pi_F^2 u' = \pi_F u' \pi_F \in \mathcal{L}(E)'_+\,.$$

Since

$$\pi_{F^\perp} u' = (1-\pi_F)u' = u'(1-\pi_F) = u'\pi_{F^\perp}\,,$$

it follows that

$$\pi_{F^\perp} u' \in \mathcal{L}(E)'_+\,.$$ ■

Corollary 6.1.7.7 *Let F be a closed vector subspace of E and take $u' \in \tau_0(\mathcal{L}(E))$ such that*

$$u'\pi_F = \pi_F u'\,.$$

Then

$$\pi_F u' = 0 \ \text{ or } \ \pi_{F^\perp} u' = 0\,.$$

Using Corollary 6.1.7.6, it follows from

$$u' = \pi_F u' + \pi_{F^\perp} u'$$

that

$$0 \leq \pi_F u' \leq u', \quad 0 \leq \pi_{F^\perp} u' \leq u'.$$

By Proposition 2.3.5.4 a $\Rightarrow$ c, there are $\alpha, \beta \in \mathbb{K}$ such that

$$\pi_F u' = \alpha u', \quad \pi_{F^\perp} u' = \beta u'.$$

By Proposition 6.1.7.3 c),

$$\alpha \pi_{F^\perp} u' = \pi_{F^\perp}(\alpha u') = \pi_{F^\perp} \pi_F u' = 0,$$

$$\beta \pi_F u' = \pi_F(\beta u') = \pi_F \pi_{F^\perp} u' = 0,$$

and the assertion now follows, since $\alpha + \beta = 1$. ∎

Proposition 6.1.7.8 (0) *If* $u \in \mathcal{L}^q(G,E)$, $v \in \mathcal{L}(E,F)$, *and* $w \in \mathcal{L}(F,G)$, *then*

$$\overset{p}{v\widetilde{u}} = \overset{p}{\widetilde{v \circ u}}, \quad \overset{p}{\widetilde{u}w} = \overset{p}{\widetilde{u \circ w}}, \quad (\overset{p}{\widetilde{u}})^* = \overset{p}{\widetilde{u^*}}.$$

Given $t \in \mathcal{L}^p(F,G)$,

$$v\widetilde{u}(t) = \widetilde{u}(t \circ v) = tr(u \circ t \circ v) = tr(v \circ u \circ t) = \widetilde{v \circ u}(t)$$

(Proposition 6.1.5.7). Given $t \in \mathcal{L}^p(E,F)$,

$$\widetilde{u}w(t) = \widetilde{u}(w \circ t) = tr(u \circ w \circ t) = \widetilde{u \circ w}(t).$$

Finally, if $t \in \mathcal{L}^p(G,E)$, then

$$\widetilde{u}^*(t) = \overline{\widetilde{u}(t^*)} = \overline{tr(u \circ t^*)} = tr(t \circ u^*) = tr(u^* \circ t) = \widetilde{u^*}(t)$$

(Corollary 6.1.5.3, Proposition 6.1.5.7). ∎

Remark. By the above proposition , the mapping $u \mapsto \widetilde{u}$ is a functor of $\mathcal{L}$–modules of $\mathcal{L}^q$ into $(\mathcal{L}^p)'$.

Corollary 6.1.7.9 (0) *The map*

$$\mathcal{L}^q(E) \longrightarrow \mathcal{L}^p(E)', \quad u \longmapsto \overset{p}{\widetilde{u}}$$

is an involutive, norm-preserving homomorphism of involutive $\mathcal{L}(E)$-modules (Corollary 6.1.7.4). It is an isometry unless $p = \infty$. ■

Proposition 6.1.7.10 (0) *Take $u \in \mathcal{L}^q(E)$ $(u \in \operatorname{Re}\mathcal{L}^q(E))$. If $\mathbb{K} = \mathbb{C}$ $(\mathbb{K} = \mathbb{R})$, then the following are equivalent:*

a) $u \geq 0$.

b) $v \in \mathcal{L}^{2p}(E,F) \Rightarrow \overset{p}{\widetilde{u}}(v^* \circ v) \in \mathbb{R}_+$.

c) $v \in \mathcal{L}^p(E_+) \Rightarrow \overset{p}{\widetilde{u}}(v) \in \mathbb{R}_+$.

d) $\langle ux|x\rangle \in \mathbb{R}_+$ *for every* $x \in E$.

e) $\overset{p}{\widetilde{u}}$ *is positive in the sense of Definition 2.3.4.1 (see Corollary 6.1.3.14).*

a $\Rightarrow$ b. We have $v^* \circ v \in \mathcal{L}^p(E)$ (Theorem 6.1.5.11),

$$v^* \circ v \geq 0, \quad u^{\frac{1}{2}} \circ v^* \circ v \circ u^{\frac{1}{2}} \geq 0$$

(Corollary 5.3.3.9), and

$$\widetilde{u}(v^* \circ v) = tr(u \circ v^* \circ v) = tr(u^{\frac{1}{2}} \circ v^* \circ v \circ u^{\frac{1}{2}}) \in \mathbb{R}_+$$

(Proposition 6.1.5.7, Corollary 6.1.5.3).

b $\Rightarrow$ c. We have that $v^{\frac{1}{2}} \in \mathcal{L}^{2p}(E)$ (Corollary 6.1.2.18), and so

$$\widetilde{u}(v) = \widetilde{u}(v^{\frac{1}{2}} v^{\frac{1}{2}}) \in \mathbb{R}_+.$$

c $\Rightarrow$ d. We have that $\langle \cdot|x\rangle x \in \mathcal{L}^p(E)_+$, and so

$$\langle ux|x\rangle = \widetilde{u}(\langle \cdot|x\rangle x) \in \mathbb{R}_+$$

(Proposition 6.1.6.2).

d $\Rightarrow$ a follows from Corollary 5.3.3.7.

b $\Rightarrow$ e is trivial.

e $\Rightarrow$ d. We have that

$$\langle ux|x\rangle = \widetilde{u}(\langle \cdot|x\rangle x) = \widetilde{u}((\langle \cdot|x\rangle x)^* \circ (\langle \cdot|x\rangle x)) \in \mathbb{R}_+$$

(Proposition 6.1.6.2). ■

Corollary 6.1.7.11 (0) *If j is the evaluation map of $\mathcal{L}^1(E)$, then*

$$ju^+ = (ju)^+, \quad ju^- = (ju)^-$$

for every $u \in \operatorname{Re}\mathcal{L}^1(E)$.

By Corollary 6.1.2.14, $u^+, u^- \in \mathcal{L}^1(E)$ and

$$\|u\|_1 = \|u^+\|_1 + \|u^-\|_1 .$$

By Proposition 6.1.7.8 and 6.1.7.10 a $\Rightarrow$ e, $ju \in \operatorname{Re}\mathcal{L}(E)'$ and $ju^+, ju^- \in \mathcal{L}(E)'_+$. Since

$$ju = ju^+ - ju^- ,$$

$$\|ju\| = \|u\|_1 = \|u^+\|_1 + \|u^-\|_1 = \|ju^+\| + \|ju^-\|$$

we see that

$$(ju)^+ = ju^+, \quad (ju)^- = ju^-$$

(Theorem 4.2.8.13). ■

Proposition 6.1.7.12 (0) *The set $\{\widetilde{\overset{0}{\langle\cdot|x\rangle x}}|x \in E\})$ is closed in the topology of pointwise convergence.*

Take

$$\varphi : E \longrightarrow \mathcal{K}(E)', \quad x \longmapsto \widetilde{\overset{0}{\langle\cdot|x\rangle x}},$$

and let $\mathfrak{F}$ be an ultrafilter on E, such that $\varphi(\mathfrak{F})$ converges to some $u' \in \mathcal{K}(E)'$ in the topology of poinwise convergence. Since the set $\mathcal{K}(E)'_+$ is closed in the topology of pointwise convergence, we have $u' \in \mathcal{K}(E)'_+$ (Corollary 5.3.3.7). By Theorem 6.1.6.6 and Proposition 6.1.7.10 e $\Rightarrow$ a, there is a $u \in \mathcal{L}^1(E)_+$ such that $\widetilde{u} = u'$. By Corollary 6.1.2.13 a $\Rightarrow$ b & c, there is an orthonormal family $(x_n)_{n\in N(u)}$ in E such that

$$u = \sum_{n\in N(u)} \theta_n(u)\langle\cdot|x_n\rangle x_n .$$

We may assume that $\theta_1(u) \neq 0$. Then

$$\theta_1(u) = \langle ux_1|x_1\rangle = \widetilde{u}(\langle\cdot|x_1\rangle x_1) =$$

$$= \lim_{x,\mathfrak{F}} \overset{0}{\widetilde{\langle\cdot|x\rangle x}}(\langle\cdot|x_1\rangle x_1) = \lim_{x,\mathfrak{F}} |\langle x_1|x\rangle|^2$$

(Proposition 6.1.6.2). Take $y \in (x_1)^\perp$. Then

$$uy = \sum_{n\in N(u)} \theta_n(u)\langle y|x_n\rangle x_n \in \{x_1\}^\perp,$$

$$0 = \langle uy|x_1\rangle = \widetilde{u}(\langle\cdot|x_1\rangle y) = \lim_{x,\mathfrak{F}} \overset{0}{\widetilde{\langle\cdot|x\rangle x}}(\langle\cdot|x_1\rangle y) =$$

$$= \lim_{x,\mathfrak{F}} \langle x|x_1\rangle\langle y|x\rangle,$$

$$\lim_{x,\mathfrak{F}} \langle y|x\rangle = 0$$

(Proposition 6.1.6.2). It follows that

$$\theta_n(u) = \langle ux_n|x_n\rangle = \widetilde{u}(\langle\cdot|x_n\rangle x_n) =$$

$$= \lim_{x,\mathfrak{F}} \overset{0}{\widetilde{\langle\cdot|x\rangle x}}(\langle\cdot|x_n\rangle x_n) = \lim_{x,\mathfrak{F}} |\langle x|x_n\rangle|^2 = 0$$

for every $n \in N(u)\backslash\{1\}$, i.e.

$$u' = \widetilde{u} = \theta_1(u)\overset{0}{\widetilde{\langle\cdot|x\rangle x}} = \overset{0}{\overbrace{\langle\cdot|\theta_1(u)^{\frac{1}{2}}x\rangle\theta_1(u)^{\frac{1}{2}}x}}. \qquad \blacksquare$$

Proposition 6.1.7.13 (0)

a) $\tau_0(\mathcal{K}(E)) = \{\overset{0}{\widetilde{\langle\cdot|x\rangle x}}|x \in E,\ \|x\| = 1\}$.

b) $\tau_0(\mathcal{K}(E))$ *is closed in* $\tau(\mathcal{K}(E))$.

a) Given $x \in E$, define

$$\widetilde{x} := \overset{0}{\widetilde{\langle\cdot|x\rangle x}}.$$

Take $x' \in \tau_0(\mathcal{K}(E))$. By Theorem 6.1.6.6, there is an $u \in \mathcal{L}^1(E)$, such that $x' = \overset{0}{\widetilde{u}}$. By Proposition 6.1.7.10 e $\Rightarrow$ a, $u \in \mathcal{L}^1(E)_+$, so by Corollary 6.1.2.13, there is an orthonormal family $(x_n)_{n\in N}$ in E and an $f \in \ell^1(N)_+$ such that

$$n \in N \Longrightarrow f(n) \neq 0\,,$$

$$u = \sum_{n \in N} f(n) \langle \cdot | x_n \rangle x_n\,.$$

Take $n \in N$. Then

$$f(n) \widetilde{x}_n \leq x'\,,$$

so that

$$x' = \widetilde{x}_n \in \{\widetilde{\langle \cdot | x \rangle x} \mid x \in E\,,\ \|x\| = 1\}\,.$$

(Proposition 2.3.5.4 a $\Rightarrow$ b).

Take $x \in E$ with $\|x\| = 1$. By Proposition 6.1.2.3 and Proposition 5.5.5.7 d),

$$\langle \cdot | x \rangle x \in \mathcal{L}^1(E)_+\,, \quad \|\langle \cdot | x \rangle x\| = 1\,.$$

Hence $\widetilde{x} \in \tau(E)$. Let $x' \in \mathcal{K}(E)'_+$ with $x' \leq \widetilde{x}$. By Theorem 6.1.6.6, there is an $u \in \mathcal{L}^1(E)$ such that $x' = \overset{0}{\widetilde{u}}$. By Proposition 6.1.7.10 e $\Rightarrow$ a $u \in \mathcal{L}^1(E)_+$, so by Corollary 6.1.2.13, there is an orthonormal family $(x_n)_{n \in N}$ in E as well as an $f \in \ell^1(N)_+$ such that

$$f(n) \neq 0 \text{ for } n \in N\,,$$

$$u = \sum_{n \in N} f(n) \langle \cdot | x_n \rangle x_n\,.$$

By Corollary 5.3.3.7, given $y \in \{x\}^\perp$,

$$\sum_{n \in N} f(n) |\langle y | x_n \rangle|^2 = \langle uy | y \rangle \leq \langle (\langle \cdot | x \rangle x) y | y \rangle = 0\,,$$

so that

$$\{x_n | n \in N\} \subset \{x\}^{\perp\perp} = \{\alpha x | \alpha \in \mathbb{K}\}\,.$$

Hence

$$\operatorname{Card} N \leq 1\,,$$

and therefore there is an $\alpha \in \mathbb{R}_+$ such that

$$u = \alpha \langle \cdot | x \rangle x\,.$$

We get

$$x' = \overset{0}{\widetilde{u}} = \alpha\widetilde{x}\,.$$

By Proposition 2.3.5.4 b $\Rightarrow$ a, $\widetilde{x} \in \tau_0(E)$.

b) follows from a) and Proposition 6.1.7.12. ∎

Corollary 6.1.7.14 (0) *Assume $E \neq \{0\}$. We identify $\mathcal{L}^1(E)$ with an involutive $\mathcal{L}(E)$–submodule of $\mathcal{L}(E)'$ via the norm–preserving homomorphism of involutive $\mathcal{L}(E)$–modules*

$$\mathcal{L}^1(E) \longrightarrow \mathcal{L}(E)', \quad u \longmapsto \overset{\infty}{\widetilde{u}}$$

(Corollary 6.1.7.9).

a) *The map*

$$\mathcal{L}(E) \longrightarrow \mathcal{L}^1(E)', \quad u \longmapsto \overset{1}{\widetilde{u}} = \langle u, \cdot\rangle|\mathcal{L}^1(E)$$

is an isometry.

b) *$\mathcal{L}(E)$ is a W^*–algebra with $\mathcal{L}^1(E)$ as predual.*

c) *Every element of $\mathcal{L}(E)$ has a left and a right carrier and a polar representation.*

d) *Take $u \in \mathcal{L}^1(E)$. The left and the right carrier and the polar representation of u as element of $\mathcal{L}(E)$ coincide with the corresponding notions for u as an element of the predual of $\mathcal{L}(E)$ (Corollary 4.4.2.17, Theorem 4.4.3.5). In particular, the moduli of u in $\mathcal{L}^1(E)$ and $\mathcal{L}(E)$ coincide.*

a) follows from Corollary 6.1.7.9.

b) follows from a).

c) By b) and Theorem 4.4.1.8 j), every element of $\mathcal{L}(E)$ has a left and right carrier. By b) and Theorem 4.4.3.1, every element of $\mathcal{L}(E)$ has a polar representation.

d) follows from b) and the definitions. ∎

Theorem 6.1.7.15 (0) *Let u be a self–normal operator on E, $j : \sigma(u) \to \mathbb{K}$ the inclusion map, and $\mathcal{B}$ the Banach algebra of bounded Borel functions on $\sigma(u)$.*

a) *There is a unique involutive unital algebra homomorphism* $\varphi : \mathcal{B} \to \mathcal{L}(E)$ *such that*

$$\varphi j = u$$

and

$$\varphi\left(\bigwedge_{n\in\mathbb{N}}^{\mathcal{B}} f_n\right) = \bigwedge_{n\in\mathbb{N}}^{\mathcal{L}(E)} (\varphi f_n)$$

for every decreasing sequence $(f_n)_{n\in\mathbb{N}}$ *in* $\mathcal{B}_+$.

b) $\varphi f = f(u)$ *for every* $f \in \mathcal{C}(\sigma(u))$. *Given* $f \in \mathcal{B}$, *define* $f(u) := \varphi f$.

c) *If* $f \in \mathcal{B}$, *then*

$$\sigma(f(u)) \subset \overline{f(\sigma(u))}, \quad \|f(u)\| \leq \|f\|_\infty, \quad \{u\}^c \subset \{f(u)\}^c.$$

d) *If* $f \in \mathcal{B}$ *and if* g *is a bounded Borel function on* $\overline{f(\sigma(x))}$, *then*

$$g(f(u)) = g \circ f(u).$$

e) *For every* $\varepsilon > 0$, *there is a finite family* $(\alpha_\iota)_{\iota\in I}$ *in* $\sigma(u)$ *as well as a family* $(p_\iota)_{\iota\in I}$ *of orthogonal projections in* E *such that*

$$\left\| u - \sum_{\iota\in I} \alpha_\iota p_\iota \right\| < \varepsilon.$$

f) *If* $\mathbb{K} = \mathbb{C}$ $(\mathbb{K} = \mathbb{R})$, *then the vector subspace of* $\mathcal{L}(E)$ *generated by the orthogonal projections is dense in* $\mathcal{L}(E)$ *(in* $\operatorname{Re}\mathcal{L}(E)$*).*

g) *Assume that* $\mathbb{K} = \mathbb{R}$ $(\mathbb{K} = \mathbb{C})$ *and let* $\mathcal{F}$ *be the hereditary closed (and involutive) subalgebra of* $\mathcal{L}(E)$ *generated by* u. *If* $f \in \mathcal{B}$ *satisfies*

$$0 \in \sigma(u) \Longrightarrow \lim_{\substack{\alpha\to 0 \\ \alpha\in\sigma(u)}} f(\alpha) = 0,$$

then $f(u) \in \mathcal{F}$.

h) *Let* $\mathbb{K} = \mathbb{R}$ $(\mathbb{K} = \mathbb{C})$. *Let* $\mathcal{F}$ *be a* σ*-faithful (involutive) closed subalgebra of* $\mathcal{L}(E)$ *containing* u, $\mathcal{G}$ *a* σ*-faithful (involutive) closed unital subalgebra of* $\mathcal{L}(E)$ *containing* u, *and put*

$$\mathcal{B}_0 := \{f \in \mathcal{B} | 0 \in \sigma(u) \Longrightarrow \lim_{\substack{\alpha\to 0 \\ \alpha\in\sigma(u)}} f(\alpha) = 0\}.$$

Then

$$\varphi(\mathcal{B}_0) \subset \mathcal{F}, \quad \varphi(\mathcal{B}) \subset \mathcal{G}.$$

i) *If*

$$F := \operatorname{Ker} u\,, \quad G := \overline{\operatorname{Im} u}\,,$$

then

$$\pi_F = e^{\sigma(u)}_{\{0\}}(u)\,, \quad \pi_G = e^{\sigma(u)}_{\sigma(u)\setminus\{0\}}(u)\,.$$

j) *If* $f \in \mathcal{B}$ *and* $\alpha \in \mathbb{K}$, *then*

$$\pi_{\operatorname{Ker} f(u)} = e^{\sigma(u)}_{\frac{-1}{f}(0)}(u)\,,$$

$$\operatorname{Ker}(\alpha 1 - u) \subset \operatorname{Ker}(f(\alpha)1 - f(u))\,.$$

In particular $\sigma_p(u)$ *is the set of atoms of* u.

a) to h). By Corollary 6.1.7.14 b), $\mathcal{L}(E)$ is a W^*-algebra, and the assertions follow from Theorem 4.4.1.8 c),f),g), Corollary 4.3.2.5, and Proposition 4.3.4.11.

i) The assertion follows from Corollary 4.3.3.14 and Propositions 5.3.4.1 and 5.3.3.13.

j) By i) and d),

$$\pi_{\operatorname{Ker} f(u)} = e_{\{0\}}(f(u)) = (e_{\{0\}} \circ f)(u) = e_{\frac{-1}{f}(0)}(u)\,.$$

Hence

$$\pi_{\operatorname{Ker}(\alpha 1-u)} = e_{\{\alpha\}}(u) \leq e_{\frac{-1}{f}(f(\alpha))}(u) = \pi_{\operatorname{Ker}(f(\alpha)1-f(u))}\,,$$

$$\operatorname{Ker}(\alpha 1 - u) \subset \operatorname{Ker}(f(\alpha)1 - f(u))$$

(Corollary 5.3.3.8.) ■

Corollary 6.1.7.16 (0) *Let* u *be a self-normal operator on* E. *Then the following are equivalent:*

a) u *is not compact.*

b) *There is an* $f \in C(\sigma(u))$ *such that*

$$0 \notin \operatorname{Supp} f$$

and $f(u) \notin \mathcal{L}_f(E)$.

c) *There is a Borel set* A *of* $\sigma(u)$ *such that* $0 \notin \overline{A}$ *and*

$$e_A^{\sigma(u)}(u) \notin \mathcal{L}_f(E)\,.$$

a $\Rightarrow$ b follows from Corollary 5.5.6.6 b $\Rightarrow$ a.

b $\Rightarrow$ c. f is a linear combination of a finite family $(f_\iota)_{\iota\in I}$ in $\mathcal{C}(\sigma(u))_+$ such that

$$f_\iota \leq |f|$$

for each $\iota \in I$. By b), there is a $\iota \in I$ such that $f_\iota(u) \notin \mathcal{L}_f(E)$. We may assume that $f_\iota \leq 1$. Then there is a Borel set A of $\sigma(u)$ such that $0 \notin \overline{A}$ and $f_\iota \leq e_A^{\sigma(u)}$. By Theorem 6.1.7.15 a),

$$0 \leq f_\iota(u) \leq e_A^{\sigma(u)}(u)\,,$$

and by Corollary 5.5.6.7, $e_A^{\sigma(u)}(u) \notin \mathcal{L}_f(E)$.

c $\Rightarrow$ a. Assume that u is compact. Then $\sigma(u)\backslash\{0\}$ is discrete (Theorem 3.1.5.1 c)) and so $e_A^{\sigma(u)}$ is continuous. This and c) contradicts Corollary 5.5.6.6 a $\Rightarrow$ b. ■

Corollary 6.1.7.17 (0) *The following are equivalent for* $u \in \mathcal{L}(E,F)$*:*

a) u *is compact.*

b) *Every closed vector subspace of* $\operatorname{Im} u$ *is finite-dimensional.*

a $\Rightarrow$ b follows from Proposition 3.1.1.16.

b $\Rightarrow$ a. Assume that u is not compact.

Case 1 $E = F$ and u positive

By Corollary 6.1.7.16 a $\Rightarrow$ c, there is a Borel set A of $\sigma(u)$ such that $0 \notin \overline{A}$ and

$$p := e_A^{\sigma(u)}(u) \notin \mathcal{L}_f(E)\,.$$

Define

$$\alpha := d_A(0) > 0\,.$$

$$G := \operatorname{Im} p\,.$$

p is an orthogonal projection in E and

$$pu = up\,, \quad \alpha^2 p \leq u^2$$

(Theorem 6.1.7.15 a)). Thus, given $x \in G$,

$$\|ux\|^2 = \langle ux|ux\rangle = \langle u^2x|x\rangle \geq \alpha^2\langle px|x\rangle = \alpha^2\langle x|x\rangle = \alpha^2\|x\|^2$$

(Corollary 5.3.3.7), so that

$$\|ux\| \geq \alpha\|x\|\,.$$

By Proposition 1.2.1.18 a),c), the map

$$G \longrightarrow G\,, \quad x \longmapsto ux$$

is injective and $u(G)$ is closed. In particular, $u(G)$ is a closed infinite–dimensional vector subspace of $\operatorname{Im} u$ and this contradicts b).

Case 2 E, F, and u are arbitrary

By Proposition 5.3.2.2 d $\Rightarrow$ a, $u \circ u^*$ is not compact. By Case 1 (and Corollary 5.3.3.9), there is an infinite–dimensional closed vector subspace of $\operatorname{Im}(u \circ u^*)$. Since

$$\operatorname{Im}(u \circ u^*) \subset \operatorname{Im} u\,,$$

this contradicts b). ■

Corollary 6.1.7.18 (0) *The following are equivalent for every* $u \in \operatorname{Sn}\mathcal{L}(E)$*:*

a) u *is atomic.*

b) *There is an orthonormal basis* A *of* E *as well as an* $f \in \ell^\infty(A)$ *such that*

$$ux = f(x)x$$

for every $x \in A$.

In particular, every compact self–normal operator is atomic and every atomic operator is determined by its eignevalues and the corresponding eigenspaces.

a $\Rightarrow$ b. For $\alpha \in \sigma_p(u)$, let A_α be an orthonormal base of $E_\alpha := \mathrm{Ker}\,(\alpha 1 - u)$ and $\pi_\alpha := \pi_{E_\alpha}$. By Corollary 5.3.4.5,

$$A := \bigcup_{\alpha \in \sigma_p(u)} A_\alpha$$

is an orthonormal set of E. By Theorem 6.1.7.15 j),

$$e_\alpha^{\sigma(u)}(u) = \pi_\alpha$$

for every $\alpha \in \sigma_p(u)$. Since u is atomic, 1_E is the supremum of $(\pi_\alpha)_{\alpha \in \sigma_p(u)}$. Take $x \in A^\perp$. Then

$$\pi_\alpha x = 0$$

for every $\alpha \in \sigma_p(u)$ (Corollary 5.5.1.16), so that

$$\left\langle \left(\bigvee_{\alpha \in B} \pi_\alpha \right) x \middle| x \right\rangle = \left\langle \sum_{\alpha \in B} \pi_\alpha x \middle| x \right\rangle = 0$$

for every $B \in \mathfrak{P}_f(\sigma_p(u))$ (Corollary 4.2.7.5). Thus,

$$\|x\|^2 = \langle 1_E x | x \rangle = \sup_B \left\langle \left(\bigvee_{\alpha \in B} \pi_\alpha \right) x \middle| x \right\rangle = 0\,,$$

where B ranges over $\mathfrak{P}_f(\sigma_p(u))$ (Theorem 5.3.3.14 b $\Rightarrow$ c). Hence, A is an orthonormal basis of E. The existence of f with the required property is obvious.

b $\Rightarrow$ a. We have

$$\mathrm{Ker}\,(\alpha 1 - u) = \overset{-1}{f}(\alpha)^{\perp\perp} =: E_\alpha$$

for every $\alpha \in \mathbb{K}$. By Theorem 6.1.7.15 j),

$$e_\alpha^{\sigma(u)}(u) = \pi_{E_\alpha}$$

for every $\alpha \in \mathbb{K}$. Since A is an orthonormal basis of E,

$$1_E = \bigvee_{\alpha \in \mathbb{K}} \pi_\alpha = \bigvee_{\alpha \in \mathbb{K}} e_\alpha^{\sigma(u)}(u)\,.$$

Hence u is atomic.

The final assertion follows from Theorem 5.5.6.1 a $\Rightarrow$ d and Proposition 5.5.5.7 a). ■

Corollary 6.1.7.19 *Let u be a self-normal operator on E and f a bounded Borel function on $\sigma(u)$.*

a) *If $\overset{-1}{f}(0)$ is not dense in $\sigma(u)$, then $f(u) \neq 0$.*

b) *If $\alpha \in \mathbb{K}$ and if for every $\varepsilon > 0$ the interior (with respect to $\sigma(u)$) of $\overset{-1}{f}(U_\varepsilon^{\mathbb{K}}(\alpha))$ is nonempty, then $\alpha \in \sigma(f(u))$.*

c) *If $\alpha \in \sigma(u)$ and if f is continuous at α, then $f(\alpha) \in \sigma(f(u))$.*

a) Assume that

$$f(u) = 0\,.$$

Then

$$|f|^2(u) = f^*(u)f(u) = 0\,,$$

so we may assume that

$$f(\sigma(u)) \subset [0,1]\,.$$

Define

$$g : \sigma(u) \longrightarrow \mathbb{K}\,, \quad \alpha \longmapsto \begin{cases} 1 & \text{if} \quad f(\alpha) \neq 0 \\ 0 & \text{if} \quad f(\alpha) = 0\,. \end{cases}$$

$$j : \sigma(u) \longrightarrow \mathbb{K}\,, \quad \alpha \longmapsto \alpha\,,$$

$$h := j - jg\,.$$

$(f^{\frac{1}{n}})_{n\in\mathbb{N}}$ is an increasing sequence of bounded Borel functions on $\sigma(u)$ with g as supremum. Since

$$f^{\frac{1}{n}}(u) = (f(u))^{\frac{1}{n}} = 0$$

for every $n \in \mathbb{N}$,

$$g(u) = 0\,.$$

$$u = j(u) - j(u)g(u) = h(u)\,,$$

$$\sigma(u) \subset \overline{h(\sigma(u))} \subset \sigma(u) \cap \overline{\overset{-1}{f}(0)} \underset{\neq}{\subset} \sigma(u)$$

(Theorem 6.1.7.15 a),c)), which is a contradiction.

b) Assume that $\alpha \notin \sigma(f(u))$. Then there is an $\varepsilon > 0$ with

$$U_{2\varepsilon}^{\mathbb{K}}(\alpha) \cap \sigma(f(u)) = \emptyset .$$

Let g be a continuous function on $\mathbb{K}$ equal to 1 on $U_{\varepsilon}^{\mathbb{K}}(\alpha)$ and equal to 0 on $\mathbb{K} \backslash U_{2\varepsilon}^{\mathbb{K}}(f(\alpha))$. Then $g = 0$ on $\sigma(f(u))$, so that

$$g \circ f(u) = g(f(u)) = 0$$

(Theorem 6.1.7.15 d)). But $g \circ f = 1$ on $\overset{-1}{f}(U_{\varepsilon}^{\mathbb{K}}(\alpha))$. Thus, by a),

$$g \circ f(u) \neq 0 .$$

Hence, $\alpha \in \sigma(f(u))$.

c) follows from b). ■

Corollary 6.1.7.20 *Let u be a self-normal operator on E, $\alpha \in \sigma(u)$, and $\varepsilon > 0$. Then there is an $x \in E$ with*

$$\|ux - \alpha x\| < \varepsilon \|x\| .$$

Define

$$f : \sigma(u) \longrightarrow \mathbb{K}, \quad \beta \longmapsto \begin{cases} 1 & \text{if} \quad |\alpha - \beta| < \frac{\varepsilon}{2} \\ 0 & \text{if} \quad |\alpha - \beta| \geq \frac{\varepsilon}{2} , \end{cases}$$

$$j : \sigma(u) \longrightarrow \mathbb{K}, \quad \alpha \longmapsto \alpha .$$

By Corollary 6.1.7.19 a),

$$f(u) \neq 0 .$$

Since $f(u)$ is an orthogonal projection in E, there is an $x \in E$ with

$$f(u)x = x , \quad \|x\| = 1 .$$

We have

$$|(j - \alpha f)f| < \frac{\varepsilon}{2}$$

on $\sigma(u)$, so that

$$\|(u - \alpha f(u))f(u)\| = \|((j - \alpha f)f)(u)\| \leq \frac{\varepsilon}{2}$$

(Theorem 6.1.7.15 a)) and

$$\|ux - \alpha x\| = \|(u - \alpha f(u))f(u)x\| \leq \frac{\varepsilon}{2}\|x\| < \varepsilon \|x\| . \quad ■$$

Corollary 6.1.7.21 *If u is a self–normal operator on E, then*

$$\sigma(u) \subset \overline{(\langle ux|x\rangle | x \in E\,,\ \|x\| = 1)}\,,$$

$$\|u\| = \sup_{x \in E^\#} |\langle ux|x\rangle|\,.$$

Take $\alpha \in \sigma(u)$ and $\varepsilon > 0$. By Corollary 6.1.7.20, there is an $x \in E$ with

$$\|ux - \alpha x\| < \varepsilon\,, \quad \|x\| = 1\,.$$

We get that

$$|\langle ux|x\rangle - \alpha| = |\langle ux - \alpha x|x\rangle| < \varepsilon\,.$$

Since ε and α are arbitrary,

$$\sigma(u) \subset \overline{\{\langle ux|x\rangle | x \in E\,,\ \|x\| = 1\}}\,.$$

The second relation follows from the first one. ■

Proposition 6.1.7.22 *Let u be a self–normal operator on the Hilbert space E. If 0 is a non–isolated point of $\sigma(u)$ then:*

a) *There is a sequence $(p_n)_{n\in\mathbb{N}}$ in $\{u\}^c \cap \Pr\mathcal{L}(E)\backslash\{0\}$ such that*

$$p_m p_n = 0$$

for all distinct $m, n \in \mathbb{N}$ and

$$u = \sum_{n\in\mathbb{N}} p_n u\,.$$

b) *There is an orthonormal sequence $(x_n)_{n\in\mathbb{N}}$ in E such that*

$$\lim_{n\to\infty} ux_n = 0\,.$$

a) Put

$$r_0 := \|u\|\,.$$

Since 0 is a non–isolated point of $\sigma(u)$, there is a decreasing sequence $(r_n)_{n\in\mathbb{N}}$ of real numbers with infimum 0 such that for every $n \in \mathbb{N}$,

$$\sigma(u) \cap A_n \neq \emptyset\,,$$

where

$$A_n := \{\alpha \in \mathbb{K} | r_n < |\alpha| \leq r_{n-1}\}.$$

Given $n \in \mathbb{N}$, put

$$p_n := e_{A_n}^{\sigma(u)}(u).$$

By Theorem 6.1.7.15 a),c), $(p_n)_{n\in\mathbb{N}}$ is a sequence in $\{u\}^c \cap \Pr \mathcal{L}(E)$ such that

$$p_m p_n = 0$$

for all distinct $m, n \in \mathbb{N}$. By Theorem 6.1.7.15 c),

$$\|u - \sum_{k=1}^{n} p_k u\| \leq r_n$$

for every $n \in \mathbb{N}$, so that

$$u = \sum_{n\in\mathbb{N}} p_n u.$$

By Corollary 6.1.7.19 a),

$$p_n \neq 0$$

for every $n \in \mathbb{N}$.

b) For $n \in \mathbb{N}$, let $x_n \in \operatorname{Im} p_n$, $\|x_n\| = 1$. Then $(x_n)_{n\in\mathbb{N}}$ is an orthonormal sequence in E with

$$\|ux_n\| = \|up_n x_n\| \leq \|up_n\|$$

for every $n \in \mathbb{N}$. Thus

$$\lim_{n\to\infty} ux_n = 0. \qquad \blacksquare$$

Theorem 6.1.7.23 (H. Weyl) *If u is a self-normal operator on an infinite-dimensional Hilbert space E then for every $\alpha \in \mathbb{K}$ the following are equivalent:*

a) $\alpha \in \sigma^e(u)$.

b) *There is an orthonormal sequence $(x_n)_{n\in\mathbb{N}}$ in E such that*

$$\lim_{n\to\infty} (ux_n - \alpha x_n) = 0.$$

We may assume that $\alpha = 0$.

a $\Rightarrow$ b. First suppose that 0 is an isolated point of $\sigma(u)$. Then $e_0(u)$ is an orthogonal projection and $u - e_0(u)$ is invertible. Since $0 \in \sigma^e(u)$, it follows that $e_0(u)$ is not compact, i.e. $\operatorname{Im} e_0(u)$ is infinite-dimensional. Since

$$ue_0(u) = 0,$$

the existence of $(x_n)_{n\in\mathbb{N}}$ is obvious.

If 0 is not an isolated point of $\sigma(u)$, then b) follows from Proposition 6.1.7.22 b).

b $\Rightarrow$ a. By Theorem 3.1.3.10, u is not Fredholm, so that by Proposition 3.1.3.25 a), $0 \in \sigma^e(u)$. ■

Corollary 6.1.7.24 *If u is a self-normal operator on an infinite-dimensional Hilbert space, then every nonisolated point of $\sigma(u)$ belongs to $\sigma^e(u)$.*

Let α be a nonisolated point of $\sigma(u)$. By Proposition 6.1.7.22 b), there is an orthonormal sequence $(x_n)_{n\in\mathbb{N}}$ in E such that

$$\lim(ux_n - \alpha x_n) = 0.$$

By Theorem 6.1.7.23 b $\Rightarrow$ a, $\alpha \in \sigma^e(u)$. ■

Proposition 6.1.7.25 *Let A be a W^*-algebra, u and v the canonical projections*

$$A \times \mathcal{L}(H) \longrightarrow A, \quad A \times \mathcal{L}(H) \longrightarrow \mathcal{L}(H),$$

respectively, and B a unital W^*-subalgebra of $A \times \mathcal{L}(H)$ (Proposition 4.4.4.21 a), Corollary 6.1.7.14 b)) such that $v(B) = \mathcal{L}(H)$. Then either $\{0\} \times \mathcal{L}(H) \subset B$ or $u|B$ is injective.

By Proposition 4.4.4.21 b), $\{0\} \times \mathcal{L}(H)$ is a W^*-subalgebra of $A \times \mathcal{L}(H)$. Hence $B \cap (\{0\} \times \mathcal{L}(H))$ is a W^*-subalgebra of $A \times \mathcal{L}(H)$ and therefore a W^*-subalgebra of B (Corollary 4.4.4.10). Hence $B \cap (\{0\} \times \mathcal{L}(H))$ is a closed ideal of B_B. By Corollary 4.4.2.16, there is a $p \in \operatorname{Pr} B$ such that

$$B \cap (\{0\} \times \mathcal{L}(H)) = pB = Bp$$

and the map

$$(1-p)B(1-p) \longrightarrow A, \quad x \longmapsto ux$$

is injective. Put

$$q := up\,, \quad r := vp\,.$$

Then for every $x \in B$,

$$(qux, rvx) = ((up)(ux), (vp)(vx)) = (u(px), v(px)) =$$

$$= px \in pB = B \cap (\{0\} \times \mathcal{L}(H))\,,$$

so that

$$qux = 0\,, \quad (0, rvx) \in B\,.$$

Assume that $r \notin \{0, 1\}$. Then there are $\xi \in \operatorname{Ker} r$, $\eta \in \operatorname{Im} r$, with $\|\xi\| = 1$, $\|\eta\| = 1$. Take $x \in B$ with

$$vx = \langle \cdot | \xi \rangle \eta\,.$$

We have

$$rvx = \langle \cdot | \xi \rangle r\eta = \langle \cdot | \xi \rangle \eta$$

(Proposition 5.3.2.13 c)). By the above ,

$$(0, \langle \cdot | \xi \rangle \eta) \in B\,.$$

Since

$$\langle \cdot | \xi \rangle \xi = (\langle \cdot | \eta \rangle \xi)(\langle \cdot | \xi \rangle \eta) = (\langle \cdot | \xi \rangle \eta)^*(\langle \cdot | \xi \rangle \eta)$$

(Proposition 5.3.2.13 b),e)) we have

$$(0, \langle \cdot | \xi \rangle \xi) = (0, \langle \cdot | \xi \rangle \eta)^*(0, \langle \cdot | \xi \rangle \eta) \in B\,.$$

From

$$\langle \cdot | \xi \rangle \xi = (1 - r) \circ \langle \cdot | \xi \rangle \xi \circ (1 - r)$$

(Proposition 5.3.2.13 c),d)) we deduce that

$$(0, \langle \cdot | \xi \rangle \xi) = (1 - p)(0, \langle \cdot | \xi \rangle \xi)(1 - p) \in (1 - p)B(1 - p)\,.$$

Since the map

$$(1-p)B(1-p) \longrightarrow A\,, \quad y \longmapsto uy$$

is injective, we obtain the contradiction

$$\langle \cdot | \xi \rangle \xi = 0\,.$$

Hence $r \in \{0,1\}$.

If $r = 0$, then $p = 0$ and $u|B$ is injective. Assume that $r = 1$ and take $x \in \mathcal{L}(H)$. There is a $y \in B$ with $x = vy$. By the above

$$(0,x) = (0, rvy) \in B\,.$$

Hence

$$\{0\} \times \mathcal{L}(H) \subset B\,. \qquad \blacksquare$$

Corollary 6.1.7.26 *Let $(H_\iota)_{\iota \in I}$ be a finite family of Hilbert spaces and A a unital W^*-subalgebra of $\prod_{\iota \in I} \mathcal{L}(H_\iota)$ such that for every $\iota \in I$ and every $x \in \mathcal{L}(H_\iota)$, there is a $y \in A$ with $y_\iota = x$. Then there is a $J \subset I$ such that the map*

$$A \longrightarrow \prod_{\iota \in J} \mathcal{L}(H_\iota)\,, \quad x \longmapsto (x_\iota)_{\iota \in J}$$

is an isomorphism of W^*-algebras.

We prove the assertion by induction on $\operatorname{Card} I$. If

$$\mathcal{L}(H_\iota) \times \prod_{\lambda \in I \setminus \{\iota\}} \{0\} \subset A$$

for every $\iota \in I$, then

$$A = \prod_{\iota \in I} \mathcal{L}(H_\iota)\,.$$

Assume that there is a $\iota \in I$ such that

$$\mathcal{L}(H_\iota) \times \prod_{\lambda \in I \setminus \{\iota\}} \{0\} \not\subset A\,.$$

By Proposition 6.1.7.25, the map

$$u : A \longrightarrow \prod_{\lambda \in I \setminus \{\iota\}} \mathcal{L}(H_\lambda)\,, \quad x \longmapsto (x_\lambda)_{\lambda \in I \setminus \{\iota\}}$$

is injective. u is obviously a unital W^*-homomorphism, whence $u(A)$ is a unital W^*-subalgebra of $\prod_{\lambda \in I \setminus \{\iota\}} \mathcal{L}(H_\lambda)$ (Corollary 4.4.4.8 b)). By the hypothesis of induction, there is a $J \subset I \setminus \{\iota\}$ such that the map

$$u(A) \longrightarrow \prod_{\lambda \in J} \mathcal{L}(H_\lambda)\,, \quad x \longmapsto (x_\lambda)_{\lambda \in J}$$

is an isomporphism of W^*-algebras. It follows that the map

$$A \longrightarrow \prod_{\lambda \in J} \mathcal{L}(H_\lambda)\,, \quad x \longmapsto (x_\lambda)_{\lambda \in J}$$

is an isomorphism of W^*-algebras. ■

Proposition 6.1.7.27 (0) *For every $\varphi \in \mathcal{L}_{\mathcal{L}(E)}(\mathcal{K}(E))$ (Example 5.6.1.5, Definition 5.6.1.7), there is a $u \in \mathcal{L}(E)$ such that*

$$\varphi v = uv$$

for every $v \in \mathcal{K}(E)$.

Let $\mathfrak{F}$ be an ultrafilter on $\mathcal{K}(E)$ which is an approximate unit of $\mathcal{K}(E)$. By Theorem 6.1.6.6, $\mathcal{L}(E)$ may be identified with the dual of $\mathcal{L}^1(E)$, so that by Alaoglu–Bourbaki Theorem (and Proposition 5.6.1.8 b)), $\varphi(\mathfrak{F})$ converges to some u in $\mathcal{L}(E)_{\mathcal{L}^1(E)}$. Take $v \in \mathcal{K}(E)$ and $w \in \mathcal{L}^1(E)$. By Proposition 6.1.7.3 a), Corollary 6.1.7.14 a), and Proposition 5.6.1.8 d),

$$\langle uv, w\rangle = \langle u, vw\rangle = \lim_{t,\mathfrak{F}} \langle \varphi t, vw\rangle =$$

$$= \lim_{t,\mathfrak{F}} \langle (\varphi t)v, w\rangle = \lim_{t,\mathfrak{F}} \langle \varphi(tv), w\rangle = \langle \varphi v, w\rangle\,,$$

so that

$$uv = \varphi v\,.$$ ■

Theorem 6.1.7.28 (0) *For $n \in \mathbb{N}$, let $\mathcal{K}(E)^n$ denote the Hilbert $\mathcal{L}(E)$-module $\ell^2(\mathbb{N}_n, \mathcal{K}(E))$ defined in Example 5.6.4.2 c). Take $m, n \in \mathbb{N}$ and for $a \in \mathcal{L}(E)_{m,n}$, put*

$$\widetilde{a} : \mathcal{K}(E)^n \longrightarrow \mathcal{K}(E)^m\,, \quad \xi \longmapsto \left(\sum_{j=1}^{n} a_{ij}\xi_j\right)_{i \in \mathbb{N}_m}\,.$$

Then for all $a \in \mathcal{L}(E)_{m,n}$,

$$\widetilde{a} \in \mathcal{L}_{\mathcal{L}(E)}(\mathcal{K}(E)^n, \mathcal{K}(E)^m)\,, \quad \widetilde{a}^* = \widetilde{a^*}\,,$$

$$\sup_{\substack{i \in \mathbb{N}_m \\ j \in \mathbb{N}_n}} \|a_{ij}\| \leq \sup_{j \in \mathbb{N}_n} \|a_{\cdot,j}\| \leq \|\widetilde{a}\| \leq \sum_{j=1}^{n} \|a_{\cdot,j}\| \leq \sum_{j=1}^{n} \left(\sum_{i=1}^{m} \|a_{ij}\|^2\right)^{\frac{1}{2}}\,,$$

and the map

$$\mathcal{L}(E)_{m,n} \longrightarrow \mathcal{L}_{\mathcal{L}(E)}(\mathcal{K}(E)^n, \mathcal{K}(E)^m), \quad a \longmapsto \widetilde{a}$$

is an isometry of $\mathcal{L}(E)$-modules (Proposition 5.6.1.8 g)). If $m = n$, then this map is an isomorphism of C^*-algebras (Theorem 5.6.1.11 a), Theorem 5.6.6.1 f)).

By Theorem 5.6.6.1 a),b),c) (and Proposition 5.2.3.22) it is sufficient to prove that the map is surjective for $m = n = 1$ and this was done in Proposition 6.1.7.27. ■

6.1.8 The Canonical Projection of the Tridual of $\mathcal{K}$

Definition 6.1.8.1 (0) *Let ρ_{EF} denote the canonical projection of the tridual of $\mathcal{K}(E,F)$ (Proposition 1.3.6.19 b)) and put*

$$\rho_E := \rho_{EE}\,.$$

Proposition 6.1.8.2 (0) *Take $E \overset{u'}{\underset{\infty}{\to}} F$.*

a) $\rho_{GE}(vu') = v\rho_{FE}u'$ *whenever* $F \overset{v}{\to} G$.

b) $\rho_{FG}(u'v) = (\rho_{FE}u')v$ *whenever* $G \overset{v}{\to} E$.

c) $\rho_{EF}(u'^*) = (\rho_{FE}u')^*$.

Let i_{EF} and j_{EF} be the evaluations of $\mathcal{K}(E,F)$ and $\mathcal{K}(E,F)'\,(=\mathcal{L}^1(F,E))$, respectively. Then i_{EF} is the inclusion map

$$\mathcal{K}(E,F) \longrightarrow \mathcal{L}(E,F)\ (=\mathcal{K}(E,F)'')$$

(Corollary 6.1.6.8) and

$$\rho_{EF} = j_{EF} \circ i'_{EF}\,.$$

a) Given $u \in \mathcal{K}(G,E)$,

$$\langle u, i'_{GE}(vu')\rangle = \langle i_{GE}u, vu'\rangle = \langle (i_{GE}u)\circ v, u'\rangle =$$

$$= \langle i_{FE}(u\circ v), u'\rangle = \langle u\circ v, i'_{FE}u'\rangle = \langle u, vi'_{FE}u'\rangle\,,$$

so that

$$i'_{GE}(vu') = vi'_{FE}u'\,.$$

By Proposition 6.1.7.8 and Corollary 6.1.6.10,

$$\rho_{GE}(vu') = j_{GE}(i'_{GE}(vu')) = j_{GE}(vi'_{FE}u') = vj_{FE}i'_{FE}u' = v\rho_{FE}u'\,.$$

b) Take $u \in \mathcal{K}(F,G)$. Then

$$\langle u, i'_{FG}(u'v)\rangle = \langle i_{FG}u, u'v\rangle = \langle v\circ i_{FG}u, u'\rangle =$$

$$= \langle i_{FE}(v\circ u), u'\rangle = \langle v\circ u, i'_{FE}u'\rangle = \langle u, (i'_{FE}u')v\rangle\,,$$

so that

$$i'_{FG}(u'v) = (i'_{FE}u')v\,.$$

By Proposition 6.1.7.8 and Corollary 6.1.6.10,

$$\rho_{FG}(u'v) = j_{FG}i'_{FG}(u'v) = j_{FG}((i'_{FE}u')v) =$$

$$= (j_{FE}i'_{FE}u')v = (\rho_{FE}u')v\,.$$

c) Given $w \in \mathcal{K}(E,F)$,

$$\langle w, i'_{EF}(u'^*)\rangle = \langle i_{EF}w, u'^*\rangle = \overline{\langle i_{FE}(w^*), u'\rangle} =$$

$$= \overline{\langle w^*, i'_{FE}u'\rangle} = \langle w, (i'_{FE}u')^*\rangle\,,$$

so that

$$i'_{EF}(u'^*) = (i'_{FE}u')^*\,.$$

By Proposition 6.1.7.8 and Corollary 6.1.6.10,

$$\rho_{EF}(u'^*) = j_{EF}i'_{EF}(u'^*) = j_{EF}((i'_{FE}u')^*) =$$

$$= (j_{FE}i'_{FE}u')^* = (\rho_{FE}u')^*\,. \qquad \blacksquare$$

Corollary 6.1.8.3 (0) *ρ_E is an involutive homomorphism of $\mathcal{L}(E)$-modules (Corollary 6.1.7.4).* ■

Theorem 6.1.8.4 (0) *Invoking the identifications of the remark of Theorem 6.1.6.6:*

a) $\rho_{EF}(\mathcal{L}(E,F)') = \mathcal{L}^1(F,E)$.

b) $\operatorname{Ker}\rho_{EF} = \mathcal{K}(E,F)^0$.

c) $\mathcal{L}(E,F)' = \mathcal{L}^1(F,E) \oplus \mathcal{K}(E,F)^0$.

d) $\|u'\| = \|\rho_{EF}u'\| + \|u' - \rho_{EF}u'\|$, *for every* $u' \in \mathcal{L}(E,F)'$.

e) $(\mathcal{L}(E,F)/\mathcal{K}(E,F))'$ *and* $\mathcal{L}(E,F)'/\mathcal{L}^1(F,E)$ *are canonically isomorphic.*

a),b),c), and e) follow from Proposition 1.3.6.19.

d) Define

$$u := \rho_{EF} u' \in \mathcal{L}^1(F,E)\,, \quad v' := u' - \rho_{EF} u' \in \mathcal{K}(E,F)^0\,.$$

Let

$$\sum_{n\in N} \theta_n \langle \cdot | y_n \rangle x_n$$

be a Schatten decomposition of u. Take $\varepsilon > 0$. There are $m \in N$, $t \in \mathcal{L}_f(E,F)$, and $w \in \mathcal{L}(E,F)$ such that

$$\sum_{\substack{n\in N \\ n>m}} \theta_n < \frac{\varepsilon}{3}\,,$$

$$\|t\| \leq 1\,, \quad \widetilde{u}(t) > \|u\|_1 - \frac{\varepsilon}{3}\,,$$

$$\|w\| \leq 1\,, \quad v'(w) > \|v'\| - \frac{\varepsilon}{3}\,.$$

Let p be the orthogonal projection of E onto the (finite-dimensional) vector subspace of E generated by

$$\{x_n | n \in N\,,\, n \leq m\} \cup t^*(F)\,,$$

q the orthogonal projection of F onto the (finite-dimensional) vector subspace of F generated by

$$\{y_n | n \in N\,,\, n \leq m\} \cup t(E)\,,$$

and put

$$\overline{w} := (1_F - q) \circ w \circ (1_E - p)\,.$$

Then

$$\|\overline{w}\| \leq \|w\| \leq 1\,,$$

$$n \in N\,,\, n \leq m \Longrightarrow \overline{w} x_n = \overline{w}^* y_n = 0\,,$$

so that

$$|\widetilde{u}(\overline{w})| = \left|\sum_{n\in N} \theta_n \widetilde{\langle \cdot | y_n \rangle x_n}(\overline{w})\right| = \left|\sum_{n\in N} \theta_n \langle \overline{w} x_n | y_n \rangle\right| \leq$$

$$\leq \sum_{\substack{n\in N \\ n\geq m}} \theta_n < \frac{\varepsilon}{3}$$

(Proposition 6.1.6.2). From

$$w - \overline{w} = w - (1_F - q) \circ w \circ (1_E - p) =$$

$$= w \circ p + q \circ w - q \circ w \circ p \in \mathcal{L}_f(E, F)\,,$$

it follows that

$$v'(w) - v'(\overline{w}) = v'(w - \overline{w}) = 0\,.$$

Thus

$$v'(\overline{w}) = v'(w) > \|v'\| - \frac{\varepsilon}{3}\,.$$

We deduce that

$$|u'(t + \overline{w})| = |\widetilde{u}(t) + \widetilde{u}(\overline{w}) + v'(\overline{w})| \geq \widetilde{u}(t) + v'(\overline{w}) - |\widetilde{u}(\overline{w})| \geq$$

$$\geq \|u\|_1 - \frac{\varepsilon}{3} + \|v'\| - \frac{\varepsilon}{3} - \frac{\varepsilon}{3} = \|u\|_1 + \|v'\| - \varepsilon\,.$$

Now

$$\operatorname{Ker} t = (\operatorname{Im} t^*)^\perp$$

(Proposition 5.3.2.4), so that for any $x \in E$,

$$\|(t + \overline{w})x\|^2 = \|tx + \overline{w}x\|^2 =$$

$$= \|(q \circ t \circ p)x + (1_F - q) \circ w \circ (1_E - p)x\|^2 =$$

$$= \|(q \circ t \circ p)x\|^2 + \|(1_F - q) \circ w \circ (1_E - p)x\|^2 \leq$$

$$\leq \|px\|^2 + \|(1_E - p)x\|^2 = \|x\|^2$$

(Pythagoras' Theorem). We deduce successively that

$$\|t + \overline{w}\| \leq 1\,,$$

$$\|u'\| \geq |u'(t + \overline{w})| \leq \|u\|_1 + \|v'\| - \varepsilon\,,$$

$$\|u'\| \geq \|u\|_1 + \|v'\| = \|\rho_{EF} u'\| + \|u' - \rho_{EF} u'\| \geq \|u'\|\,,$$

$$\|u\| = \|\rho_{EF} u'\| + \|u' - \rho_{EF} u'\|\,.$$

■

Corollary 6.1.8.5

a) *Let* $\mathbb{K} = \mathbb{C}$ $(\mathbb{K} = \mathbb{R})$. *An element* $u' \in \mathcal{L}(E)'$ $(u' \in \operatorname{Re}\mathcal{L}(E)')$ *is order continuous iff*

$$u' = \rho_E u' .$$

b) *If* $\mathbb{K} = \mathbb{C}$, *then*

$$\mathcal{L}(E)^\pi = \{u' \in \mathcal{L}(E)' | u' = \rho_E u'\} =$$

$$= \{u' | u' \text{ is an order continuous linear form on } \mathcal{L}(E)\} .$$

a) First suppose that

$$u' = \rho_E u' .$$

By Theorem 6.1.8.4 a), $u' \in \mathcal{L}^1(E)$. Hence by Proposition 6.1.6.12, u' is order continuous.

Now suppose that u' is order continuous. Put

$$v' := u' - \rho_E u' .$$

By the above, v' is order continuous and by Theorem 6.1.8.4 b), $v' \in \mathcal{K}(E)^0$. Moreover, $v' \in \operatorname{Re}\mathcal{L}(E')$ if $\mathbb{K} = \mathbb{R}$ (Corollary 6.1.8.3). Take $u \in \mathcal{L}(E)_+$. By Corollary 5.3.3.17, there is an upward directed set $\mathcal{F}$ of $\mathcal{K}(E)_+$ with supremum u. Then

$$v'(u) = \lim_{v,\mathfrak{F}} v'(v) = 0 ,$$

where $\mathfrak{F}$ denotes the upper section filter of $\mathcal{F}$. By Theorem 4.2.2.9 a) (and Theorem 5.3.1.13), v' vanishes on $\operatorname{Re}\mathcal{L}(E)$. Hence $v' = 0$ and

$$u' = \rho_E u' .$$

b) Let u' be an order continuous linear form on $\mathcal{L}(E)$. By Proposition 1.7.2.4 b) (and Theorem 4.2.2.9 a)), u' is continuous. By a),

$$u' = \rho_E u' .$$

Now take $u' \in \mathcal{L}(E')$ with

$$u' = \rho_E u' .$$

By Theorem 6.1.8.4 a), this is equivalent to $u' \in \mathcal{L}^1(E)$. Since every element of $\mathcal{L}^1(E)$ is a linear combination of positive elements of $\mathcal{L}^1(E)$, it follows from a) that $u' \in \mathcal{L}(E)^\pi$.

The inclusion

$$\mathcal{L}(E)^\pi \subset \{u' | u' \text{ is an order continuous linear form on } \mathcal{L}(E)\}$$

is trivial. ■

Corollary 6.1.8.6 (0) *If* $u' \in \operatorname{Re}\mathcal{L}(E)'$, *then*

$$\rho_E u'^+ = (\rho_E u')^+, \quad \rho_E u'^- = (\rho_E u')^-,$$

$$u'^+ - \rho_E u'^+ = (u' - \rho_E u')^+, \quad u'^- - \rho_E u'^- = (u' - \rho_E u')^-.$$

First suppose that u' is positive. Then

$$\|u'\| = \langle 1_E, u' \rangle = \langle 1_E, \rho_E u' \rangle + \langle 1_E, u' - \rho_E u' \rangle \leq \|\rho_E u'\| + \|u' - \rho_E u'\| = \|u'\|$$

(Corollary 4.1.2.7 c), Theorem 6.1.8.4 d)), so

$$\|\rho_E u'\| = \langle 1_E, \rho_E u' \rangle,$$

$$\|u' - \rho_E u'\| = \langle 1_E, u' - \rho_E u' \rangle.$$

Since $\rho_E u'$ and $u' - \rho_E u'$ are selfadjoint (Corollary 6.1.8.3), it follows from Corollary 4.1.2.7 c), that $\rho_E u'$ and $u' - \rho_E u'$ are positive.

Now suppose that u' is selfadjoint. By Theorem 6.1.8.4 d),

$$\|u'\| = \|\rho_E u'\| + \|u' - \rho_E u'\|,$$

$$\|u'^+\| = \|\rho_E u'^+\| + \|u'^+ - \rho_E u'^+\|,$$

$$\|u'^-\| = \|\rho_E u'^-\| + \|u'^- - \rho_E u'^-\|.$$

Thus

$$\|\rho_E u'\| + \|u' - \rho_E u'\| = \|u'\| = \|u'^+\| + \|u'^-\| =$$

$$= \|\rho_E u'^+\| + \|\rho_E u'^-\| + \|u'^+ - \rho_E u'^+\| + \|u'^- - \rho_E u'^-\|$$

(Theorem 4.2.8.13). Since

$$\|\rho_E u'\| \leq \|\rho_E u'^+\| + \|\rho_E u'^-\|\,,$$

$$\|u' - \rho_E u'\| \leq \|u'^+ - \rho_E u'^+\| + \|u'^- - \rho_E u'^-\|\,,$$

it follows that

$$\|\rho_E u'\| = \|\rho_E u'^+\| + \|\rho_E u'^-\|\,,$$

$$\|u' - \rho_E u'\| = \|u'^+ - \rho_E u'^+\| + \|u'^- - \rho_E u'^-\|\,.$$

By the first part of the proof

$$\rho_E u'^+ = (\rho_E u')^+\,, \quad \rho_E u'^- = (\rho_E u')^-\,,$$

$$u'^+ - \rho_E u'^+ = (u' - \rho_E u')^+\,, \quad u'^- - \rho_E u'^- = (u' - \rho_E u')^-\,. \qquad \blacksquare$$

Corollary 6.1.8.7 (0) *Let u', v' be positive linear forms on $\mathcal{L}(E)$ such that $u' \leq v'$. Then*

$$u' \in \mathcal{L}^1(E)\,, \quad \textit{whenever } v' \in \mathcal{L}^1(E)\,,$$

and

$$u' \in \mathcal{K}(E)^0\,, \quad \textit{whenever } v' \in \mathcal{K}(E)^0\,.$$

Since $v' - u'$ is positive, we have, by Corollary 6.1.8.6,

$$u' - \rho_E u' \geq 0\,,$$

$$\rho_E u' - u' + (v' - \rho_E v') = (v' - u') - \rho_E(v' - u') \geq 0\,,$$

$$\rho_E u' \geq 0\,, \quad \rho_E v' - \rho_E u' = \rho_E(v - u') \geq 0\,.$$

By Theorem 6.1.8.4 a),b), it follows from

$$v' \in \mathcal{L}^1(E)$$

that

$$\rho_E v' = v'\,,$$

and hence that

$$u' = \rho_E u' \in \mathcal{L}^1(E)\,.$$

Moreover, if

$$v' \in \mathcal{K}(E)^0\,,$$

then

$$\rho_E v' = 0\,,$$

so that

$$\rho_E u' = 0\,,$$

and hence

$$u' \in \mathcal{K}(E)^0\,. \qquad \blacksquare$$

Corollary 6.1.8.8 (0) *If* $E \overset{u'}{\underset{\infty}{\to}} F$, $v \in \mathcal{K}(F,G)$, *and* $w \in \mathcal{K}(G,E)$, *then*

$$vu' = \rho_{GE}(vu') \in \mathcal{L}^1(E,G)\,,$$

$$u'w = \rho_{GF}(u'w) \in \mathcal{L}^1(G,F)\,.$$

By Theorem 6.1.8.4 b),

$$u' - \rho_{FE}u' \in \operatorname{Ker}\rho_{FE} = \mathcal{K}(F,E)^0\,.$$

If $u \in \mathcal{L}(G,E)$, then $u \circ v \in \mathcal{K}(F,E)$, so that

$$0 = \langle u \circ v\,,\, u' - \rho_{FE}u'\rangle = \langle u \circ v, u'\rangle - \langle u \circ v, \rho_{FE}u'\rangle = \langle u, vu'\rangle - \langle u, v\rho_{FE}u'\rangle\,,$$

$$\langle u, vu'\rangle = \langle u, v\rho_{FE}u'\rangle = \langle u, \rho_{GE}(vu')\rangle$$

(Proposition 6.1.8.2 a)). Since u is arbitrary,

$$vu' = \rho_{GE}(vu') \in \mathcal{L}^1(E,G)$$

(Theorem 6.1.8.4 a)). It follows that

$$(u'w)^* = w^*u'^* = \rho_{GF}(w^*u'^*)$$

(Proposition 6.1.7.3 i)),

$$u'w = \rho_{FG}(u'w) \in \mathcal{L}^1(G,F)$$

(Proposition 6.1.8.2 c), Proposition 6.1.7.3 h), Theorem 6.1.8.4 a)). $\blacksquare$

Proposition 6.1.8.9 *Let $\mathfrak{E}$ and $\mathfrak{F}$ be the set of finite-dimensional vector subspaces of E and F, respectively, ordered by inclusion, and let $\mathfrak{G}$ and $\mathfrak{H}$ be the upper section filters of $\mathfrak{E}$ and $\mathfrak{F}$, respectively. Then, for each $u' \in \mathcal{L}(E,F)'$,*

$$\rho_{EF}u' = \lim_{G\mathfrak{G}} \pi_G u' = \lim_{H,\mathfrak{H}} u'\pi_H = \lim_{(G,H),\mathfrak{G}\times\mathfrak{H}} \pi_G u'\pi_H ,$$

$$u' - \rho_{EF}u' = \lim_{G,\mathfrak{G}} \pi_{G^\perp} u' = \lim_{H,\mathfrak{H}} u'\pi_{H^\perp} = \lim_{(G,H),\mathfrak{G}\times\mathfrak{H}} \pi_{G^\perp} u'\pi_{H^\perp}$$

in the topology of pointwise convergence.

Let

$$\sum_{n\in N} \theta_n \langle \cdot | y_n \rangle x_n$$

be a Schatten decomposition of $\rho_{EF}u'$ (Theorem 6.1.8.4 a)). Take $v \in \mathcal{L}(E,F)$ and $\varepsilon > 0$. Choose $m \in N$ with

$$2\|v\| \sum_{\substack{n\in N \\ n>m}} \theta_n < \varepsilon .$$

Let G_0 be the vector subspace of E generated by $\{x_n | n \in N \cap \mathbb{N}_m\}$. Given any $G \in \mathfrak{E}$, with $G \supset G_0$,

$$|\langle v, \rho_{EF}u' \rangle - \langle v, \pi_G u' \rangle| = |\langle v, \rho_{EF}u' \rangle - \langle v\pi_G, u' \rangle| =$$

$$= \left| \sum_{n\in N} \theta_n \langle vx_n | y_n \rangle - \sum_{n\in N} \theta_n \langle v\pi_G x_n | y_n \rangle \right| =$$

$$= \left| \sum_{\substack{n\in N \\ n>m}} \theta_n \langle vx_n | y_n \rangle - \sum_{\substack{n\in N \\ n>m}} \theta_n \langle v\pi_G x_n | y_n \rangle \right| \leq$$

$$\leq \sum_{\substack{n\in N \\ n>m}} \theta_n |\langle vx_n | y_n \rangle| + \left| \sum_{\substack{n\in N \\ n>m}} \theta_n \langle v\pi_G x_n | y_n \rangle \right| \leq 2\|v\| \sum_{\substack{n\in N \\ n>m}} \theta_n < \varepsilon ,$$

(Proposition 6.1.6.2), so that

$$\rho_{EF}u' = \lim_{G,\mathfrak{G}} \pi_G u'$$

in the topology of pointwise convergence. We deduce that

$$\rho_{FE}(u'^*) = \lim_{H,\mathfrak{H}} \pi_H(u'^*)\,,$$

and so

$$\rho_{EF}u' = (\rho_{FE}(u'^*))^* = (\lim_{H,\mathfrak{H}} \pi_H(u'^*))^* = \lim_{H,\mathfrak{H}}(\pi_H(u'^*))^* = \lim_{H,\mathfrak{H}} u'\pi_H$$

is the topology of pointwise convergence (Proposition 6.1.8.2 c), Proposition 6.1.7.3 h)).

Take $v \in \mathcal{L}(E,F)$ and $\varepsilon > 0$. Since

$$\rho_{EF}\rho_{EF}u' = \rho_{EF}u'$$

there is a $(G_0, H_0) \in \mathfrak{E} \times \mathfrak{F}$ such that

$$\left|\langle v, \pi_G u' - \rho_{EF}u'\rangle\right| < \frac{\varepsilon}{4}\,, \quad |\langle v, u'\pi_H - \rho_{EF}u'\rangle| < \frac{\varepsilon}{4}$$

$$\left|\langle v, \pi_G\rho_{EF}u' - \rho_{EF}u'\rangle\right| < \frac{\varepsilon}{4}$$

for all $(G,H) \in \mathfrak{E} \times \mathfrak{F}$ with

$$G_0 \subset G\,, \quad H_0 \subset H\,.$$

Take such a (G,H). Then

$$|\langle v, \pi_G u'\pi_H - \rho_{EF}u'\rangle| \leq$$

$$\leq |\langle v, \pi_G u'\pi_H - \pi_G\rho_{EF}u'\rangle| + |\langle v, \pi_G\rho_{EF}u' - \rho_{EF}u'\rangle| \leq$$

$$\leq |\langle v, u'\pi_H - \rho_{EF}u'\rangle| + \frac{\varepsilon}{4} < \frac{\varepsilon}{4} + \frac{\varepsilon}{4} = \frac{\varepsilon}{2} < \varepsilon$$

$$|\langle v, \pi_{G^\perp}u' - (u' - \rho_{EF}u')\rangle| = |\langle v, \rho_{EF}u' - \pi_G u'\rangle| < \frac{\varepsilon}{4} < \varepsilon\,,$$

$$|\langle v, u'\pi_{H^\perp} - (u' - \rho_{EF}u')\rangle| = |\langle v, \rho_{EF}u' - u'\pi_H\rangle| < \frac{\varepsilon}{4} < \varepsilon\,,$$

$$|\langle v, \pi_{G^\perp}u'\pi_{H^\perp} - (u' - \rho_{EF}u')\rangle| =$$

$$= |\langle v, (1-\pi_G)u'(1-\pi_H) - u' + 2\rho_{EF}u' - \rho_{EF}u'\rangle| \leq$$

$$\leq |\langle v, \rho_{EF}u' - \pi_G u'\rangle| + |\langle v, \rho_{EF}u' - u'\pi_H\rangle| + |\langle v, \pi_G u'\pi_H - \rho_{EF}u'\rangle| <$$

$$< \frac{\varepsilon}{4} + \frac{\varepsilon}{4} + \frac{\varepsilon}{2} = \varepsilon .$$

Hence

$$\rho_{EF}u' = \lim_{(G,H),\mathfrak{G}\times\mathfrak{H}} \pi_G u'\pi_H ,$$

$$u' - \rho_{EF}u' = \lim_{G,\mathfrak{G}} \pi_{G^\perp}u' = \lim_{H,\mathfrak{H}} u'\pi_{H^\perp} = \lim_{(G,H),\mathfrak{G}\times\mathfrak{H}} \pi_{G^\perp}u'\pi_{H^\perp}$$

in the topology of pointwise convergence. ■

Theorem 6.1.8.10 (0) *Let j denote the evaluation map of $\mathcal{L}^1(E)$ and given $x \in E$, define*

$$\widetilde{x} := j(\langle\cdot|x\rangle x) .$$

a) $\tau_0(\mathcal{L}(E)) \subset \mathcal{L}^1(E) \cup \mathcal{K}(E)^0$.

b) $\tau_0(\mathcal{L}(E)) \cap \mathcal{L}^1(E) = \{\widetilde{x} | x \in E, \|x\| = 1\}$.

c) $\tau(\mathcal{L}(E))$ *is the closed convex hull of* $\{\widetilde{x} | x \in E, \|x\| = 1\}$ *with respect to the topology of pointwise convergence.*

d) *If we endow* $\tau(\mathcal{L}(E)) \cap \mathcal{L}^1(E)$ *with the topology induced by* $\tau(\mathcal{L}(E))$, *then* $\tau_0(\mathcal{L}(E)) \cap \mathcal{L}^1(E)$ *is a closed set of* $\tau(\mathcal{L}(E)) \cap \mathcal{L}^1(E)$.

e) *The topologies induced on* $\tau_0(\mathcal{L}(E)) \cap \mathcal{L}^1(E)$ *by* $\tau(\mathcal{L}(E))$, $\mathcal{L}^1(E)$, *and* $\mathcal{L}(E)$ *coincide and the map*

$$\{x \in E | \|x\| = 1\} \longrightarrow \tau_0(\mathcal{L}(E)) \cap \mathcal{L}^1(E), \quad x \longmapsto \widetilde{x}$$

is a homeomorphism.

a) Take $u' \in \tau_0(\mathcal{L}(E))$. By Corollary 6.1.8.6, $\rho_E u'$ and $u' - \rho_E u'$ are positive. Hence

$$\rho_E u' = \langle 1, \rho_E u'\rangle u' , \quad u' - \rho_E u' = \langle 1, u' - \rho_E u'\rangle u'$$

(Proposition 2.3.5.4 a $\Rightarrow$ b, Corollary 2.3.4.7, Theorem 5.3.1.13). By Theorem 6.1.8.4 a),c), either $\rho_E u'$ or $u' - \rho_E u'$ vanishes. Hence

$$u' \in \mathcal{L}^1(E) \cup \mathcal{K}(E)^0$$

(Theorem 6.1.8.4 a),b)).

b) Take $x \in E$ with $\|x\| = 1$. By Proposition 6.1.7.10 a $\Rightarrow$ e (and Corollary 5.3.3.7), $\widetilde{x}$ is a positive linear form on $\mathcal{L}(E)$. Since

$$\|\widetilde{x}\| = \|\langle \cdot | x \rangle x\|_1 = 1$$

(Theorem 6.1.6.6, Proposition 6.1.2.3),

$$\widetilde{x} \in \tau(\mathcal{L}(E)) \cap \mathcal{L}^1(E).$$

Take $u', v' \in \tau(\mathcal{L}(E))$ and $\alpha, \beta \in]0,1[$ with

$$\alpha + \beta = 1, \quad \alpha u' + \beta v' = \widetilde{x}.$$

By Corollary 6.1.8.7, $u', v' \in \mathcal{L}^1(E)$, so that

$$u'|\mathcal{K}(E), v'|\mathcal{K}(E) \in \tau(\mathcal{K}(E)).$$

By Proposition 6.1.7.13 a),

$$u'|\mathcal{K}(E) = v'|\mathcal{K}(E) = \widetilde{x}|\mathcal{K}(E).$$

Since $\mathcal{K}(E)$ is dense in $\mathcal{L}(E)_{\mathcal{L}^1(E)}$ (Corollary 1.3.6.5),

$$u' = v' = \widetilde{x}$$

and $\widetilde{x} \in \tau_0(\mathcal{L}(E))$.

Now suppose $u \in \tau_0(\mathcal{L}(E)) \cap \mathcal{L}^1(E)$. By Proposition 6.1.7.10 e $\Rightarrow$ a, $u \in \mathcal{L}^1(E)_+$, and so by Corollary 6.1.2.13, there are an orthonormal family $(x_n)_{n \in N}$ in E and an $f \in \ell^1(N)_+$, such that

$$u = \sum_{n \in N} f(n) \langle \cdot | x_n \rangle x_n.$$

Choose $n \in N$ with $f(n) \neq 0$. Then

$$f(n)\widetilde{x}_n \leq u,$$

so

$$f(n)\widetilde{x}_n = f(n)u$$

(Proposition 2.3.5.4 a $\Rightarrow$ b), and

$$u = \widetilde{x}_n.$$

c) We have

$$\|u^*u\| = \|u\|^2 = \sup_{\substack{x\in E\\ \|x\|=1}} \|ux\|^2 = \sup_{\substack{x\in E\\ \|x\|=1}} \langle ux|ux\rangle = \sup_{\substack{x\in E\\ \|x\|=1}} \langle u^*ux|x\rangle =$$

$$= \sup_{\substack{x\in E\\ \|x\|=1}} \langle u^*u, \widetilde{x}\rangle$$

for every $u \in \mathcal{L}(E)$ (Proposition 6.1.6.2). By Proposition 2.3.5.13, $\tau(\mathcal{L}(E))$ is the closed convex hull of $\{\widetilde{x}|x \in E\,,\ \|x\| = 1\}$ with respect to the topology of pointwise convergence.

d) Let $\mathfrak{F}$ be a filter on $\tau(\mathcal{L}(E)) \cap \mathcal{L}^1(E)$ converging to a $u' \in \tau(\mathcal{L}(E)) \cap \mathcal{L}^1(E)$ such that

$$\{\widetilde{x}|x \in E\,,\ \|x\| = 1\} \in \mathfrak{F}\,.$$

$\mathfrak{F}$ converges to u' in the topologly of pointwise convergence on $\mathcal{K}(E)$. By Proposition 6.1.7.12, there is an $x \in E$ such that

$$u'|\mathcal{K}(E) = \widetilde{\langle\cdot|x\rangle x}\,.$$

Since $u' \in \mathcal{L}^1(E)$ it follows

$$u' = \widetilde{x}$$

(Theorem 6.1.8.4 a),b)). Since $u' \in \tau(\mathcal{L}(E))$, we deduce that

$$\|x\|^2 = \langle 1, \widetilde{x}\rangle = \langle 1, u'\rangle = 1\,.$$

The assertion now follows from b).

e) Let us denote the mentioned topologies by $\mathfrak{T}_1$, $\mathfrak{T}_2$, and $\mathfrak{T}_3$. The inclusions

$$\mathfrak{T}_1 \subset \mathfrak{T}_2\,, \quad \mathfrak{T}_3 \subset \mathfrak{T}_2$$

are obvious. Take $x, y \in E$ with $\|x\| = \|y\| = 1$. Then

$$\|\widetilde{x} - \widetilde{y}\| = \|\langle\cdot|x\rangle x - \langle\cdot|y\rangle y\|_1 \leq$$

$$\leq \|\langle\cdot|x\rangle x - \langle\cdot|x\rangle y\|_1 + \|\langle\cdot|x\rangle y - \langle\cdot|y\rangle y\|_1 =$$

$$= \|\langle\cdot|x\rangle(x - y)\|_1 + \|\langle\cdot|x - y\rangle y\|_1 \leq 2\|x - y\|$$

(Proposition 6.1.2.3). Hence $\mathfrak{T}_2 \subset \mathfrak{T}_3$.

Let $x \in E$ with $\|x\| = 1$ and let $\mathfrak{F}$ be a filter on $\{y \in E | \|y\| = 1\}$ such that

$$\lim_{y,\mathfrak{F}} \widetilde{y} = \widetilde{x}$$

in $\mathfrak{T}_1$ Since $\langle \cdot | x \rangle x \in \mathcal{L}(E)$ it follows

$$1 = \widetilde{x}(\langle \cdot | x \rangle x) = \lim_{y,\mathfrak{F}} \widetilde{y}(\langle \cdot | x \rangle x) = \lim |\langle x | y \rangle|^2$$

(Proposition 6.1.6.2). Take $y \in E$ with $\|y\| = 1$. Since

$$\|y - \langle y | x \rangle x\|^2 + |\langle x | y \rangle|^2 = 1\,,$$

it follows

$$\lim_{y,\mathfrak{F}} \|y - \langle y | x \rangle x\| = 0\,.$$

Since

$$\|x - y\| \leq |1 - \langle y | x \rangle| \|x\| + \|\langle y | x \rangle x - y\|$$

we deduce

$$\lim_{y,\mathfrak{F}} \|x - y\| = 0\,.$$

Hence $\mathfrak{T}_3 \subset \mathfrak{T}_1$. ■

Corollary 6.1.8.11 *$\tau(\mathcal{L}(E)) \cap \mathcal{K}(E)^0$ is a convex set of $\mathcal{L}(E)'$ and a closed nowhere dense set of $\tau(\mathcal{L}(E))$.*

$\mathcal{K}(E)^0$ is a vector subspace of $\mathcal{L}(E)'$ closed in the topology of pointwise convergence. Hence $\tau(\mathcal{L}(E)) \cap \mathcal{K}(E)^0$ is a convex set of $\mathcal{L}(E)'$ and a closed set of $\tau(\mathcal{L}(E))$. By Theorem 6.1.8.10 b),c), $\tau(\mathcal{L}(E)) \cap \mathcal{L}^1(E)$ is a dense set of $\tau(\mathcal{L}(E))$. Since it is contained in $\tau(\mathcal{L}(E)) \backslash \mathcal{K}(E)^0$ (Theorem 6.1.8.4 c)), the set $\tau(\mathcal{L}(E)) \cap \mathcal{K}(E)^\circ$ is a nowhere dense set of $\tau(\mathcal{L}(E))$. ■

6.1.9 Integral Operators on Hilbert Spaces

Proposition 6.1.9.1 (0) *Let T be a σ-compact (i.e. the union of a sequence of compact sets) Hausdorff space, μ a positive Radon measure on T, and A an orthonormal basis of $L^2(\mu)$. Take $f \in \ell^2(A)$, and put*

$$k := \sum_{x \in A} f(x) x \otimes \overline{x} \quad (\text{in } L^2(\mu \otimes \mu)).$$

a) $\|k(t,\cdot)\|_{2,\mu}^2 = \sum_{x \in A} |f(x)x(t)|^2$ *for μ-almost every $t \in T$.*

b) *If T is compact,* Supp $\mu = T$, *and k is continuous, then*

$$\|k(t,\cdot)\|_{2,\mu}^2 = \sum_{x \in A} |f(x)x(t)|^2$$

for every $t \in T$.

c) *If $A \subset C(T)$,* Supp $\mu = T$, *and there is a μ-null set S such that*

$$\sup_{t \in T \setminus S} \|k(t,\cdot)\|_{2,\mu} < \infty,$$

then for every $y \in L^2(\mu)$, $(f(x)\langle y|x\rangle x)_{x \in A}$ is summable in $C(T)$.

a) (resp. b)). By Proposition 3.1.6.16 (by Corollary 3.1.6.4)

$$kx = f(x)x \quad \mu\text{-almost everwhere } (kx = f(x)x)$$

for every $x \in A$. Since $\{x \in A | f(x) \neq 0\}$ is countable, we deduce

$$\sum_{x \in A} |f(x)x(t)|^2 = \sum_{x \in A} |kx(t)|^2 = \sum_{x \in A} |\int k(t,s)x(s)d\mu(s)|^2 =$$

$$= \sum_{x \in A} |\langle k(t,\cdot)|\overline{x}\rangle|^2 = \|k(t,\cdot)\|_{2,\mu}^2$$

for μ-almost every $t \in T$ (for every $t \in T$), by Parseval's Equality.

c) Take $\varepsilon > 0$. Put

$$\alpha := \sup_{t \in T \setminus S} \|k(t,\cdot)\|_{2,\mu}.$$

There is a $B \in \mathfrak{P}_f(A)$ such that

$$\sum_{x \in A \setminus B} |\langle y|x\rangle|^2 \leq \frac{\varepsilon^2}{1+\alpha^2}.$$

We deduce, by a), that for every $C \in \mathfrak{P}_f(A \backslash B)$,

$$\sum_{x \in C} |f(x)\langle y|x\rangle x(t)| \leq (\sum_{x \in C} |\langle y|x\rangle|^2)^{\frac{1}{2}} \, (\sum_{x \in C} |f(x)x(t)|^2)^{\frac{1}{2}} \leq \frac{\varepsilon\alpha}{\sqrt{1+\alpha^2}} < \varepsilon$$

for μ-almost every $t \in T$. The above inequality holds for every $t \in T$ since the functions in C are continuous and since $\operatorname{Supp} \mu = T$. It follows that the family $(f(x)\langle y|x\rangle x)_{x \in A}$ is summable in $\mathcal{C}(T)$ (Proposition 1.1.6.6). ■

Theorem 6.1.9.2 (0) *Let $(T, \mathfrak{T}, \mu)$ be a σ-finite measure space and take $k \in \mathcal{L}^2(\mu \otimes \mu)$. Assume that the operator*

$$\widetilde{k} : L^2(\mu) \longrightarrow L^2(\mu), \quad x \longmapsto kx$$

is self-normal (Proposition 3.1.6.17 a)). Then $\widetilde{k}$ is a Hilbert–Schmidt operator, and there is an orthonormal basis A of $L^2(\mu)$ and an $f \in \mathbb{K}^A$ such that

$$\widetilde{k}x = f(x)x$$

for every $x \in A$. If A and f have the above properties, then:

a) $f \in \ell^2(A)$ *and*

$$\widetilde{k} = \sum_{x \in A} f(x)\langle \cdot|x\rangle x \qquad (\text{in } \mathcal{L}^2(L^2(\mu))).$$

b) *If $p \in [1, \infty[\cup \{0\}$ and $f \in \ell^p(A)$, then $k \in \mathcal{L}^p(L^2(\mu))$,*

$$\widetilde{k} = \sum_{x \in A} f(x)\langle \cdot|x\rangle x \qquad (\text{in } \mathcal{L}^p(L^2(\mu))),$$

and

$$\|\widetilde{k}\|_p = \|f\|_p.$$

c) $k = \sum_{x \in A} f(x)x \otimes \overline{x}$ *(in $L^2(\mu \otimes \mu)$).*

d) $\sigma_p(\widetilde{k}) = f(A)$.

e) $ky = \sum_{x \in A} f(x)\langle y|x\rangle x$ *for every $y \in L^2(\mu)$.*

f) *If $\alpha \in \mathbb{K} \backslash \{0\}$, then*

$$\operatorname{Im}(\alpha 1 - \widetilde{k}) = \overline{f}^{-1}(\alpha)^{\perp},$$

and

$$\frac{1}{\alpha}y+\frac{1}{\alpha}\sum_{y\in A\setminus \overset{-1}{f}(\alpha)}\frac{f(x)\langle y|x\rangle}{\alpha-f(x)}x\in\overbrace{\alpha 1-\widetilde{k}}^{-1}(y)$$

for every $y\in \mathrm{Im}\,(\alpha 1-\widetilde{k})$.

g) *if* T *is a* σ*-compact Hausdorff space,* μ *is a Radon measure on* T *with* $\mathrm{Supp}\,\mu=T$, $A\subset\mathcal{C}(T)$, *and there is a* μ*-null set* S *such that*

$$\sup_{t\in T\setminus S}\|k(t,\cdot)\|_{2,\mu}<\infty,$$

then the families in e) and f) are summable in $\mathcal{C}(T)$.

By Proposition 6.1.4.9 a), $\widetilde{k}$ is a Hilbert–Schmidt operator. By Corollary 6.1.3.18, there is an orthonormal basis A of $L^2(\mu)$ and an $f\in\mathbb{K}$ such that

$$\widetilde{k}x=f(x)x$$

for every $x\in A$.

a),b),d),e), and f) follow from Corollary 6.1.3.18 a),b),c),d).

c) follows from a) and Proposition 6.1.4.9 a),c).

g) By Proposition 6.1.9.1 c), $(f(x)\langle y|x\rangle x)_{x\in A}$ is summable in $\mathcal{C}(T)$. Take $\alpha\in\mathbb{K}\setminus\{0\}$. Put

$$B:=A\setminus\overset{-1}{f}(\alpha).$$

Then

$$\inf_{x\in B}|\alpha-f(x)|>0,$$

$$\left(\frac{f(x)}{\alpha-f(x)}\right)_{x\in B}\in\ell^2(B).$$

By the above considerations, the family $\left(\frac{f(x)\langle y|x\rangle}{\alpha-f(x)}x\right)_{x\in B}$ is summable in $\mathcal{C}(T)$. ■

Proposition 6.1.9.3 (0) *Let* T *be a Hausdorff space and* μ *a positve Radon measure on* T *such that* $\mathrm{Supp}\,\mu=T$. *Let* k *be a continuous function in* $\mathcal{L}^2(\mu\otimes\mu)$. *Then the following are equivalent:*

a) *The operator*

$$L^2(\mu) \longrightarrow L^2(\mu), \quad x \longmapsto kx$$

(Proposition 3.1.6.17 a)) is selfadjoint (positive).

b) *For every finite family* $(t_\iota)_{\iota\in I}$ *in* T, *the sesquilinear form*

$$\mathbb{K}^I \times \mathbb{K}^I \longrightarrow \mathbb{K}, \quad (\alpha, \beta) \longmapsto \sum_{\iota,\lambda\in I} k(t_\iota, t_\lambda)\alpha_\iota\overline{\beta}_\iota$$

is Hermitian (positive).

Case 1 Selfadjointness

By Proposition 5.3.2.1, a) is equivalent to

$$s, t \in T \Longrightarrow k(s,t) = \overline{k(t,s)},$$

and by Example 2.3.3.6 this is equivalent to b).

Case 2 Positivity

a $\Rightarrow$ b. By Corollary 5.3.3.7,

$$\sum_{\iota,\lambda\in I} \int k(s,t)x_\iota(t)\overline{x_\lambda(s)}d(\mu\otimes\mu)(s,t) =$$

$$= \left\langle k(\sum_{\iota\in I} x_\iota) \middle| \sum_{\iota\in I} x_\iota \right\rangle \in \mathbb{R}_+$$

for every family $(x_\iota)_{\iota\in I}$ in $L^2(\mu)$. It follows (by approximation) that

$$\sum_{\iota,\lambda\in I} k(t_\iota, t_\lambda)\alpha_\iota\overline{\alpha_\lambda} \in \mathbb{R}_+$$

for every $(\alpha_\iota)_{\iota\in I} \in \mathbb{K}^I$, i.e. the sesquilinear form of b) is positive.

b $\Rightarrow$ a. Take $x \in L^2(\mu)$. Let K be a compact set of T such that the restrictions of x and kx to K are continuous. By b),

$$\langle e_K kx | e_K x\rangle \in \mathbb{R}_+$$

(using approximation). By the Vitali–Lusin Theorem (of integration theory), we see further that

$$\langle kx|x\rangle \in \mathbb{R}_+ .$$

By Corollary 5.3.3.7, the operator of a) is positive. ■

Theorem 6.1.9.4 (0) *Let T be a compact space and μ a positive Radon measure on T with $\operatorname{Supp}\mu = T$. Choose $k \in \mathcal{C}(T \times T)$ such that the operator*

$$\widetilde{k} : L^2(\mu) \longrightarrow L^2(\mu), \quad x \longmapsto kx$$

(Proposition 3.1.6.17 a)) is positive. Let A be an orthonormal basis of $L^2(\mu)$, and take $f \in \mathbb{K}^A$ such that

$$x \in A \Longrightarrow \widetilde{k}x = f(x)x$$

(Theorem 6.1.9.2).

a) $A\backslash f^{-1}(0) \subset \mathcal{C}(T)$, $\sigma_p(\widetilde{k}) = f(A)$.

b) $f \in \ell^1(A)$.

c) $k = \sum_{x\in A} f(x)x \otimes \overline{x}$ *(in $\mathcal{C}(T \otimes T)$)*.

d) $\widetilde{k}$ *is nuclear,*

$$\widetilde{k} = \sum_{x\in A} f(x)\langle\cdot|x\rangle x \quad \textit{(in } \mathcal{L}^1(L^2(\mu))\textit{)}$$

and

$$\|\widetilde{k}\|_1 = \int k(t,t)d\mu(t) = \|f\|_1 \quad \text{(J. Mercer, 1909).}$$

a) By Corollary 3.1.6.4,

$$f(x)x = \widetilde{k}x \in \mathcal{C}(T)$$

for every $x \in A$, so that

$$A\backslash f^{-1}(0) \subset \mathcal{C}(T).$$

By Proposition 5.5.5.7 c),

$$\sigma_p(\widetilde{k}) = f(A).$$

b) Step 1 $t \in T \Rightarrow \sum_{x\in A} f(x)|x(t)|^2 \leq k(t,t)$.

Take $B \in \mathfrak{P}_f(A)$. Define

$$h := \sum_{x\in B} f(x)x \otimes \overline{x},$$

$$\widetilde{h} := \sum_{x \in B} f(x) \langle \cdot | x \rangle x \,.$$

Since $\widetilde{k}$ is positive, f is positive, and so $\widetilde{k} - \widetilde{h}$ is positive (Theorem 6.1.9.2 c)). It follows that

$$0 \leq k(t,t) - h(t,t)$$

(Proposition 6.1.9.3 a $\Rightarrow$ b) and

$$\sum_{x \in B} f(x) |x(t)|^2 = h(t,t) \leq k(t,t) \,.$$

Since B is arbitrary, it follows that

$$\sum_{x \in A} f(x) |x(t)|^2 \leq k(t,t) \,.$$

Step 2 $\quad f \in \ell^1(A)\,,\ \sum\limits_{x \in A} f(x) = \int (\sum\limits_{x \in A} f(x) |x(t)|^2) d\mu(t)$

By Step 1,

$$\sum_{x \in A} f(x) = \sum_{x \in A} f(x) \int |x(t)|^2 d\mu(t) =$$

$$= \int \left(\sum_{x \in A} f(x) |x(t)| \right) d\mu(t) \leq \int k(t,t) d\mu(t) \,.$$

c) Step 1 $\quad s \in T \Rightarrow (f(x) x(s) \overline{x})_{x \in A}$ is summable in $\mathcal{C}(T)$.

Choose $\varepsilon > 0$. By Step 1 in b), there is a $B \in \mathfrak{P}_f(A \backslash B)$ such that

$$\sum_{x \in A \backslash B} f(x) |x(s)|^2 < \frac{\varepsilon}{1 + \|k\|_\infty} \,.$$

Let $C \in \mathfrak{P}_f(A \backslash B)$ and $t \in T$. Then, by Step 1 in b),

$$\left| \sum_{x \in C} f(x) x(s) \overline{x(t)} \right| \leq \left(\sum_{x \in C} f(x) |x(s)|^2 \right)^{\frac{1}{2}} \left(\sum_{x \in C} f(x) |x(t)|^2 \right)^{\frac{1}{2}} \leq$$

$$\leq \frac{\varepsilon \sqrt{k(t,t)}}{\sqrt{1 + \|k\|_\infty}} < \varepsilon \,.$$

Now the assertion follows from Proposition 1.1.6.6.

Step 2 $\begin{cases} k_0 : T \times T \to \mathbb{K}, \quad (s,t) \mapsto \sum\limits_{x \in A} f(x) x(s) \overline{x(t)} \\ \text{is continuous in every variable} \end{cases}$

The assertion follows from Step 1.

Step 3 $k = k_0$

By Proposition 6.1.4.9 c),

$$k = \sum_{x \in A} f(x) x \otimes \overline{x} \quad (\text{in } L^2(\mu \otimes \mu)).$$

Hence

$$k = k_0 \quad \mu \otimes \mu\text{-a.e.}$$

By continuity, at every $s \in T$, the assertions

$$k(s, \cdot) = k_0(s, \cdot),$$

$$k(s, \cdot) = k_0(s, \cdot), \quad \mu\text{-a.e.}$$

are equivalent. By Fubini's Theorem, we deduce that

$$\mu(\{s \in T | k(s, \cdot) \neq k_0(s, \cdot)\}) = 0.$$

Hence

$$k = k_0.$$

Step 4 $k = \sum\limits_{x \in A} f(x) x \otimes \overline{x} \quad (\text{in } \mathcal{C}(T \times T))$.

By Step 3,

$$k(t,t) = k_0(t,t) = \sum_{x \in A} f(x) |x(t)|^2$$

for every $t \in T$. By Dini's Theorem, this sum converges uniformly. Take $\varepsilon > 0$. There is a $B \in \mathfrak{P}_f(A)$ such that

$$\sum_{x \in A \setminus B} f(x) |x|^2 < \varepsilon.$$

We get that

$$\sum_{x\in A\setminus B} |f(x)x(s)\overline{x(t)}| \leq \left(\sum_{x\in A\setminus B} f(x)|x(s)|^2\right)^{\frac{1}{2}} \left(\sum_{x\in A\setminus B} f(x)|x(t)|^2\right)^{\frac{1}{2}} < \varepsilon$$

for all $s,t \in T$. Hence the family $(f(x)x \otimes \overline{x})_{x\in A}$ is summable in $\mathcal{C}(T\times T)$, and therefore

$$k = \sum_{x\in A} f(x)x \otimes \overline{x} \quad (\text{in } \mathcal{C}(T\times T)).$$

d) By Step 2 in b) and Corollary 6.1.2.13 d), $\widetilde{k}$ is nuclear and

$$\|\widetilde{k}\|_1 = \sum_{x\in A} f(x) = \int \left(\sum_{x\in A} f(x)|x(t)|^2\right) d\mu(t).$$

By Step 3 in c),

$$\|\widetilde{k}\|_1 = \int k(t,t)d\mu(t).$$

By Theorem 6.1.3.12 c),

$$\widetilde{k} = \sum_{x\in A} f(x)\langle\cdot|x\rangle x \quad (\text{in } \mathcal{L}^1(L^2(\mu))). \qquad \blacksquare$$

6.2 Selfadjoint Linear Differential Equations

This section contains examples of selfadjoint ordinary linear differential equations as well as examples of selfadjoint linear partial differential equations of second order on compact Riemannian manifolds.

6.2.1 Selfadjoint Boundary Value Problems

We use the notation from Subsection 3.2.1 throughout this subsection and assume that u is injective (except in Proposition 6.2.1.1). We also use the following additional notation:

a) λ is Lebesgue measure on $[a,b]$.

b) r is a strictly positive continuous function on $[a,b]$ (we shall work in the Hilbert space $L^2(r\cdot\lambda)$ and $\frac{1}{r}u$ will be our basic differential equation).

c) g is the Green function of the given boundary value problem (Theorem 3.2.1.3). g is assumed to be selfadjoint, so that u is selfadjoint (Corollary 3.2.2.6 d $\Rightarrow$ c).

d) $v : L^2(r\cdot\lambda) \longrightarrow L^2(r\cdot\lambda)\,, \quad x \longmapsto \int_a^b g(\cdot,t)x(t)r(t)dt\,.$

e) Given $\gamma,\delta \in \mathbb{K}$, define

$$u_\delta : \mathcal{D} \longrightarrow \mathcal{C}\,, \quad x \longmapsto ux + \delta rx\,,$$

$$\mathcal{F}_{\delta,\gamma} := \{x \in \mathcal{D} | u_\delta x = \gamma r x\}\,,$$

$$\mathcal{F}_\gamma := \mathcal{F}_{0,\gamma}\,,$$

$$A_\delta := \{\gamma \in \mathbb{K} | \mathcal{F}_{\delta,\gamma} \neq \{0\}\}\,, \quad A := A_0\,.$$

If u_δ is injective, then g_δ denotes the Green function associated to u_δ (and to the given boundary value problem) and we put

$$v_\delta : L^2(r\cdot\lambda) \longrightarrow L^2(r\cdot\lambda)\,, \quad x \longmapsto \int_a^b g_\delta(\cdot,t)x(t)r(t)dt\,.$$

Proposition 6.2.1.1 (5) *In this proposition we do not assume u to be injective. If $\gamma,\delta \in \mathbb{K}$, then*

$$\mathcal{F}_{\delta,\gamma} = \mathcal{F}_{\gamma-\delta}$$

and

$$A_\delta = A + \delta\,.$$

Take $x \in \mathcal{D}$. Then

$$(x \in \mathcal{F}_{\delta,\gamma}) \Longleftrightarrow (u_\delta x = \gamma r x) \Longleftrightarrow (ux + \delta r x = \gamma r x) \Longleftrightarrow$$

$$\Longleftrightarrow (ux = (\gamma - \delta) r x) \Longleftrightarrow (x \in \mathcal{F}_{\gamma-\delta})\,.$$

Hence

$$\mathcal{F}_{\delta,\gamma} = \mathcal{F}_{\gamma-\delta}\,.$$

It follows that

$$(\gamma \in A_\delta) \Longleftrightarrow (\mathcal{F}_{\delta,\gamma} \neq \{0\}) \Longleftrightarrow (\mathcal{F}_{\gamma-\delta} \neq \{0\}) \Longleftrightarrow (\gamma - \delta \in A)\,,$$

$$A_\delta = A + \delta\,. \qquad \blacksquare$$

Remark. In fact this proposition is contained in Proposition 3.2.2.1 a),b) applied for $\frac{1}{r}u$ instead of u.

Proposition 6.2.1.2 (5)

a) v *is injective and selfadjoint.*

b) $\begin{cases} x \in \mathcal{D} \Rightarrow v(\frac{1}{r}ux) = -x \\ x \in \mathcal{C} \Rightarrow vx \in \mathcal{D}\,,\ \frac{1}{r}uvx = -x\,. \end{cases}$

c) $\gamma \in \mathbb{K}\backslash\{0\} \Rightarrow \mathcal{F}_\gamma = \{x \in L^2(r\cdot\lambda) | vx = -\frac{1}{\gamma}x\}\,.$

d) $\sigma_p(v) = \{-\frac{1}{\gamma} | \gamma \in A\}\,.$

e) $A \subset \mathbb{R}\,.$

f) $x, y \in \mathcal{D}\,,\ \delta \in \mathbb{K} \Rightarrow \langle \frac{1}{r}u_\delta x | y\rangle = \langle x | \frac{1}{r}u_{\bar{\delta}} y\rangle\,.$

b) By Corollary 3.2.1.5,

$$v(\frac{1}{r}ux) = \int_a^b g(\cdot, t)ux(t)dt = -x$$

for every $x \in \mathcal{D}$. By Proposition 3.2.1.4, $vx \in \mathcal{D}$ and

$$\frac{1}{r}uvx = \frac{1}{r}u(\int_a^b g(\cdot,t)x(t)r(t)dt) = -x$$

for every $x \in \mathcal{C}$.

a) By Proposition 5.3.2.1, v is selfadjoint. By b), $\mathcal{D} \subset \operatorname{Im} v$, so that $\operatorname{Im} v$ is dense in $\mathcal{L}^2(r\cdot\lambda)$. By Proposition 5.3.2.4,

$$\operatorname{Ker} v = (\operatorname{Im} v)^{\perp} = \{0\}.$$

Hence v is injective.

c) Take $x \in L^2(r\cdot\lambda)$ such that

$$vx = -\frac{1}{\gamma}x.$$

By Corollary 3.1.6.4, $x \in \mathcal{C}$, so by b), $x \in \mathcal{D}$. It follows by b), that

$$\{x \in L^2(r\cdot\lambda) | vx = -\frac{1}{\gamma}x\} = \{x \in \mathcal{D} | vx = -\frac{1}{\gamma}x\} =$$

$$= \{x \in \mathcal{D} | ux = \gamma r x\} \in \mathcal{F}_\gamma.$$

d) follows from c).

e) follows from d) and a).

f) By a) and b),

$$\langle \frac{1}{r}u_\delta x | y\rangle = \langle \frac{1}{r}ux + \delta x | y\rangle = \langle \frac{1}{r}ux | y\rangle + \delta\langle x | y\rangle =$$
$$= -\langle \frac{1}{r}ux | v(\frac{1}{r}uy)\rangle + \delta\langle x | y\rangle = -\langle v(\frac{1}{r}ux) | \frac{1}{r}uy\rangle + \delta\langle x | y\rangle =$$

$$= \langle x | \frac{1}{r}uy\rangle + \langle x | \bar{\delta} y\rangle = \langle x | \frac{1}{r}u_{\bar{\delta}}\rangle. \qquad \blacksquare$$

Theorem 6.2.1.3 (5) *v is an injective selfadjoint Hilbert–Schmidt operator and there is an orthonormal basis B of $L^2(r\cdot\lambda)$ and an $f \in \mathbb{K}^B$ such that*

$$vx = f(x)x$$

for every $x \in B$. If B and f have the above properties, then:

a) *$f \in \ell^2(B)$ and*

$$v = \sum_{x\in B} f(x)\langle\cdot|x\rangle x \quad (\text{in } \mathcal{L}^2(L^2(r\cdot\lambda))).$$

b) $g = \sum_{x \in B} f(x) x \otimes \overline{x} \quad (in\ L^2((r \cdot \lambda) \otimes (r \cdot \lambda)))$.

c) $\sigma_p(v) = f(B)$.

d) f *is real and vanishes nowhere.*

e) $x \in B \Rightarrow x \in \mathcal{D},\ ux = -\frac{1}{f(x)} r x$.

f) $\|v\| = \|f\|_\infty = \sup_{\gamma \in A} \frac{1}{|\gamma|}, \quad \|v\|_2 = \|f\|_2$.

g) $y \in L^2(r \cdot \lambda) \Rightarrow vy = \sum_{x \in B} f(x) \langle y|x \rangle x \quad (in\ \mathcal{C})$.

h) *If* $\alpha \in \mathbb{K} \setminus \{0\}$, *then*

$$\operatorname{Im}(\alpha 1 - v) = \overset{-1}{f}(\alpha)^\perp$$

and

$$\frac{1}{\alpha} y + \frac{1}{\alpha} \sum_{y \in B \setminus \overset{-1}{f}(\alpha)} \frac{f(x) \langle y|x \rangle}{\alpha - f(x)} x \in \overbrace{\alpha 1 - v}^{-1}(y)$$

for every $y \in \operatorname{Im}(\alpha 1 - v)$, *where the above series converges in* $\mathcal{C}$.

i) *The following are equivalent for every* $y \in L^2(r \cdot \lambda)$:

i$_1$) $y \in \mathcal{D}$.

i$_2$) $\left(\frac{\langle y|x \rangle}{f(x)} \right)_{x \in B} \in \ell^2(B)$,

$$\sum_{x \in B} \frac{\langle y|x \rangle}{f(x)} x \in \mathcal{C} \quad (where\ the\ sum\ is\ considered\ in\ L^2(r \cdot \lambda)).$$

If these conditions are fulfilled then:

i$_3$) $v \left(\sum_{x \in B} \frac{\langle y|x \rangle}{f(x)} x \right) = y$.

i$_4$) $y = \sum_{x \in B} \langle y|x \rangle x \quad (in\ \mathcal{C})$.

i$_5$) $uy = -r \sum_{x \in B} \frac{\langle y|x \rangle}{f(x)} x \quad (in\ L^2(r \cdot \lambda))$.

j) *If* $y \in v(\mathcal{D})$ *then* $(\frac{\langle y|x \rangle}{f(x)} x)_{x \in B}$ *is a summable family in* $\mathcal{C}$.

v is injective and selfadjoint (Proposition 6.2.1.2 a)). By Theorem 6.1.9.2, v is a Hilbert–Schmidt operator and there are an orthonormal basis B of $L^2(r\cdot\lambda)$ and an $f \in \ell^2(B)$ such that

$$vx = f(x)x$$

for every $x \in B$.

a) follows from Theorem 6.1.9.2 a).

b) follows from Theorem 6.1.9.2 c).

c) follows from Theorem 6.1.9.2 d).

d) follows from c) and Proposition 6.2.1.2 a).

e) By d), Corollary 3.1.6.4, and Proposition 6.2.1.2 b), $x \in \mathcal{D}$ and

$$ux = \frac{1}{f(x)}uvx = -\frac{1}{f(x)}rx\,.$$

f) By Theorem 6.1.9.2 b),

$$\|v\| = \|f\|_\infty\,, \quad \|v\|_2 = \|f\|_2\,.$$

By c) and Proposition 6.2.1.2 d),

$$\|f_\infty\| = \sup_{\gamma\in A}\frac{1}{|\gamma|}\,.$$

g) and h) follow from e) and Theorem 6.1.9.2 e),f),g).

$\mathrm{i}_1 \Rightarrow \mathrm{i}_2$ & i_3. By Proposition 6.2.1.2 b), $y \in \operatorname{Im} v$. By Theorem 5.5.5.4 a $\Rightarrow$ i,

$$\left(\frac{\langle y|x\rangle}{f(x)}\right)_{x\in B} \in \ell^2(B)\,, \quad v\left(\sum_{x\in B}\frac{\langle y|x\rangle}{f(x)}x\right) = y\,.$$

Since v is injective and $v(\mathcal{C}) = \mathcal{D}$ (Proposition 6.2.1.2 a),b)), it follows

$$\sum_{x\in B}\frac{\langle y|x\rangle}{f(x)}x \in \mathcal{C}\,.$$

$\mathrm{i}_2 \Rightarrow \mathrm{i}_1$ & i_3 & i_4. By g),

$$v\left(\sum_{x\in B}\frac{\langle y|x\rangle}{f(x)}x\right) = \sum_{x\in B}\frac{\langle y|x\rangle}{f(x)}f(x)x = \sum_{x\in B}\langle y|x\rangle x \ \text{ (in } \mathcal{C})\,.$$

i_3) and i_4) follow from the Fourier expansion of y. By i_3) and Proposition 6.2.1.2 b), $y \in \mathcal{D}$.

i_2 & $\mathrm{i}_3 \Rightarrow \mathrm{i}_5$. By Proposition 6.2.1.2 b),

$$uy = uv\left(\sum_{x\in B}\frac{\langle y|x\rangle}{f(x)}x\right) = -r\sum\frac{\langle y|x\rangle}{f(x)}x \text{ (in } L^2(r\cdot\lambda))\,.$$

j) There is a $z \in \mathcal{D}$ with $vz = y$. By d) and Proposition 6.2.1.2 a),

$$\frac{\langle y|x\rangle}{f(x)} = \frac{\langle vz|x\rangle}{f(x)} = \frac{\langle z|vx\rangle}{f(x)} = \langle z|x\rangle$$

for every $x \in B$. Hence by i), $\left(\frac{\langle y|x\rangle}{f(x)}x\right)_{x\in B}$ is a summable family in $\mathcal{C}$. ■

Proposition 6.2.1.4 (5) *Take $\delta \in \mathbb{R}$. Let B be an orthonormal basis of $L^2(r\cdot\lambda)$ and $f \in \mathbb{K}^B$ such that*

$$vx = f(x)x$$

for every $x \in B$ (Theorem 6.2.1.3) and put

$$h := \frac{\delta f - 1}{f}$$

(Theorem 6.2.1.3 d)).

a) $x \in \mathcal{F}_{\delta,h(x)}$ *whenever* $x \in B$.

b) $h(B) = A_\delta = A + \delta$.

c) *If u_δ is injective, then g_δ and v_δ are selfadjoint, $1-\delta f$ vanishes nowhere,*

$$v_\delta x = \frac{f(x)}{1-\delta f(x)}x$$

for every $x \in B$, and

$$\sigma_p(v_\delta) = \{-\frac{1}{\gamma}|\gamma \in A + \delta\}\,.$$

a) By Proposition 6.2.1.2 b),

$$ux = -\frac{1}{f(x)}rx\,.$$

It follows that

$$u_\delta x = -\frac{1}{f(x)}rx + \delta rx = \frac{\delta f(x) - 1}{f(x)}rx = h(x)x\,,$$

$$x \in \mathcal{F}_{\delta,h(x)}\,.$$

b) By a) and Proposition 6.2.1.1,

$$h(B) \subset A_\delta = A + \delta .$$

Take $\gamma \in A + \delta$. Then $\gamma - \delta \in A$. By Proposition 6.2.1.2 d) and Theorem 6.2.1.3 c), there is an $x \in B$ such that

$$f(x) = \frac{1}{\delta - \gamma} .$$

Thus

$$h(x) = \frac{\delta f(x) - 1}{f(x)} = \delta - \frac{1}{f(x)} = \delta - (\delta - \gamma) = \gamma .$$

Hence

$$\gamma \in h(B)$$

and

$$A + \delta \subset h(B) .$$

c) By Proposition 3.2.2.2 (applied to the differential operator $\frac{1}{r}u$), g_δ is selfadjoint and so v_δ is also selfadjoint (Proposition 5.3.2.1). Since u_δ is injective, $0 \notin A_\delta$. By b), h vanishes nowhere. Take $x \in B$. By a), $x \in \mathcal{F}_{\delta,h(x)}$. Thus

$$v_\delta = -\frac{1}{h(x)}x = \frac{f(x)}{1 - \delta f(x)}x$$

(Proposition 6.2.1.2 c)) and

$$\sigma_p(v_\delta) = \{-\frac{1}{\gamma} | \gamma \in A + \delta\}$$

by b) and Proposition 6.2.1.2 d). ■

Proposition 6.2.1.5 (5) *Take $\delta \in \mathbb{K}$ and $y \in L^2(r\cdot\lambda)$. Let B be an orthonormal basis of $L^2(r\cdot\lambda)$ and $f \in \mathbb{K}^B$ such that $x \in \mathcal{D}$ and*

$$u_\delta x = f(x)rx$$

for every $x \in B$.

a) *For every $x \in B$,*

$$\delta - f(x) \in \mathbb{R}\backslash\{0\}$$

and

$$vx = \frac{1}{\delta - f(x)}x .$$

b) $\overset{-1}{f}(C)$ *is finite whenever* C *is a bounded set of* $\mathbb{K}$.

c) *The following are equivalent:*

c_1) $y \in \mathcal{D}$.

c_2) $(f(x)\langle y|x\rangle)_{x\in B} \in \ell^2(B)$,

$$\sum_{x\in B} f(x)\langle y|x\rangle x \in \mathcal{C} \quad \textit{(where the sum is considered in } L^2(r\cdot\lambda)).$$

If these conditions are fulfilled then:

c_3) $v\left(\sum_{x\in B} f(x)\langle y|x\rangle x\right) = \delta v y - y$.

c_4) $y = \sum_{x\in B}\langle y|x\rangle x$ *(in* $\mathcal{C}$*)*.

c_5) $u_\delta y = r\sum_{x\in B} f(x)\langle y|x\rangle x$ *(in* $L^2(r\cdot\lambda)$*)*.

d) *If* $y \in v(\mathcal{D})$ *then* $(f(x)\langle y|x\rangle x)_{x\in B}$ *is summable in* $\mathcal{C}$.

e) *If* $(f(x)\langle y|x\rangle)_{x\in B} \in \ell^2(B)$ *and if the coefficients of* u *are* $\mathcal{C}^\infty$*-functions, then*

$$u_\delta y = r\sum_{x\in B}\langle y|x\rangle f(x)x$$

in the sense of L^2*-distributions.*

a) We have

$$ux = (f(x) - \delta)rx$$

so that

$$f(x) - \delta \in A\backslash\{0\} \subset \mathbb{R}\backslash\{0\}$$

(Proposition 6.2.1.2 d),e)). By Proposition 6.2.1.2 b),

$$-x = v\left(\frac{1}{r}ux\right) = (f(x) - \delta)vx,$$

so that

$$vx = \frac{1}{\delta - f(x)}x.$$

b) Put

$$\alpha := \sup_{\beta \in C} |\beta| < \infty .$$

Then

$$|\delta - f(x)| \leq |\delta| + \alpha ,$$

$$\frac{1}{|\delta - f(x)|} \geq \frac{1}{|\delta| + \alpha}$$

for every $x \in \overset{-1}{f}(C)$. By a) and Theorem 6.2.1.3 a),

$$\frac{1}{(|\delta| + \alpha)^2} \operatorname{Card} \overset{-1}{f}(C) \leq \sum_{x \in \overset{-1}{f}(C)} \frac{1}{|\delta - f(x)|^2} < \infty .$$

Hence $\overset{-1}{f}(C)$ is finite.

c) First assume that c_1) holds. By a) and Theorem 6.2.1.3 $i_1 \Rightarrow i_2$ & i_3 & i_4 & i_5,

$$((\delta - f(x))\langle y|x\rangle)_{x \in B} \in \ell^2(B) ,$$

$$\sum_{x \in B} (\delta - f(x))\langle y|x\rangle x \in \mathcal{C} ,$$

$$v\left(\sum_{x \in B} (\delta - f(x))\langle y|x\rangle x\right) = y ,$$

$$y = \sum_{x \in B} \langle y|x\rangle x \quad (\text{in } \mathcal{C}) ,$$

$$u_\delta y = uy + \delta r y = -r \sum_{x \in B} (\delta - f(x))\langle y|x\rangle x + \delta r y =$$

$$= r \sum_{x \in B} f(x)\langle y|x\rangle x \quad (\text{in } L^2(r \cdot \lambda)) .$$

From this it follows easily

$$(f(x)\langle y|x\rangle)_{x \in B} \in \ell^2(B) ,$$

$$\sum_{x\in B} f(x)\langle y|x\rangle x \in \mathcal{C}\,,$$

$$v\left(\sum_{x\in B} f(x)\langle y|x\rangle x\right) = \delta vy - y\,.$$

Now assume c_2) holds. By Theorem 6.2.1.3 g),

$$\sum_{x\in B} \frac{f(x)\langle y|x\rangle}{\delta - f(x)} x = v\left(\sum_{x\in B} f(x)\langle y|x\rangle x\right) \in \mathcal{C}\,,$$

$$\sum_{x\in B} \frac{\langle y|x\rangle}{\delta - f(x)} x = vy \in \mathcal{C}\,.$$

Hence

$$\sum_{x\in B}\langle y|x\rangle x = \delta \sum_{x\in B} \frac{\langle y|x\rangle}{\delta - f(x)} x - \sum_{x\in B} \frac{f(x)\langle y|x\rangle}{\delta - f(x)} x \in \mathcal{C}\,,$$

$$\sum_{x\in B}(\delta - f(x))\langle y|x\rangle x \in \mathcal{C}\,.$$

Since

$$((\delta - f(x))\langle y|x\rangle)_{x\in B} \in \ell^2(B)$$

it follows $y \in \mathcal{D}$ by Theorem 6.2.1.3. $i_2 \Rightarrow i_1$.

d) By a) and Theorem 6.2.1.3 j), $((\delta - f(x))\langle y|x\rangle x)_{x\in B}$ is summable in $\mathcal{C}$. By $c_1 \Rightarrow c_4$, $(f(x)\langle y|x\rangle x)_{x\in B}$ is summable in $\mathcal{C}$.

e) Since u is selfadjoint, u_δ is also selfadjoint. Let φ be a $\mathcal{C}^\infty$-function on $]a,b[$ with compact support. For every $x \in B$,

$$\langle x|\frac{1}{r}u_\delta\varphi\rangle = \int_a^b x\overline{u_\delta\varphi}dt = \int_a^b (u_\delta x)\overline{\varphi}dt =$$

$$= f(x)\int_a^b x\overline{\varphi}r\,dt = f(x)\langle x|\varphi\rangle\,.$$

Hence (Theorem 5.5.1.5 a $\Rightarrow$ c)

$$\int_a^b y\overline{u_\delta\varphi}dt = \langle y|\frac{1}{r}u_\delta\varphi\rangle = \sum_{x\in B}\langle y|x\rangle\langle x|\frac{1}{r}u_\delta\varphi\rangle =$$

$$= \sum_{x\in B} \langle y|x\rangle f(x)\langle x|\varphi\rangle = \langle \sum_{x\in B} \langle y|x\rangle f(x)x|\varphi\rangle =$$

$$= \int_a^b r \left(\sum_{x\in B} \langle y|x\rangle f(x)x \right) \overline{\varphi} dt \,.$$

Since φ is arbitrary,

$$u_\delta y = r \sum_{x\in B} \langle y|x\rangle f(x)x$$

in the sense of L^2–distributions. ■

Theorem 6.2.1.6 (5) *Assume that A is bounded below. Let B be an orthonormal basis of $L^2(r\cdot\lambda)$ and $f \in \mathbb{K}^B$ such that*

$$vx = f(x)x$$

for every $x \in B$.

a) $f \in \ell^1(B)$.

b) *v is nuclear,*

$$\|v\|_1 = \|f\|_1 \,,$$

and

$$v = \sum_{x\in B} f(x)\langle\cdot|x\rangle x \quad (\text{in } \mathcal{L}^1(L^2(r\cdot\lambda))) \,.$$

c) *If $y \in \mathcal{D}$, then*

$$\left(\frac{\langle y|x\rangle}{\sqrt{|f(x)|}} \right)_{x\in B} \in \ell^1(B) \,.$$

d) $g = \sum_{x\in B} f(x)x \otimes \overline{x} \quad (\text{in } \mathcal{C}([a,b]\times[a,b]))$.

e) $f^{-1}(\mathbb{R}_+)$ *is finite.*

a & b & d & e. By hypothesis, there is a $\delta \in \mathbb{R}$ such that $A + \delta \subset]0,\infty[$. Put

$$h := \frac{\delta f - 1}{f}$$

(Theorem 6.2.1.3 d)). Then, by Proposition 6.2.1.1,

$$A_\delta = A + \delta \subset]0, \infty[.$$

Thus u_δ is injective. By Proposition 6.2.1.4 c), v_δ is selfadjoint, $1-\delta f$ vanishes nowhere,

$$v_\delta x = \frac{f(x)}{1 - \delta f(x)} x$$

for each $x \in B$, and

$$\sigma_p(-v_\delta) = -\sigma_p(v_\delta) = \{\frac{1}{\gamma} | \gamma \in A + \delta\} \subset]0, \infty[.$$

Hence $-v_\delta$ is positive.

Define

$$f_\delta := \frac{f}{1 - \delta f} .$$

By the above considerations,

$$v_\delta x = f_\delta(x) x$$

for each $x \in B$. By Theorem 6.1.9.4 a),b),c),

$$B \subset \mathcal{C} , \quad f_\delta \in \ell^1(B) ,$$

$$g_\delta = \sum_{x \in B} f_\delta(x) x \otimes \overline{x} \quad (\text{in } \mathcal{C}([a,b] \times [a,b])) ,$$

$$\sup_{x \in B} \| f_\delta(x) x \otimes \overline{x} \|_\infty < \infty .$$

It follows from

$$1 + \delta f_\delta = 1 + \frac{\delta f}{1 - \delta f} = \frac{1}{1 - \delta f}$$

that $1 + \delta f_\delta$ vanishes nowhere. Define

$$\varepsilon := \inf |1 + \delta f_\delta| > 0 .$$

Then it follows from

$$f = \frac{f_\delta}{1 + \delta f_\delta} ,$$

that

$$|f| \leq \frac{1}{\varepsilon}|f_\delta|\,,$$

$$f \in \ell^1(B)\,.$$

Hence the series

$$\sum_{x\in B} f(x) f_\delta(x) x \otimes \overline{x}$$

is absolutely convergent. From

$$f = f_\delta(1 - \delta f) = f_\delta - \delta f f_\delta$$

it follows that the series

$$\sum_{x\in B} f(x) x \otimes \overline{x}$$

converges in $\mathcal{C}([a,b]\times[a,b])$. Thus, by 6.2.1.3 b),

$$g = \sum_{x\in B} f(x) x \otimes \overline{x} \quad (\text{in } \mathcal{C}([a,b]\times[a,b]))\,.$$

By Theorem 6.1.9.2 b), v is nuclear.

$$v = \sum_{x\in B} f(x)\langle\cdot|x\rangle x \quad (\text{in } \mathcal{L}^1(L^2(r\cdot\lambda)))$$

and

$$\|v\|_1 = \|f\|_1\,.$$

Since $-v_\delta$ is positive, f_δ is strictly negative. From

$$f = \frac{f_\delta}{1 + \delta f_\delta}$$

it follows

$$f(x) \in \mathbb{R}_+ \Longrightarrow 1 + \delta f_\delta(x) < 0\,.$$

Since $f_\delta \in \ell^1(B)$, $\overset{-1}{f}(\mathbb{R}_+)$ is finite.

c) By Theorem 6.2.1.3 $\mathrm{i}_1 \Rightarrow \mathrm{i}_2$,

$$\left(\frac{\langle y|x\rangle}{f(x)}\right)_{x\in B} \in \ell^2(B)\,.$$

By a), $f \in \ell^1(B)$ and so $\sqrt{|f|} \in \ell^2(B)$. Hence

$$\left(\frac{\langle y|x\rangle}{\sqrt{|f(x)|}}\right)_{x\in B} \in \ell^1(B)\,. \qquad \blacksquare$$

Corollary 6.2.1.7 (5) *Take $\delta \in \mathbb{R}$ and $y \in \mathcal{D}$. Let B be an orthonormal basis of $L^2(r\cdot\lambda)$ and $f \in \mathbb{K}^B$ with $B \subset \mathcal{D}$ and*

$$u_\delta x = f(x)rx$$

for every $x \in B$. If A is bounded below then:

a) $\sum\limits_{\substack{x\in B\\ f(x)\neq 0}} \frac{1}{|f(x)|} < \infty$.

b) $(\sqrt{|f(x)|}\langle y|x\rangle)_{x\in B} \in \ell^1(B)$.

By Proposition 6.2.1.5 a),

$$vx = \frac{1}{\delta - f(x)}x$$

for every $x \in B$.

a) By Theorem 6.2.1.6 a),

$$\sum_{x\in B} \frac{1}{|\delta - f(x)|} < \infty.$$

Hence

$$\sum_{\substack{x\in B\\ f(x)\neq 0}} \frac{1}{|f(x)|} < \infty.$$

b) By Theorem 6.2.1.6 c),

$$\left(\sqrt{|f(x)-\delta|}\langle y|x\rangle\right)_{x\in B} \in \ell^1(B).$$

Hence by a),

$$\left(\sqrt{|f(x)|}\langle y|x\rangle\right)_{x\in B} \in \ell^1(B).$$ ■

6.2.2 The Regular Sturm–Liouville Theory

In this subsection, we set $\mathbb{K} = \mathbb{R}$ and use the following notation:

a) a, b, α, β are real numbers with $a < b$.

b) λ is Lebesgue measure on $[a, b]$.

c) p is a strictly positive continuously differentiable function on $[a, b]$.

d) q is a continuous function on $[a, b]$.

e) r is a strictly positive continuous function on $[a, b]$.

f) $\mathcal{D}$ is the set of twice continuously differentiable functions x on $[a, b]$ such that

$$\begin{cases} x(a)\sin\alpha - x'(a)\cos\alpha = 0 \\ x(b)\sin\beta + x'(b)\cos\beta = 0\,. \end{cases}$$

g) $u : \mathcal{D} \longrightarrow \mathcal{C}([a, b])\,, \quad x \longmapsto -(px')' + qx\,.$

h) $\mathcal{F}_\gamma := \{x \in \mathcal{D} | ux = \gamma r x\}$ for every $\gamma \in \mathbb{R}$.

i) $A := \{\gamma \in \mathbb{R} | \mathcal{F}_\gamma \neq \{0\}\}$.

Proposition 6.2.2.1 (5) *For any $\gamma \in \mathbb{R}$, $\mathcal{F}_\gamma$ is a one-dimensional vector subspace of $\mathcal{D}$.*

The Wronskian determinant of any two functions of $\mathcal{D}$ vanishes at a and b. Hence, given any $x, y \in \mathcal{F}_\gamma$, the Wronskian determinant of x and y vanishes identically, and x and y are linearly dependent. ■

Proposition 6.2.2.2 (5) *Let y, z be twice continuously differentiable functions on $[a, b]$, which are not identically 0, such that*

$$-(py')' + qy = 0\,, \quad y(a)\sin\alpha - y'(a)\cos\alpha = 0\,,$$

$$-(pz')' + qz = 0\,, \quad z(b)\sin\beta + z'(b)\cos\beta = 0\,.$$

If u is injective (equivalently, if $0 \notin A$) then:

a) $p(yz' - zy')$ *is a constant* δ, *and* $\delta \neq 0$.

b) *The function*

$$g : [a,b] \times [a,b] \longrightarrow \mathbb{R}, \quad (s,t) \longmapsto \begin{cases} \frac{1}{\delta} y(s) z(t) & if \quad s \leq t \\ \frac{1}{\delta} y(t) z(s) & if \quad s \geq t \end{cases}$$

is the Green function of the differential operator u *(for the given boundary value problem) (Theorem 3.2.1.3).*

c) *If we define*

$$v : L^2(r \cdot \lambda) \longrightarrow L^2(r \cdot \lambda), \quad x \longmapsto \int_a^b g(\cdot, t) x(t) r(t) dt,$$

then v *is an injective selfadjoint Hilbert–Schmidt operator and*

c_1) $\begin{cases} x \in \mathcal{D} \Rightarrow v(\frac{1}{r} ux) = -x \\ x \in \mathcal{C}([a,b]) \Rightarrow vx \in \mathcal{D}, \ \frac{1}{r} uvx = -x. \end{cases}$

c_2) $\gamma \in \mathbb{R} \backslash \{0\} \Rightarrow \mathcal{F}_\gamma = \{x \in L^2(r \cdot \lambda) | vx = -\frac{1}{\gamma} x\}$.

c_3) $\sigma_p(v) = \{-\frac{1}{\gamma} | \gamma \in A\}$.

c_4) $\|v\| = \sup_{\gamma \in A} \frac{1}{|\gamma|}, \quad \|v\|_2 = \left(\sum_{\gamma \in A} \frac{1}{\gamma^2} \right)^{\frac{1}{2}}$.

a) We have

$$(p(yz' - zy'))' = (pz')'y + pz'y' - (py')'z - py'z' =$$

$$= qzy - qyz = 0$$

i.e. $p(yz' - zy')$ is a constant δ. $yz' - zy'$ is the Wronskian determinant of y and z. $\delta = 0$ implies that y and z are linearly dependent, i.e., $y, z \in \mathcal{D}$, which contradicts the injectivity of u.

b) Take $t \in]a,b[$. $g(\cdot, t)$ is continuous on $[a,b]$ and twice continuously differentiable on $[a,b] \backslash \{t\}$. Moreover,

$$\lim_{\substack{s \to t \\ s<t}} g'(s,t) - \lim_{\substack{s \to t \\ s>t}} g'(s,t) = \frac{1}{\delta} y'(t) z(t) - \frac{1}{\delta} y(t) z'(t) = \frac{1}{-p(t)},$$

$$-(pg'(\cdot, t))' + qg(\cdot, t) = 0 \text{ on } [a,b] \backslash \{t\},$$

and

$$\begin{cases} g(a,t)\sin\alpha - g'(a,t)\cos\alpha = 0 \\ g(b,t)\sin\beta - g'(b,t)\cos\beta = 0\,. \end{cases}$$

Hence g is the Green function of u for the given boundary value problem (Definition 3.2.1.2).

c) By b) and Proposition 6.2.1.2 a), v is injective and selfadjoint, and by Theorem 6.1.9.2, it is a Hilbert–Schmidt operator.

c_1) By Corollary 3.2.1.5,

$$v(\frac{1}{r}ux) = \int_a^b g(\cdot,t)ux(t)dt = -x$$

for every $x \in \mathcal{D}$. By Proposition 3.2.1.4, $vx \in \mathcal{D}$ and

$$\frac{1}{r}uvx = \frac{1}{r}u\left(\int_a^b g(\cdot,t)x(t)r(t)dt\right) = -x$$

for every $x \in \mathcal{C}([a,b])$.

c_2) Choose $x \in L^2(r\cdot\lambda)$ such that

$$vx = -\frac{1}{\gamma}x\,.$$

By Corollary 3.1.6.4, $x \in \mathcal{C}([a,b])$, and so, by c_1), $x \in \mathcal{D}$. It follows by c_1), that

$$\{x \in L^2(r\cdot\lambda) | vx = -\frac{1}{\gamma}x\} = \{x \in \mathcal{D} | vx = -\frac{1}{\gamma}x\} =$$

$$= \{x \in \mathcal{D} | ux = \gamma r x\} = \mathcal{F}_\gamma$$

c_3) follows from c_2).

c_4) Since v is selfadjoint, we get

$$\|v\| = \sup_{\gamma\in A}\frac{1}{|\gamma|}, \quad \|v\|_2 = \left(\sum_{\gamma\in A}\frac{1}{\gamma^2}\right)^{\frac{1}{2}}$$

by c_3) and Theorem 6.1.9.2 d). ■

Lemma 6.2.2.3 (5) *Take $\delta > 0$. If x is a continuously differentiable function on $[a,b]$, then*

$$\|x\|_\infty^2 \leq \delta \int_a^b x'^2 dt + \Big(\frac{1}{b-a} + (2+\frac{1}{\delta})^2\Big)^2 \int_a^b x^2 dt\,.$$

We may assume that

$$\int_a^b x^2 dt = 1\,,$$

$$\gamma := \|x\|_\infty > \frac{1}{b-a} + (2+\frac{1}{\delta})^2\,.$$

Define

$$B := \Big\{ t \in [a,b] \;\Big|\; |x(t)| \geq \gamma^{\frac{1}{2}} \Big\}\,,$$

$$C := \Big\{ t \in [a,b] \;\Big|\; |x'(t)| \geq \gamma^{\frac{3}{2}} \Big\}\,.$$

Then

$$\gamma\lambda(B) \leq \int_B x^2 dt \leq 1\,,$$

so that

$$\lambda(B) \leq \frac{1}{\gamma} < b-a\,.$$

Hence, there are $t_1, t_2 \in [a,b]$ with

$$|x(t_1)| = \gamma\,, \quad |x(t_2)| = \gamma^{\frac{1}{2}}\,, \quad |t_1 - t_2| \leq \frac{1}{\gamma}\,.$$

Then

$$\gamma - \gamma^{\frac{1}{2}} \leq |x(t_1) - x(t_2)| = \Big|\int_{t_1}^{t_2} x' dt\Big| \leq \gamma^{\frac{3}{2}}|t_1 - t_2| + \int_C |x'| dt \leq$$

$$\gamma^{\frac{1}{2}} + \gamma^{-\frac{3}{2}} \int_C x'^2 dt\,,$$

and therefore

$$\delta \int_a^b x'^2 dt \geq \delta \int_C x'^2 dt \geq \delta\gamma^{\frac{3}{2}}(\gamma - 2\gamma^{\frac{1}{2}}) = \gamma^2\delta(\gamma^{\frac{1}{2}} - 2) \geq$$

$$\geq \gamma^2\delta(2 + \frac{1}{\delta} - 2) = \gamma^2 = \|x\|_\infty^2\,. \qquad \blacksquare$$

Lemma 6.2.2.4 (5) *Take* $x \in \mathcal{D}\backslash\{0\}$ *and* $\gamma \in \mathbb{R}$ *such that*

$$ux = \gamma r x\,.$$

Define

$$\delta := \inf_{t\in[a,b]} p(t)\,, \quad \delta' := \inf_{t\in[a,b]} r(t)\,,$$

$$\dot{\alpha} = \begin{cases} \alpha & if \ \cos\alpha \neq 0 \\ 0 & if \ \cos\alpha = 0\,, \end{cases} \qquad \dot{\beta} = \begin{cases} \beta & if \ \cos\beta \neq 0 \\ 0 & if \ \cos\beta = 0\,, \end{cases}$$

and assume that q *is positive.*

a) $\gamma \int_a^b rx^2dt = p(a)x^2(a)tg\dot{\alpha} + p(b)x^2(b)tg\dot{\beta} + \int_a^b (px'^2 + qx^2)dt\,.$

b) $\alpha, \beta \in [0, \frac{\pi}{2}] + \pi\mathbb{Z} \Rightarrow \gamma > 0\,.$

c) $\gamma \geq -\frac{1}{\delta'}(p(a)|tg\dot{\alpha}| + p(b)|tg\dot{\beta}|)\times \times \Big(\frac{1}{b-a} + \Big(2 + \frac{p(a)|tg\dot{\alpha}| + p(b)|tg\dot{\beta}|}{\delta}\Big)^2\Big)^2\,.$

a) We have

$$\gamma \int_a^b rx^2dt = \int_a^b x(\gamma r x)dt = \int_a^b x(-(px')' + qx)dt =$$

$$= -\int_a^b x(px')'dt + \int_a^b qx^2dt =$$

$$= x(a)p(a)x'(a) - x(b)p(b)x'(b) + \int_a^b x'(px')dt + \int_a^b qx^2dt =$$

$$= p(a)x^2(a)tg\dot{\alpha} + p(b)x^2(b)tg\dot{\beta} + \int_a^b (px'^2 + qx^2)dt\,.$$

b) follows from a).

c) By b), we may assume that

$$|\sin\alpha| + |\sin\beta| \neq 0\,, \quad |\cos\alpha| + |\cos\beta| \neq 0\,.$$

Put

$$\varepsilon := \frac{\delta}{p(a)|tg\dot{\alpha}| + p(b)|tg\dot{\beta}|}\,.$$

By Lemma 6.2.2.3,

$$\|x\|_\infty^2 \leq \varepsilon \int_a^b x'^2 dt + \Big(\frac{1}{b-a} + (2+\frac{1}{\varepsilon})^2\Big)^2 \int_a^b x^2 dt \leq$$

$$\leq \varepsilon \int_a^b x'^2 dt + \frac{1}{\delta'}\Big(\frac{1}{b-a} + (2+\frac{1}{\varepsilon})^2\Big)^2 \int_a^b r x^2 dt\,,$$

and so, by a),

$$\gamma \int_a^b r x^2 dt \geq -\Big(p(a)|tg\dot{\alpha}| + p(b)|tg\dot{\beta}|\Big) \times$$

$$\times\Big(\varepsilon \int_a^b x'^2 dt + \frac{1}{\delta'}\Big(\frac{1}{b-a} + (2+\frac{1}{\varepsilon})^2\Big)^2 \int_a^b r x^2 dt\Big) + \delta \int_a^b x'^2 dt =$$

$$= -\frac{1}{\delta'}(p(a)|tg\dot{\alpha}| + p(b)|tg\dot{\beta}|)\Big(\frac{1}{b-a} + (2+\frac{1}{\varepsilon})^2\Big)^2 \int_a^b r x^2 dt\,,$$

$$\gamma \geq -\frac{1}{\delta'}\Big(p(a)|tg\dot{\alpha}| + p(b)|tg\dot{\beta}|\Big) \times$$

$$\times\Big(\frac{1}{b-a} + \Big(2 + \frac{p(a)|tg\dot{\alpha}| + p(b)|tg\dot{\beta}|}{\delta}\Big)^2\Big)^2. \qquad \blacksquare$$

Theorem 6.2.2.5 (5) (Sturm, 1836; Liouville, 1837)

a) *A is infinite and bounded below,*

$$\sum_{\gamma \in A\setminus\{0\}} \frac{1}{|\gamma|} < \infty\,,$$

and there is an orthonormal basis $(x_\gamma)_{\gamma\in A}$ of $L^2(r\cdot\lambda)$ such that $x_\gamma \in \mathcal{F}_\gamma$ for every $\gamma \in A$.

b) *If $x \in \mathcal{D}$, then*

$$(\gamma\langle x|x_\gamma\rangle)_{\gamma\in A} \in \ell^2(A)\,, \quad (\sqrt{|\gamma|}\langle x|x_\gamma\rangle)_{\gamma\in A} \in \ell^1(A)\,,$$

$$\sum_{\gamma\in A}\gamma\langle x|x_\gamma\rangle x_\gamma\in\mathcal{C}\quad(\textit{where the sum is considered in } L^2(r\cdot\lambda))\,,$$

$$x=\sum_{\gamma\in A}\langle x|x_\gamma\rangle x_\gamma\quad(\textit{in } \mathcal{C}([a,b]))\,,)$$

$$ux=r\sum_{\gamma\in A}\gamma\langle x|x_\gamma\rangle x_\gamma\quad(\textit{in } L^2(r\cdot\lambda))\,.$$

c) *If u is injective (equivalently, if $0\notin A$), g is the Green function of u (for the given boundary value problem), and*

$$v: L^2(r\cdot\lambda)\longrightarrow L^2(r\cdot\lambda)\,,\quad x\longmapsto\int_a^b g(\cdot,t)x(t)r(t)dt\,,$$

then:

c_1) *v is nuclear and*

$$\|v\|_1=\sum_{\gamma\in A}\frac{1}{|\gamma|}\,;$$

c_2) $v=-\sum_{\gamma\in A}\frac{1}{\gamma}\langle\cdot|x_\gamma\rangle x_\gamma\quad(\textit{in } \mathcal{L}^1(L^2(r\cdot\lambda)))\,.$

c_3) $g=-\sum_{\gamma\in A}\frac{1}{\gamma}x_\gamma\otimes x_\gamma\quad(\textit{in } \mathcal{C}([a,b]\times[a,b]))\,.$

c_4) $x\in L^2(r\cdot\lambda)\Rightarrow vx=-\sum_{\gamma\in A}\frac{\langle x|x_\gamma\rangle}{\gamma}x_\gamma\quad(\textit{ in } \mathcal{C}([a,b]))\,.$

d) *If q is positive and if $\alpha,\beta\in[0,\frac{\pi}{2}]+\pi\mathbb{Z}$, then $A\subset]0,\infty[$.*

e) *If*

$$\alpha,\beta\in\frac{\pi}{2}\mathbb{Z}\,,$$

then

$$\Big(\frac{\|x_\gamma\|_\infty}{\sqrt{|\gamma|}}\Big)_{\gamma\in A\setminus\{0\}}\in\ell^\infty(A\setminus\{0\})\,,$$

and

$$\Big(\sqrt{|\gamma|}\langle x|x_\gamma\rangle\|x\|_\infty\Big)_{\gamma\in A}\in\ell^2(A)\,,$$

$$(\langle x|x_\gamma\rangle\|x_\gamma\|_\infty)_{\gamma\in A}\in\ell^1(A)$$

for every $x\in\mathcal{D}$.

a) Given $\gamma, \delta \in \mathbb{R}$, define

$$u_\delta : \mathcal{D} \longrightarrow \mathcal{C}([a,b]), \quad x \longmapsto ux + \delta rx,$$

$$\mathcal{F}_{\delta,\gamma} := \{x \in \mathcal{D} | u_\delta x = \gamma r x\},$$

$$A_\delta := \{\gamma \in \mathbb{R} | \mathcal{F}_{\delta,\gamma} \neq \{0\}\}.$$

If $x \in \mathcal{D}$, then

$$(x \in \mathcal{F}_{\delta,\gamma}) \Longleftrightarrow (u_\delta x = \gamma r x) \Longleftrightarrow (ux = (\gamma - \delta) r x) \Longleftrightarrow (x \in \mathcal{F}_{\gamma-\delta}),$$

and so

$$\mathcal{F}_{\delta,\gamma} = \mathcal{F}_{\gamma-\delta}.$$

If $\gamma \in \mathbb{R}$, then

$$(\gamma \in A_\delta) \Longleftrightarrow (\mathcal{F}_{\delta,\gamma} \neq \{0\}) \Longleftrightarrow (\mathcal{F}_{\gamma-\delta} \neq \{0\}) \Longleftrightarrow (\gamma - \delta \in A) \Longleftrightarrow (\gamma \in A + \delta),$$

so that

$$A_\delta = A + \delta.$$

Take $\delta = \mathbb{R}$ such that

$$q + \delta r \geq 0.$$

By Lemma 6.2.2.4 c), A_δ is bounded below. Hence A is bounded below, and there is an $\eta \in \mathbb{R}$, such that

$$A_\eta \subset]0, \infty[.$$

Let g_η be the Green function of u_δ and consider

$$v_\eta : L^2(r \cdot \lambda) \longrightarrow L^2(r \cdot \lambda), \quad x \longmapsto \int_a^b g_\eta(\cdot, t) x(t) r(t) dt.$$

By Proposition 6.2.2.2 c), v_η is selfadjoint and

$$\sigma_p(-v_\eta) = -\sigma_p(v_\eta) = \left\{ \frac{1}{\gamma} \,\middle|\, \gamma \in A_\eta \right\} \subset]0, \infty[.$$

Hence $-v_\eta$ is positive. By Theorem 6.1.9.4 (and Theorem 6.1.9.2), v_η is nuclear and there is an orthonormal basis B of $L^2(r \cdot \lambda)$ and an $f \in \ell^1(B)$ such that

$$-v_\eta = \sum_{y \in B} f(y) \langle \cdot | y \rangle y \quad (\text{in } \mathcal{L}^1(L^2(r \cdot \lambda)))\,,$$

$$f(B) = \sigma_p(-v_\eta) \subset]0, \infty[\,,$$

$$B \subset \mathcal{C}([a,b])\,,$$

$$-g_\eta = \sum_{y \in B} f(y) y \otimes y \quad (\text{in } \mathcal{C}([a,b] \times [a,b]))\,.$$

By Proposition 6.2.2.2 c_3),

$$f(B) = \sigma_p(-v_\eta) = -\sigma_p(v_\eta) = \left\{ \frac{1}{\gamma} \,\middle|\, \gamma = A_\eta \right\}$$

and by Proposition 6.2.2.1, the mapping

$$B \longrightarrow A_\eta\,, \quad y \longmapsto \frac{1}{f(y)}$$

is bijective. Since

$$A_\eta = A + \eta\,,$$

the mapping

$$h : B \longrightarrow A\,, \quad y \longmapsto \frac{1}{f(y)} - \eta$$

is bijective. Since $L^2(r \cdot \lambda)$ is infinite–dimensional, B, and therefore A, is infinite. Given $\gamma \in A$, define

$$x_\gamma := h^{-1}(\gamma)\,.$$

$(x_\gamma)_{\gamma \in A}$ is an orthonormal basis of $L^2(r \cdot \lambda)$, and

$$v_\eta = -\sum_{\gamma \in A} \frac{1}{\gamma + \eta} \langle \cdot | x_\gamma \rangle x_\gamma \quad (\text{in } \mathcal{L}^1(L^2(r \cdot \lambda)))\,.$$

If $\gamma \in A$, then

$$v_\eta x_\gamma = -\frac{1}{\gamma + \eta} x_\gamma\,,$$

and so, by Proposition 6.2.2.2 c_2),

$$x_\gamma \in \mathcal{F}_{\eta, \gamma + \eta} = \mathcal{F}_\gamma\,.$$

We have that

$$\sum_{\gamma\in A}\frac{1}{\gamma+\eta}=\sum_{y\in B}f(y)<\infty\,,$$

so that

$$\sum_{\gamma\in A\setminus\{0\}}\frac{1}{|\gamma|}<\infty\,.$$

b) By a) and Proposition 6.2.1.5 $c_1 \Rightarrow c_2$ & c_4 & c_5,

$$(\gamma\langle x|x_\gamma\rangle)_{\gamma\in A}\in\ell^2(A)\,,$$

$$\sum_{\gamma\in A}\gamma\langle x|x_\gamma\rangle x_\gamma\in\mathcal{C}$$

(where the sum is considered in $L^2(r\cdot\lambda)$),

$$x=\sum_{\gamma\in A}\langle x|x_\gamma\rangle x_\gamma\quad(\text{in }\mathcal{C})\,,$$

$$ux=r\sum_{\gamma\in A}\gamma\langle y|x_\gamma\rangle x_\gamma\quad(\text{in }L^2(r\cdot\lambda))\,.$$

By Proposition 6.2.1.7 b),

$$(\sqrt{|\gamma|}\langle y|x_\gamma\rangle)_{\gamma\in A}\in\ell^1(A)\,.$$

c) By a) and Proposition 6.2.1.5 a),

$$vx_\gamma=-\frac{1}{\gamma}x_\gamma$$

for every $\gamma\in A$. By Theorem 6.2.1.6 a),b),d), v is nuclear,

$$\|v\|_1=\sum_{\gamma\in A}\frac{1}{|\gamma|}\,,$$

$$v=-\sum_{\gamma\in A}\frac{1}{\gamma}\langle\cdot|x_\gamma\rangle x_\gamma\quad(\text{in }\mathcal{L}^1(L^2(r\cdot\lambda)))\,,$$

$$g=-\sum_{\gamma\in A}\frac{1}{\gamma}x_\gamma\otimes x_\gamma\quad(\text{in }\mathcal{C}([a,b]\times[a,b]))\,.$$

By Theorem 6.2.1.3 g),

$$vx = \sum_{\gamma \in A} \frac{1}{\gamma} \langle x | x_\gamma \rangle x_\gamma \quad (\text{in } \mathcal{C})$$

for every $x \in L^2(\lambda)$.

d) follows from Lemma 6.2.2.4 b).

e) Take $\delta > 0$ such that $q + \delta r$ is strictly positive. Put

$$\delta_1 := \inf_{t \in [a,b]} p(t) > 0\,, \quad \delta_2 := \inf_{t \in [a,b]} (q(t) + \delta r(t))\,.$$

Let $\gamma \in A$. By Lemma 6.2.2.3,

$$\|x_\gamma\|_\infty^2 \leq \delta \int_a^b {x'_\gamma}^2 dt + \left(\frac{1}{b-a} + \left(2 + \frac{1}{\delta}\right)^2 \right)^2 \int_a^b x_\gamma^2 dt \leq$$

$$\leq \frac{\delta}{\delta_1} \int_a^b p {x'_\gamma}^2 dt + \left(\frac{1}{b-a} + \left(2 + \frac{1}{\delta}\right)^2 \right)^2 \frac{1}{\delta_2} \int_a^b (q + \delta r) x_\gamma^2 dt \leq$$

$$\leq \left(\frac{\delta}{\delta_1} + \left(\frac{1}{b-a} + \left(2 + \frac{1}{\delta}\right)^2 \right)^2 \frac{1}{\delta_2} \right) \int_a^b (p {x'_\gamma}^2 dt + (q + \delta r) x_\gamma^2) dt\,.$$

We have

$$-(px'_\gamma)' + (q + \delta r) x_\gamma = (\gamma + \delta) r x_\gamma\,,$$

and so, by Lemma 6.2.2.4 a),

$$\gamma + \delta = (\gamma + \delta) \int_a^b r x_\gamma^2 dt = \int_a^b (p {x'_\gamma}^2 dt + (q + \delta r) x_\gamma^2) dt\,.$$

Thus

$$\|x_\gamma\|_\infty^2 \leq \left(\frac{\delta}{\delta_1} + \left(\frac{1}{b-a} + \left(2 + \frac{1}{\delta}\right)^2 \right)^2 \frac{1}{\delta_2} \right) (\gamma + \delta)\,.$$

It follows that

$$\left(\frac{\|x_\gamma\|_\infty}{\sqrt{|\gamma|}} \right)_{\gamma \in A \setminus \{0\}} \in \ell^\infty(A \setminus \{0\})\,.$$

By b),

$$\left(\sqrt{|\gamma|} \langle x | x_\gamma \rangle \|x_\gamma\|_\infty \right)_{\gamma \in A} \in \ell^2(A)\,,$$

$$(\langle x | x_\gamma \rangle \|x_\gamma\|_\infty)_{\gamma \in A} \in \ell^1(A)\,. \qquad \blacksquare$$

6.2.3 Selfadjoint Linear Differential Equations on $\mathbb{T}$

In this subsection we use the following notational conventions.

a) Given $j \in \mathbb{N} \cup \{0\}$, write $\mathcal{C}^{(j)}$ for the set of j-times continuously differentiable functions on $\mathbf{T}$.

b) n is an even natural number.

c) $(q_j)_{j \in \{0\} \cup \mathbb{N}_{\frac{n}{2}-1}}$ is a family in $\mathcal{C}^{(0)}$ such that $q_j \in \mathcal{C}^{(j)}$ for every $j \in \mathbb{N}_{\frac{n}{2}-1}$.

d) $u : \mathcal{C}^{(n)} \longrightarrow \mathcal{C}^{(0)}, \quad x \longmapsto (-1)^{\frac{n}{2}} x^{(n)} + \sum_{j=0}^{\frac{n}{2}-1} \left(q_j x^{(j)}\right)^{(j)}$.

e) λ is the Haar measure on $\mathbf{T}$ with $\lambda(\mathbf{T}) = 1$.

f) Given $\gamma \in \mathbb{K}$,

$$\mathcal{F}_\gamma := \{x \in \mathcal{C}^{(n)} | ux = \gamma x\}.$$

g) $A := \{\gamma \in \mathbb{K} | \mathcal{F}_\gamma \neq \{0\}\}$.

By Theorem 3.2.2.15 b $\Rightarrow$ c, u is selfadjoint. If u is injective, then we write g for the Green function associated to u (Theorem 3.2.1.3, Proposition 3.2.2.10 g)) and define

$$v : L^2(\lambda) \longrightarrow L^2(\lambda), \quad x \longmapsto \int g(\cdot, t) x(t) d\lambda(t).$$

Proposition 6.2.3.1 (5) *A is a subset of $\mathbb{R}$ which is bounded below.*

We may assume that $\mathbb{K} = \mathbb{C}$. By Propositions 3.2.2.10 f) and 6.2.1.1, we may assume u to be injective. By Proposition 6.2.1.2 e), $A \subset \mathbb{R}$ and by Example 3.2.2.12, A is bounded below. ■

Proposition 6.2.3.2 (5) *If u is injective, then:*

a) *v and g are selfadjoint and v is injective.*

b) $\begin{cases} v(ux) = -x \text{ for every } x \in \mathcal{C}^{(n)} \\ vx \in \mathcal{C}^{(n)} \text{ and } uvx = -x \text{ for every } x \in \mathcal{C}^{(0)} \end{cases}$.

c) $\mathcal{F}_\gamma = \{x \in L^2(\lambda) | vx = -\frac{1}{\gamma}x\}$ *for every* $\gamma \in \mathbb{K}\backslash\{0\}$.

d) $\sigma_p(v) = \{-\frac{1}{\gamma} | \gamma \in A\}$.

By Theorem 3.2.2.15 b $\Rightarrow$ d, g is selfadjoint. The assertions now follow from Proposition 6.2.1.2 and Proposition 3.2.2.10. ■

Theorem 6.2.3.3 (5) *If u is injective, then v is an injective, selfadjoint, nuclear operator and there is an orthonormal basis B of $L^2(\lambda)$ and an $f \in \mathbb{K}^B$ such that*

$$vx = f(x)x$$

for every $x \in B$. If B and f have the above properties, then:

a) $f \in \ell^1(B)$ *and*

$$v = \sum_{x \in B} f(x)\langle \cdot | x\rangle x \quad (\text{ in } \mathcal{L}^1(L^2(\lambda)))\,.$$

b) $\|v\|_1 = \|f\|_1$.

c) $\sigma_p(v) = f(B)$.

d) f *is real and vanishes nowhere.*

e) *If* $x \in B$, *then* $x \in \mathcal{C}^{(n)}$ *and* $ux = -\frac{1}{f(x)}x$.

f) $g = \sum_{x \in B} f(x)x \otimes \overline{x} \quad (\text{in } \mathcal{C}(\mathbf{T} \times \mathbf{T}))$.

g) *If* $y \in L^2(\lambda)$, *then* $vy = \sum_{x \in B} f(x)\langle y | x\rangle x \quad (\text{in } \mathcal{C})$.

h) *If* $y \in \mathcal{C}^{(n)}$, *then*

$$\left(\frac{\langle y|x\rangle}{f(x)}\right)_{x \in B} \in \ell^2(B)\,, \quad \left(\frac{\langle y|x\rangle}{\sqrt{|f(x)|}}\right)_{x \in B} \in \ell^1(B)\,,$$

$$\sum_{x \in B} \frac{\langle y|x\rangle}{f(x)}x \in \mathcal{C}^{(0)} \quad (\text{where the sum is considered in } L^2(\lambda))\,,$$

$$y = \sum_{x \in B} \langle y|x\rangle x \quad (\text{in } \mathcal{C})\,,$$

$$uy = -\sum_{x \in B} \frac{\langle y|x\rangle}{f(x)}x \quad (\text{in } L^2(r \cdot \lambda))\,.$$

i) *If* $\alpha \in \mathbb{K}\backslash\{0\}$, *then*

$$\operatorname{Im}(\alpha 1 - v) = \overset{-1}{f}(\alpha)^{\perp}$$

and

$$\frac{1}{\alpha}y + \frac{1}{\alpha}\sum_{y \in A\backslash \overset{-1}{f}(\alpha)} \frac{f(x)\langle y|x\rangle}{\alpha - f(x)}x \in \overbrace{\alpha 1 - v}^{-1}(y)$$

for every $y \in \operatorname{Im}(\alpha 1 - v)$, *where the above series converges in* $\mathcal{C}$.

By Proposition 6.2.3.1, A is lower bounded and by Proposition 6.2.3.2 a), v is injective and selfadjoint. The assertions follow from the Theorems 6.2.1.3 and 6.2.1.6 and from Proposition 3.2.2.10 e). ■

6.2.4 Associated Parabolic and Hyperbolic Evolution Equations

We continue with the notational conventions from Subsection 6.2.1 in this subsection, replacing the assumption "u is injective" with the assumption "A is a lower bounded subset of $\mathbb{R}$". By Theorem 6.2.2.5 a) and Proposition 6.2.3.1, A is bounded below as a subset of $\mathbb{R}$ whenever the hypotheses of Section 6.2.2 or of Section 6.2.3 are fulfilled. Let B denote an orthonormal basis of $L^2(r\cdot\lambda)$ and f a function $B \to \mathbb{K}$ for which $x \in \mathcal{D}$ and

$$ux = f(x)rx$$

for every $x \in B$ (by Theorem 6.2.1.3 and Propositions 6.2.1.1 and 6.2.1.4 a), there always are such B and f).

Proposition 6.2.4.1 (5) *Let S, T be topological spaces, $(x_\iota)_{\iota\in I}$ a summable family in $\mathcal{C}(S)$ and $(y_\iota)_{\iota\in I}$ a bounded family in $\mathcal{C}(T)$. Then $(x_\iota \otimes y_\iota)_{\iota\in I}$ is summable in $\mathcal{C}(S \times T)$.*

Take $\varepsilon > 0$. Put

$$\delta := \sup_{\iota\in I} \|y_\iota\| .$$

There is a $J \in \mathfrak{P}_f(I)$ such that

$$\Big\| \sum_{\iota\in K} x_\iota \Big\| < \frac{\varepsilon}{4(1+\delta)}$$

for every $K \in \mathfrak{P}_f(I\backslash J)$ (Proposition 1.1.6.6). By Proposition 1.1.1.5,

$$\Big| \sum_{\iota\in K} x_\iota(s) y_\iota(t) \Big| < \varepsilon$$

for all $K \in \mathfrak{P}_{f(I\backslash J)}$ and $(s,t) \in S \times T$. Hence

$$\Big\| \sum_{\iota\in K} x_\iota \otimes y_\iota \Big\| < \varepsilon$$

for each $K \in \mathfrak{P}_f(J\backslash J)$. By Proposition 1.1.6.6, $(x_\iota \otimes y_\iota)_{\iota\in I}$ is summable in $\mathcal{C}(S \times T)$. ■

Corollary 6.2.4.2 *Let T be a topological space, $(x_\iota)_{\iota\in I}$ a summable family in $\mathcal{C}(T)$, and $(y_\iota)_{\iota\in I}$ a bounded family in $\mathcal{C}(T)$. Then $(x_\iota y_\iota)_{\iota\in I}$ is a summable family in $\mathcal{C}(T)$.*

By Proposition 6.2.4.1, the family $(x_\iota \otimes y_\iota)_{\iota\in I}$ is summable in $\mathcal{C}(T\times T)$. Hence the family of restrictions of the functions to the diagonal of $T\times T$ is summable. Thus $(x_\iota y_\iota)_{\iota\in I}$ is summable in $\mathcal{C}(T)$. ■

Theorem 6.2.4.3 (5) *Take $c\in]0,\infty]$ and $p,q\in\mathcal{C}([0,c[)$ such that p vanishes nowhere. Define*

$$P:[0,c[\longrightarrow \mathbb{K}\,,\quad s\longmapsto \int_0^s \frac{d\sigma}{p(\sigma)}\,,$$

$$Q:[0,c[\longrightarrow \mathbb{K}\,,\quad s\longmapsto \int_0^s \frac{q(\sigma)d\sigma}{p(\sigma)}$$

and assume that

$$\sup_{s\in[s_0,c[} \operatorname{re} P(s)<0$$

for every $s_0\in]0,c[$ and that

$$\inf_{s\in[0,c[} \operatorname{re} Q(s)>-\infty\,.$$

Take $h\in\mathcal{D}$. For each $x\in B$, define

$$h_x:[a,b]\times[0,c[\longrightarrow \mathbb{K}\,,\quad (t,s)\longmapsto \langle h|x\rangle x(t)e^{f(x)P(s)-Q(s)}\,.$$

Put

$$B_0:=\{x\in B|f(x)\in\mathbb{R}_+\}\,.$$

a) *$\{x\in B|f(x)<\alpha\}$ is finite for every $\alpha\in\mathbb{R}$. In particular, $B\backslash B_0$ is finite.*

b) *$(h_x)_{x\in B_0}$ is a summable family in $\mathcal{C}([a,b]\times[0,c[)$. We set (by a))*

$$y:[a,b]\times[0,c[\longrightarrow \mathbb{K}\,,\quad (t,s)\longmapsto \sum_{x\in B} h_x(t,s)\,.$$

c) *Given $s_0\in]0,c[$, the family $(f(x)h_x|([a,b]\times[s_o,c[))_{x\in B_0}$ is summable in $\mathcal{C}([a,b]\times[s_0,c[)$. Define*

$$z:[a,b]\times]0,c[\longrightarrow \mathbb{K}\,,\quad (t,s)\longmapsto \sum_{x\in B} f(x)h_x(t,s)\,.$$

d) *y and z are continuous.*

e) *If* $A \subset \mathbb{R}_+$, *then* y *in bounded and* $z|([a,b]\times[s_0,c])$ *is bounded for every* $s_0 \in]0,c[$.

f) $y(\cdot,0) = h$.

g) $y(t,\cdot)$ *is differentiable on* $]0,c[$ *for every* $t \in [a,b]$ *and*

$$p\frac{\partial y}{\partial s} + qy = z$$

on $[a,b]\times]0,c[$.

h) $y(\cdot,s) \in \mathcal{D}$ *for all* $s \in [0,c[$ *and*

$$r(p\frac{\partial y}{\partial s} + qy) = uy$$

on $[a,b]\times]0,c[$.

a) By hypothesis, there is a $\delta \in \mathbb{R}$ such that $A+\delta \subset]0,\infty[$. By Proposition 6.2.1.1,

$$A_\delta = A + \delta \subset]0,\infty[\,.$$

Hence u_δ is injective.

Take $x \in B$ with $f(x) < \alpha$. Then

$$f(x) + \delta < \alpha + \delta\,,$$

$$x \in \overbrace{f+\delta}^{-1}(]0,\alpha+\delta[)\,.$$

Hence

$$\{x \in B | f(x) < \alpha\} \subset \overbrace{f+\delta}^{-1}(]0,\alpha+\delta[)\,,$$

so by Proposition 6.2.1.5 b), $\{x \in B|f(x) < \alpha\}$ is finite.

b & c. By Proposition 6.2.1.5 $c_1 \Rightarrow c_4$, $(\langle h|x\rangle x)_{x\in B}$ is summable in $\mathcal{C}([a,b])$. Take $s_0 \in]0,c[$. Since $\left(e^{f(x)P-Q}\right)_{x\in B_0}$ (resp. $\left(f(x)e^{f(x)P-Q}\right)_{x\in B_0}$) is a bounded family in $\mathcal{C}([0,c[)$ (resp. a bounded family in $\mathcal{C}([s_0,c[)$), it follows from Proposition 6.2.4.1, that $(h_x)_{x\in B_0}$ is summable in $\mathcal{C}([a,b]\times[0,c[)$ (resp. $(f(x)h_x)_{x\in B_0}$ is summable in $\mathcal{C}([a,b]\times[s_0,\infty[)$.

d & e follows from a),b), and c).

f) follows from Proposition 6.2.1.5 $c_1 \Rightarrow c_4$.

g) Given $x \in B$,

$$p\frac{\partial h_x}{\partial s} + qh_x = ph_x\Big(\frac{f(x)}{p} - \frac{q}{p}\Big) + qh_x = f(x)h_x\,.$$

By b) and c), it follows that $y(t,\cdot)$ is differentiable on $]0,c[$ for every $t \in [a,b]$ and that

$$p\frac{\partial y}{\partial s} + qy = z$$

on $[a,b]\times]0,c[$.

h) Let δ be a real number such that u_δ is injective. Then, given $x \in B$,

$$v_\delta((f(x)+\delta)x) = v_\delta\Big(\frac{1}{r}(ux + \delta rx)\Big) = v_\delta\Big(\frac{1}{r}u_\delta x\Big) = -x$$

(Proposition 6.2.1.2 b)). Using a),b), and c), we deduce that

$$v_\delta(z(\cdot,s) + \delta y(\cdot,s)) = v_\delta\Big(\sum_{x\in B}(f(x)+\delta)h_x(\cdot,s)\Big) =$$

$$= \sum_{x\in B}\langle h|x\rangle e^{f(x)P(s)-Q(s)}v_\delta((f(x)+\delta)x) =$$

$$= -\sum_{x\in B}\langle h|x\rangle e^{f(x)P(s)-Q(s)}x = -\sum_{x\in B}h_x(\cdot,s) = -y(\cdot,s)\,.$$

By d) and Proposition 6.2.1.2 b), $y(\cdot,s) \in \mathcal{D}$ and

$$u_\delta y(\cdot,s) = rz(\cdot,s) + \delta_r y(\cdot,s)\,.$$

Hence by g),

$$uy = rz = r\Big(\frac{\partial y}{\partial s} + qy\Big)$$

on $[a,b]\times]0,\infty[$. ■

Theorem 6.2.4.4 (5) *Take $c \in]0,\infty[$, $p,q \in \mathcal{C}([0,c])$, $f_1, f_2 \in \mathcal{D}$, and $\beta,\gamma \in \mathbb{K}^4$. Let C denote the set of those $\alpha \in \mathbb{R}$ for which there are uniquely twice differentiable functions y_α and z_α on $[0,c]$ such that*

$$\begin{cases} y_\alpha'' + py_\alpha' + (q \pm \alpha)y_\alpha = 0 \\ \beta_1 y_\alpha(0) + \beta_2 y_\alpha'(0) + \beta_3 y_\alpha(c) + \beta_4 y_\alpha'(c) = 1 \\ \gamma_1 y_\alpha(0) + \gamma_2 y_\alpha'(0) + \gamma_3 y_\alpha(c) + \delta_4 y_\alpha'(c) = 0\,, \end{cases}$$

$$\begin{cases} z_\alpha'' + pz_\alpha' + (q \pm \alpha)z_\alpha = 0 \\ \beta_1 z_\alpha(0) + \beta_2 z_\alpha'(0) + \beta_3 z_\alpha(c) + \beta_4 z_\alpha'(c) = 0 \\ \gamma_1 z_\alpha(0) + \gamma_2 z_\alpha'(0) + \gamma_3 z_\alpha'(c) + \delta_4 z_\alpha'(c) = 1\,. \end{cases}$$

We assume that $A \subset C$, *that the coefficients of* u *are* $\mathcal{C}^\infty$*-functions, and that there are* α_0, $\delta \in \mathbb{R}_+$ *with*

$$\|y_\alpha\| \leq \delta\,, \quad \|z_\alpha\| \leq \delta\,, \quad \|y'_\alpha\| \leq \delta\sqrt{\alpha}\,, \quad \|z'_\alpha\| \leq \delta\sqrt{\alpha}$$

whenever $\alpha \in C$ *and* $\alpha \geq \alpha_0$.

a) *There is a unique* $y \in \mathcal{C}([a,b] \times [0,c])$ *such that*

$$z : [0,c] \longrightarrow L^2(r\cdot\lambda)\,, \quad s \longmapsto y(\cdot,s)$$

is twice differentiable, $(f(x)\langle z(s)|x\rangle)_{x\in B} \in \ell^2(B)$ *for every* $s \in [0,c]$,

$$\begin{cases} \beta_1 z(0) + \beta_2 z'(0) + \beta_3 z(c) + \beta_4 z'(c) = f_1 \\ \gamma_1 z(0) + \gamma_2 z'(0) + \gamma_3 z(c) + \gamma_4 z'(c) = f_2\,, \end{cases}$$

and

$$r(z'' + pz' + qz) \pm uz = 0$$

on $]0,c[$, *where* uz *is understood in the sense of* L^2*-distributions.*

Given $x \in B$, *define*

$$h_x : [a,b] \times [0,c] \longrightarrow \mathbb{K}\,, \quad (t,s) \longmapsto (\langle f_1|x\rangle y_{f(x)}(s) + \langle f_2|x\rangle z_{f(x)}(s))x(t)\,.$$

Then $(h_x)_{x\in B}$ *is summable in* $\mathcal{C}([a,b] \times [0,c])$ *and*

$$y = \sum_{x\in B} h_x\,.$$

b) *If* p *and* q *are* $\mathcal{C}^\infty$*- functions and* $\mathbb{K} = \mathbb{R}$, *then*

$$r\left(\frac{\partial^2 y}{\partial s^2} + p\frac{\partial y}{\partial s} + qy\right) \pm uy = 0$$

on $]a,b[\times]0,c[$ *in the sense of distributions.*

c) *If the families* $(y'_\alpha)_{\alpha\in C,\alpha\geq\alpha_0}$, $(z'_\alpha)_{\alpha\in c,\alpha\geq\alpha_0}$ *are bounded in* $\mathcal{C}([0,c])$, *then the family* $\left(\frac{\partial h_x}{\partial s}\right)_{x\in B}$ *is summable in* $\mathcal{C}([a,b] \times [0,c])$, $y(t,\cdot)$ *is differentiable for every* $t \in [a,b]$ *and*

$$\frac{\partial y}{\partial s} = \sum_{x\in B} \frac{\partial h_x}{\partial s} \in \mathcal{C}([a,b] \times [0,c])\,.$$

d) *If* uf_1, $uf_2 \in \mathcal{D}$, *then the families* $\left(\frac{\partial h_x}{\partial x}\right)_{x\in B}$, $\left(\frac{\partial^2 h_x}{\partial s^2}\right)_{x\in B}$ *are summable in* $\mathcal{C}([a,b]\times[0,c])$, $y(\cdot,s)\in\mathcal{D}$ *for every* $s\in[0,c]$, $y(t,\cdot)$ *is twice differentiable for every* $t\in[a,b]$ *and*

$$\frac{\partial y}{\partial s}=\sum_{x\in B}\frac{\partial h_x}{\partial s}\in\mathcal{C}([a,b]\times[0,c])\,,$$

$$\frac{\partial^2 y}{\partial s^2}=\sum_{x\in B}\frac{\partial^2 h_x}{\partial s^2}\in\mathcal{C}([a,b]\times[0,c])\,.$$

a) Step 1 $\left\{\begin{array}{l}\text{The families } (y_{f(x)})_{x\in B}\,,\ (z_{f(x)})_{x\in B}\,,\ \left(\dfrac{y'_{f(x)}}{\sqrt{|f(x)|}}\right)_{x\in B,f(x)\neq 0}\,,\\ \left(\dfrac{z'_{f(x)}}{\sqrt{|f(x)|}}\right)_{x\in B,f(x)\neq 0} \text{ are bounded in } \mathcal{C}([0,c])\end{array}\right.$

By Proposition 6.2.1.5 b), $\overset{-1}{f}(]-\infty,\alpha_0])$ is finite and so the assertion follows directly from the hypotheses.

Step 2 $(h_x)_{x\in B}$ is a summable family in $\mathcal{C}([a,b]\times[0,c])$

By Proposition 6.2.1.5 $c_1 \Rightarrow c_4$, the families $(\langle f_1|x\rangle x)_{x\in B}$, $(\langle f_2|x\rangle x)_{x\in B}$ are summable in $\mathcal{C}([a,b])$. By Step 1 and Proposition 6.2.4.1, $(h_x)_{x\in B}$ is summable in $\mathcal{C}([a,b]\times[0,c])$.

We put

$$y:=\sum_{x\in B}h_x\,,$$

$$z:[0,c]\longrightarrow L^2(r\cdot\lambda)\,,\quad s\longmapsto y(\cdot,s)\,.$$

Step 3 $\left\{\begin{array}{l}\text{If } s\in[0,c]\,,\ \text{then } (\sqrt{|f(x)|}\,h'_x(\cdot,s))_{x\in B}\\ \text{and } (f(x)h_x(\cdot,s))_{x\in B} \text{ are summable in } L^2(r\cdot\lambda)\end{array}\right.$

By Proposition 6.2.1.5 $c_1 \Rightarrow c_2$,

$$(f(x)\langle f_1|x\rangle)_{x\in B}\,,\quad (f(x)\langle f_2|x\rangle)_{x\in B}\in\ell^2(B)$$

and the assertion follows from Step 1 and Corollary 5.2.2.5.

Define

$$\widetilde{y} : [0,c] \longrightarrow L^2(r\cdot\lambda)\,, \quad s \longmapsto \sum_{x\in B} f(x) h_x(\cdot, s)\,.$$

Step 4 $\begin{cases} (f(x)\|\langle z|x\rangle\|)_{x\in B} \in \ell^2(B)\,,\ z \text{ is twice differentiable,} \\ \text{and} \quad z'' + pz' + qz = \mp\widetilde{y} \end{cases}$

By Proposition 6.2.1.5 $c_1 \Rightarrow c_2$,

$$(f(x)\langle f_1|x\rangle)_{x\in B}\,, \quad (f(x)\langle f_2|x\rangle)_{x\in B} \in \ell^2(B)\,.$$

Since

$$|\langle z(s)|x\rangle| \le |\langle f_1|x\rangle|\delta + |\langle f_2|x\rangle|\delta$$

for every $s \in [0,c]$, and for every $x \in B$ with $f(x) \ge \alpha_0$,

$$(f(x)|\langle z|x\rangle|)_{x\in B} \in \ell^2(B)\,.$$

Take $x \in B$. Then

$$\frac{d}{ds}\langle z|x\rangle = \langle f_1|x\rangle y'_{f(x)} + \langle f_2|x\rangle z'_{f(x)}\,,$$

$$\frac{d^2}{ds^2}\langle z|x\rangle = \langle f_1|x\rangle y''_{f(x)} + \langle f_2|x\rangle z''_{f(x)}\,,$$

so that if $f(x) \ge \alpha_0$,

$$\|\frac{d}{ds}\langle z|x\rangle\| \le \delta\sqrt{|f(x)|}\,(|\langle f_1|x\rangle| + |\langle f_2|x\rangle|)\,,$$

$$\|\frac{d^2}{ds^2}\langle z|x\rangle\| \le 2\delta(|\langle f_1|x\rangle| + |\langle f_2|x\rangle|)(\|p\|\sqrt{|f(x)|} + \|q\| + |f(x)|)\,.$$

It follows that

$$\left(\left\|\frac{d}{ds}\langle z|x\rangle\right\|\right)_{x\in B}\,, \left(\left\|\frac{d^2}{ds^2}\langle z|x\rangle\right\|\right)_{x\in B} \in \ell^2(B)\,.$$

By Proposition 5.5.1.20, z is twice differentiable and

$$z'' + pz' + qz =$$

$$= \sum_{x\in B}\langle f_1|x\rangle\left(y''_{f(x)} + py'_{f(x)} + qy_{f(x)}\right)x +$$

$$+\sum_{x\in B}\langle f_2|x\rangle\Big(z''_{f(x)}+pz'_{f(x)}+qz_{f(x)}\Big)x=$$

$$=\mp\sum_{x\in B}\langle f_1|x\rangle f(x)y_{f(x)}x\mp\sum_{x\in B}\langle f_2|x\rangle f(x)z_{f(x)}x=$$

$$=\mp\sum_{x\in B}f(x)h_x=\mp\widetilde{y}\,.$$

Step 5 $\begin{cases} r(z''+pz'+qz)\pm uz=0\,, \text{ with } uz \\ \text{in the sense of } L^2\text{–distributions} \end{cases}$

By Proposition 6.2.1.5 e), for every $s\in[0,c]$

$$uz(s)=r\sum_{x\in B}\langle z(s)|x\rangle f(x)x=r\sum_{x\in B}f(x)h_x(\cdot,s)=r\widetilde{y}(s)$$

in the sense of L^2–distributions. By Step 4,

$$uz(s)=\mp r(z''+pz'+qz)\,.$$

Step 6 The Boundary Condition

We have

$$\beta_1z(0)+\beta_2z'(0)+\beta_3z(c)+\beta_4z'(c)=$$

$$=\beta_1\sum_{x\in B}h_x(\cdot,0)+\beta_2\sum_{x\in B}\frac{\partial}{\partial s}h_x(\cdot,0)+\beta_3\sum_{x\in B}h_x(\cdot,c)+\beta_4\sum_{s\in B}\frac{\partial}{\partial s}h_x(\cdot,c)=$$

$$=\sum_{x\in B}\langle f_1|x\rangle\Big(\beta_1y_{f(x)}(0)+\beta_2y'_{f(x)}(0)+\beta_3y_{f(x)}(c)+\beta_4y'_{f(x)}(c)\Big)x+$$

$$+\sum_{x\in B}\langle f_2|x\rangle\Big(\beta_1z_{f(x)}(0)+\beta_2z'_{f(x)}(0)+\beta_3z_{f(x)}(c)+\beta_4z'_{f(x)}(c)\Big)x=$$

$$=\sum_{x\in B}\langle f_1|x\rangle x=f_1\,,$$

$$\gamma_1z(0)+\gamma_2z'(0)+\gamma_3z(c)+\gamma_4z'(c)=f_2\,.$$

Step 7 Uniqueness

Assume that η has the described properties. Define

$$\zeta : [0,c] \longrightarrow L^2(r\cdot\lambda)\,, \quad s \longmapsto \eta(\cdot,s)\,.$$

Take $s \in [0,c]$. Then

$$\zeta(s) = \sum_{x\in B} \langle \zeta(s)|x\rangle x\,.$$

By Proposition 6.2.1.5 e),

$$u\zeta(s) = r\sum_{x\in B} \langle \zeta(s)|x\rangle f(x)x$$

in the sense of distributions. Hence

$$\sum_{x\in B} \langle \zeta(s)|x\rangle f(x)x = \frac{1}{r}u\zeta(s) = \mp(\zeta''(s) + p(s)\zeta'(s) + q(s)\zeta(s))\,.$$

It follows that for every $x \in B$,

$$\langle \zeta''(s) + p(s)\zeta'(s) + q(s)\zeta(s)|x\rangle = \mp f(x)\langle \zeta(s)|x\rangle\,.$$

Thus

$$\frac{d^2}{ds^2}\langle \zeta|x\rangle + p\frac{d}{ds}\langle \zeta|x\rangle + (q \pm f(x))\langle \zeta|x\rangle = 0\,,$$

$$\langle \zeta|x\rangle = \langle z|x\rangle\,.$$

Hence $\zeta = z$ and $\eta = y$.

b) Let w and w^* be the differential operators

$$\psi \longmapsto \frac{\partial^2\psi}{\partial s^2} + p\frac{\partial\psi}{\partial s} + q\psi\,,$$

$$\psi \longmapsto \frac{\partial^2\psi}{\partial s^2} - \frac{\partial}{\partial s}(p\psi) + q\psi$$

on $]0,c[$, respectively. Let φ be a $\mathcal{C}^\infty$–function on $]a,b[\times]0,c[$ with compact carrier and take $x \in B$. Then

$$\int_a^b x(t)u\varphi(t,s)dt = \langle x|\frac{1}{r}u\varphi(\cdot,s)\rangle = \langle \frac{1}{r}ux|\varphi(\cdot,s)\rangle =$$

$$= f(x)\langle x|\varphi(\cdot,s)\rangle = f(x)\int_a^b \varphi(t,s)x(t)r(t)dt$$

for every $s \in]0,c[$ (Proposition 6.2.1.2 f)) and

$$\int_0^c w^*\varphi(t,s)y_{f(x)}(s)ds = \int_0^c \varphi(t,s)wy_{f(x)}(s)ds = \mp f(x)\int_0^c \varphi(t,s)y_{f(x)}(s)ds\,,$$

$$\int_0^c w^*\varphi(t,s)z_{f(z)}(s)ds = \int_0^c \varphi(t,s)wz_{f(x)}(s)ds = \mp f(x)\int_0^c \varphi(t,s)z_{f(x)}(s)ds$$

for every $t \in]a,b[$. It follows that

$$\iint_{]a,b[\times]0,c[} h_x(rw^*\varphi \pm u\varphi)dsdt =$$

$$= \int_a^b \langle f_1|x\rangle x(t)r(t)\left(\int_0^c w^*\varphi(t,s)y_{f(x)}(s)ds\right)dt +$$

$$+ \int_a^b \langle f_2|x\rangle x(t)r(t)\left(\int_0^c w^*\varphi(t,s)z_{f(x)}(s)ds\right)dt \pm$$

$$\pm \int_0^c (\langle f_1|x\rangle y_{f(x)}(s) + \langle f_2|x\rangle z_{f(x)}(s))\left(\int_a^b x(t)u\varphi(t,s)dt\right)ds =$$

$$= \mp f(x)\int_a^b \langle f_1|x\rangle x(t)r(t)\left(\int_0^c \varphi(t,s)y_{f(x)}(s)ds\right)dt \mp$$

$$\mp f(x)\int_a^b \langle f_2|x\rangle x(t)r(t)\left(\int_0^c \varphi(t,s)z_{f(x)}(s)ds\right)dt \pm$$

$$\pm f(x)\int_0^c (\langle f_1|x\rangle y_{f(x)}(s) + \langle f_2|x\rangle z_{f(x)}(s))\left(\int_a^b \varphi(t,s)x(t)r(t)dt\right)ds = 0\,.$$

Hence

$$\iint_{]a,b[\times]0,c[} y(rw^*\varphi \pm u\varphi)dsdt =$$

$$= \sum_{x\in B}\iint_{]a,b[\times]0,c[} h_x(rw^*\varphi \pm u\varphi)dsdt = 0$$

and

$$r\left(\frac{\partial^2 y}{\partial s^2}+p\frac{\partial y}{\partial s}+qy\right)\pm uy=0$$

on $]a,b[\times]0,c[$ in the sense of distributions.

c) By Proposition 6.2.1.5 b), $\overset{-1}{f}(]-\infty,\alpha_0])$ is finite and the families $(\langle f_1|x\rangle x)_{x\in B}$, $(\langle f_2|x\rangle x)_{x\in B}$ are summable in $\mathcal{C}([a,b])$. Hence the families

$$(y'_{f(x)})_{x\in B}\,,\ (z'_{f(x)})_{x\in B}$$

are bounded. By Proposition 6.2.4.1, the family $\left(\frac{\partial h_x}{\partial s}\right)_{x\in B}$ is summable in $\mathcal{C}([a,b]\times[0,c])$. By Proposition 1.1.6.26, $y(t,\cdot)$ is differentiable for every $t\in[a,b]$ and

$$\frac{\partial y}{\partial s}=\sum_{x\in B}\frac{\partial h_x}{\partial s}\in\mathcal{C}([a,b]\times[0,c])\,.$$

d) Take $\delta\in\mathbb{R}$ such that u_δ is injective and take $k\in\{1,2\}$. Then

$$v_\delta x=-\frac{1}{\delta+f(x)}x$$

for every $x\in B$ (Proposition 6.2.1.5 a)) and

$$f_k=-v_\delta(\frac{1}{r}u_\delta f_k)\in v_\delta(\mathcal{D})$$

(Proposition 6.2.1.2 b)). By Theorem 6.2.1.3 j), $((\delta+f(x))\langle f_k|x\rangle x)_{x\in B}$ is summable in $\mathcal{C}$. Hence $(f(x)\langle f_k|x\rangle x)_{x\in B}$ is summable in $\mathcal{C}$ and therefore $(f(x)h_x(\cdot,s))_{x\in B}$ is summable in $\mathcal{C}$ for every $s\in[0,c]$. By Proposition 6.2.1.5 $c_2\Rightarrow c_1$, $y(\cdot,s)\in\mathcal{D}$ for every $s\in[0,c]$.

By Step 1 of a), the families

$$\left(\frac{y'_{f(x)}}{\sqrt{|f(x)|}}\right)_{x\in B,f(x)\neq 0}\,,\quad\left(\frac{z'_{f(x)}}{\sqrt{|f(x)|}}\right)_{x\in B,f(x)\neq 0}\,,$$

$$\left(\frac{y''_{f(x)}}{f(x)}\right)_{x\in B,f(x)\neq 0}\,,\quad\left(\frac{z''_{f(x)}}{f(x)}\right)_{x\in B,f(x)\neq 0}$$

are bounded in $\mathcal{C}([0,c])$. Hence by Proposition 6.2.4.1, the families $\left(\frac{\partial h_x}{\partial s}\right)_{x\in B}$, $\left(\frac{\partial^2 h_x}{\partial s^2}\right)_{x\in B}$ are summable in $\mathcal{C}([a,b]\times[0,c])$. It follows that $y(t,\cdot)$ is twice differentiable for every $t\in[a,b]$ and

$$\frac{\partial y}{\partial s} = \sum_{x \in B} \frac{\partial h_x}{\partial s} \in \mathcal{C}([a,b] \times [0,c])\,,$$

$$\frac{\partial^2 y}{\partial s^2} = \sum_{x \in B} \frac{\partial^2 h_x}{\partial s^2} \in \mathcal{C}([a,b] \times [0,c])\,. \qquad \blacksquare$$

Lemma 6.2.4.5 (5) *Take $n \in \mathbb{N}$ and $c \in]0,\infty[$. Let $(a_{ij})_{i,j \in \mathbb{N}_n}$ be a family in $\mathcal{C}([0,c])$ and $(x_i)_{i \in \mathbb{N}_n}$ a family of differentiable functions on $[0,c]$ such that*

$$x_i' = \sum_{j=1}^{n} a_{ij} x_j$$

for every $i \in \mathbb{N}_n$. Then

$$\|x_i\|_\infty \leq e^{c \sum\limits_{j,k=1}^{n} \|a_{jk}\|_\infty} \sum_{j=1}^{n} |x_j(0)|$$

for every $i \in \mathbb{N}_n$.

Take $\mathbb{K}^n$ with the usual scalar product and define

$$x : [0,c] \longrightarrow \mathbb{K}^n\,, \quad t \longmapsto (x_i(t))_{i \in \mathbb{N}_n}\,,$$

$$a : [0,c] \longrightarrow \mathbb{K}_{n,n}\,, \quad t \longmapsto [a_{ij}(t)]_{i,j \in \mathbb{N}_n}\,.$$

Then

$$\frac{d}{dt}\|x(t)\|^2 = \frac{d}{dt}\langle x(t)|x(t)\rangle = \langle x'(t)|x(t)\rangle + \langle x(t)|x'(t)\rangle =$$

$$= 2\mathrm{re}\,\langle x'(t)|x(t)\rangle = 2\mathrm{re}\,\langle a(t)x(t)|x(t)\rangle \leq$$

$$\leq 2\|a(t)x(t)\|\,\|x(t)\| \leq 2\|a(t)\|\,\|x(t)\|^2\,.$$

It follows that

$$\|x(t)\|^2 \leq e^{2\int_0^t \|a(s)\|ds}\|x(0)\|^2 \leq e^{2c \sum\limits_{j,k=1}^{n} \|a_{jk}\|_\infty} \left(\sum_{j=1}^{n} |x_j(0)|\right)^2$$

and therefore

$$\|x_i\|_\infty \leq e^{c \sum\limits_{j,k=1}^{\infty} \|a_{jk}\|_\infty} \sum_{j=1}^{n} |x_j(0)|$$

for every $i \in \mathbb{N}_n$. $\blacksquare$

Lemma 6.2.4.6 (5) *Take* $c \in]0, \infty[$ *and* $p, q \in \mathcal{C}([0,c])$. *For* $\alpha \in \mathbb{R}_+$, *let* x_α *be the twice differentiable function on* $[0,c]$ *such that*

$$x_\alpha'' + px_\alpha' + (q + \alpha^2)x_\alpha = 0$$

and

$$x_\alpha(0) = 1\,, \quad x_\alpha'(0) = 0 \qquad (x_\alpha(0) = 0\,, \quad x_\alpha'(0) = 1)\,.$$

Then there is a $\beta \in \mathbb{R}_+$ *such that*

$$\|x_\alpha\| \leq \beta\,, \quad \|x_\alpha'\| \leq \alpha\beta\,,$$

$$\|x_\alpha''\| \leq (\alpha^2 + \|p\|\alpha + \|q\|)\beta$$

for every $\alpha \geq 1$.

Let $\alpha \in [1, \infty[$ be fixed. Let y, z be the solution on $[0,c]$ of the system of differential equations

$$\begin{cases} y' = (-p\sin^2 \alpha t + \frac{q}{\alpha}\sin\alpha t\cos\alpha t)y + (p\sin\alpha t\cos\alpha t + \frac{q}{\alpha}\sin^2\alpha t)z \\ z' = (p\sin\alpha t\cos\alpha t - \frac{q}{\alpha}\cos^2\alpha t)y - (p\cos^2\alpha t + \frac{q}{\alpha}\sin\alpha t\cos\alpha t)z \end{cases} \qquad (*)$$

satisfying the initial conditions

$$y(0) = 1\,, \quad z(0) = 0 \qquad (y(0) = 0\,, \quad z(0) = \frac{1}{\alpha})\,.$$

(This system was obtained by the method known as "variation of constants".) We put

$$x = y\cos\alpha t + z\sin\alpha t\,.$$

Then

$$y'\cos\alpha t + z'\sin\alpha t = 0\,,$$

so that

$$x' = -\alpha y\sin\alpha t + \alpha z\cos\alpha t\,.$$

Furthermore,

$$-y'\sin\alpha t + z'\cos\alpha t = p(y\sin\alpha t - z\cos\alpha t) - \frac{q}{\alpha}(y\cos\alpha t + z\sin\alpha t) =$$

$$= -\frac{1}{\alpha}(px' + qx)\,,$$

so that

$$x'' = \alpha(-y' \sin \alpha t + z' \cos \alpha t) - \alpha^2(y \cos \alpha t + z \sin \alpha t) =$$

$$= -px' - qx - \alpha^2 x\,,$$

$$x'' + px' + (q + \alpha^2)x = 0$$

and

$$x(0) = 1\,, \quad x'(0) = 0 \qquad (x(0) = 0\,, \quad x'(0) = 1)\,.$$

Hence $x_\alpha = x$ and therefore

$$x_\alpha = y \cos\, \alpha t + z \sin \alpha t\,,$$

$$x'_\alpha = -\alpha y \sin \alpha t + \alpha z \cos \alpha t\,.$$

By Lemma 6.2.4.5, there is an upper bound for $\|y\|$ and $\|z\|$ independent of $\alpha \geq 1$. The desired inequalities follow. ■

Theorem 6.2.4.7 (5) *Take $c \in]0, \infty[$. Let p, q be continuous functions on $[0, c[$ and take $f_1, f_2 \in \mathcal{D}$. For $\alpha \in \mathbb{R}$, let y_α and z_α be the twice differentiable functions on $[0, c[$ satisfying*

$$y''_\alpha + py'_\alpha + (q + \alpha)y_\alpha = 0\,, \quad y_\alpha(0) = 1\,, \quad y'_\alpha(0) = 0\,,$$

$$z''_\alpha + pz'_\alpha + (q + \alpha)z_\alpha = 0\,, \quad z_\alpha(0) = 0\,, \quad z'_\alpha(0) = 1\,.$$

If the coefficients of u are $\mathcal{C}^\infty$-functions, then:

a) *There is a unique continuous function y on $[a, b] \times [0, c[$ such that*

$$z : [0, c[\longrightarrow L^2(r \cdot \lambda)\,, \quad s \longmapsto y(\cdot, s)$$

is twice differentiable, $\langle f(x)\langle z(s)|x\rangle\rangle_{x \in B} \in \ell^2(B)$ for every $s \in [0, c[$,

$$z(0) = f_1\,, \quad z'(0) = f_2$$

and

$$r(z'' + pz' + qz) + uz = 0$$

on $[0, c[$, *where* uz *is understood in the sense of* L^2*–distributions.*

Given $x \in B$, *define*

$$h_x : [a, b] \times [0, c[\longrightarrow \mathbb{K}\,,\ (t, s) \longmapsto \Big(\langle f_1|x\rangle y_{f(x)}(s) + \langle f_2|x\rangle z_{f(z)}(s)\Big)x(t)\,.$$

Then $(h_x)_{x\in B}$ *is summable in* $\mathcal{C}([a, b] \times [0, d])$ *for every* $d \in]0, c[$ *and*

$$y(t, s) = \sum_{x\in B} h_x(t, s)$$

for all $(t, s) \in [a, b] \times [0, c[$.

b) *If* uf_1, $uf_2 \in \mathcal{D}$, *then for each* $d \in]0, c[$, *the familes* $\left(\frac{\partial h_x}{\partial s}\right)_{x\in B}$, $\left(\frac{\partial^2 h_x}{\partial s^2}\right)_{x\in B}$ *are summable in* $\mathcal{C}([a, b] \times [0, d])$, $y(\cdot, s) \in \mathcal{D}$ *for every* $s \in [0, c[$, $y(t, \cdot)$ *is twice differentiable for every* $t \in [a, b]$ *and*

$$\frac{\partial y}{\partial s} = \sum_{x\in B} \frac{\partial h_x}{\partial s}\,, \quad \frac{\partial^2 y}{\partial s^2} = \sum_{x\in B} \frac{\partial^2 h_x}{\partial s^2}\,.$$

By Lemma 6.2.4.6, if $d \in]0, c[$, then there is a $\delta \in \mathbb{R}_+$ such that

$$\|y_\alpha\| \le \delta\,, \quad \|z_\alpha\| \le \delta\,, \quad \|y'_\alpha\| \le \delta\sqrt{\alpha}\,, \quad \|z'_\alpha\| \le \delta\sqrt{\alpha}$$

for every $\alpha \ge 1$, where the norms are cconsidered on $[0, d]$. The assertion now follows from Theorem 6.2.4.4 a),d). ■

Lemma 6.2.4.8 (5) *Take* $c \in]0, \infty[$ *and* $\beta, \gamma \in [0, \frac{\pi}{2}]$. *Let* q *be a continuous, positive real function on* $[0, c]$ *and* x *a twice differentiable real function on* $[0, c]$ *such that*

$$x'' = qx\,,$$

$$x(0)\sin\beta - x'(0)\cos\beta = 0\,,$$

$$x(c)\sin\gamma + x'(c)\cos\gamma = 1\,.$$

a) x *and* x' *are increasing and positive.*

b) *If* $\beta \neq \frac{\pi}{2}$, *then* $x(0) > 0$.

c) *If* $\beta = \frac{\pi}{2}$, *then* $x'(0) > 0$.

Case 1 $\beta \neq \frac{\pi}{2}$

$x(0) < 0$ implies that $x'(0) \leq 0$, so that x and x' are decreasing. Then

$$x(c) \leq x(0) < 0\,, \quad x'(c) \leq x'(0) \leq 0\,,$$

so that

$$1 = x(c)\sin\gamma + x'(c)\cos\gamma \leq 0\,,$$

which is a contradiction.

$x(0) = 0$ implies that $x'(0) = 0$, and so x vanishes identically and we again obtain a contradiction.

Hence $x(0) > 0$. It follows that $x'(0) \geq 0$ and that x and x' are increasing.

Case 2 $\beta = \frac{\pi}{2}$

We have $x(0) = 0$. $x'(0) < 0$ implies that x and x' are decreasing, so that

$$x(c) \leq x(0) = 0\,, \quad x'(c) \leq x'(0) < 0$$

and we get the contradiction

$$1 = x(c)\sin\gamma + x'(c)\cos\gamma \leq 0\,.$$

If $x'(0) = 0$ then x vanishes identically, which also leads to a contradiction. Hence $x'(0) > 0$ and x and x' are increasing. ■

Lemma 6.2.4.9 (5) *Take* $c \in]0, \infty[$, $\varepsilon > 0$, $\beta \in [0, \frac{\pi}{2}]$, *and* $\gamma \in [0, \frac{\pi}{2}[$. *Then there is a* $\delta > 0$ *such that for each continuous real function* q *on* $[0, c]$ *and for each twice differentiable real function* x *on* $[0, c]$ *such that*

$$q \geq \varepsilon\,, \quad x'' = qx\,,$$

$$\begin{cases} x(0)\sin\beta - x'(0)\cos\beta = 0 \\ x(c)\sin\gamma + x'(c)\cos\gamma = 1\,, \end{cases}$$

we have

$$\|x\|_\infty \leq \delta\,, \quad \|x'\|_\infty \leq \delta\,.$$

By Lemma 6.2.4.8 a), x and x' are increasing and positive, so that

$$\|x\|_\infty = x(c)\,, \quad \|x'\|_\infty = x'(c)\,.$$

Step 1 $\beta = \gamma = 0 \Rightarrow \|x\|_\infty \leq c + \frac{1}{\varepsilon}\,,\ \|x'\|_\infty = 1$

We have

$$x'(0) = 0\,, \quad x'(c) = 1\,,$$

so that

$$\|x'\|_\infty = 1$$

and

$$1 = x'(c) - x'(0) = \int_0^c x''(t)dt = \int_0^c q(t)x(t)dt \geq \varepsilon x(0)\,,$$

$$x(0) \leq \frac{1}{\varepsilon}\,,$$

$$\|x\|_\infty = x(c) = x(0) + \int_0^c x'(t)dt \leq \frac{1}{\varepsilon} + c\,.$$

Step 2 $\beta \in]0, \frac{\pi}{2}[\,,\ \gamma = 0 \Rightarrow \|x\|_\infty \leq c + \operatorname{ctg}\beta\,,\ \|x'\|_\infty = 1$

We have (Lemma 6.2.4.8 a),b)) that

$$0 < x(0) \leq x(c)\,,$$

$$0 \leq x'(0) \leq x'(c) = 1\,.$$

Hence

$$\|x'\|_\infty = 1\,,$$

$$x(0) = x'(0)\operatorname{ctg}\beta \leq \operatorname{ctg}\beta\,.$$

It follows that

$$\|x\|_\infty = x(c) = x(0) + \int_0^c x'(t)dt \leq \operatorname{ctg}\beta + c\,.$$

Step 3 $\beta \neq \frac{\pi}{2}$, $\gamma \neq 0 \Rightarrow \|x\|_\infty \leq \frac{1}{\sin\gamma}$, $\|x'\|_\infty \leq \frac{1}{\cos\gamma}$

We have (Lemma 6.2.4.8 a),b))

$$x'(c) \geq x'(0) \geq 0\,,$$

$$0 < x(0) \leq x(c) \leq \frac{1}{\sin\gamma}\,,$$

$$x'(c) \leq \frac{1}{\cos\gamma}\,.$$

Hence

$$\|x\|_\infty \leq \frac{1}{\sin\gamma}\,, \quad \|x'\|_\infty \leq \frac{1}{\cos\gamma}\,.$$

Step 4 $\beta = \frac{\pi}{2} \Rightarrow \|x\|_\infty \leq \frac{c}{\cos\gamma}$, $\|x'\|_\infty \leq \frac{1}{\cos\gamma}$

We have

$$\|x'\|_\infty = x'(c) \leq \frac{1}{\cos\gamma}\,,$$

$$\|x\|_\infty = x(c) = \int_0^c x'(t)dt \leq cx'(c) \leq \frac{c}{\cos\gamma}\,. \qquad \blacksquare$$

Proposition 6.2.4.10 (5) *Take $\beta, \gamma \in [0, \frac{\pi}{2}]$. Let $c \in]0, \infty[$ and q a continuous real function on $[0, c]$. Given $\alpha > \sqrt{\|q\|}$, let x_α be the twice differentiable real function on $[0, c]$ such that*

$$x''_\alpha + (q - \alpha^2)x_\alpha = 0\,,$$

$$\begin{cases} x_\alpha(0)\sin\beta - x'_\alpha(0)\cos\beta = 0 \\ x_\alpha(c)\sin\gamma + x'_\alpha(c)\cos\gamma = 1 \end{cases}$$

(Theorem 6.2.2.5 d), Corollary 3.2.1.7 a $\Rightarrow$ c). Then there is a $\delta \in \mathbb{R}_+$ such that

$$\|x_\alpha\| \leq \delta\,, \quad \textit{and} \quad \|x'_\alpha\| \leq \alpha\delta$$

for every $\alpha > \sqrt{\|q\| + 1}$.

By Lemma 6.2.4.9, we may assume that $\gamma = \frac{\pi}{2}$. By Lemma 6.2.4.8 a), x_α and x'_α are increasing and positive. It follows that

$$\|x_\alpha\| = x_\alpha(c) = 1\,, \quad \|x'_\alpha\| = x'_\alpha(c)\,.$$

Put

$$\alpha_0 := \sqrt{\|q\|}\,,$$

$$\varphi_\alpha : [0,c] \longrightarrow \mathbb{R}\,, \quad t \longmapsto \sin\beta\,\mathrm{sh}(\alpha-\alpha_0)t + (\alpha-\alpha_0)\cos\beta\,\mathrm{ch}(\alpha-\alpha_0)t\,,$$

and

$$y_\alpha := \frac{x_\alpha}{\varphi_\alpha}$$

for every $\alpha > 0$, where

$$y_\alpha(0) = \frac{x'_\alpha(0)}{\alpha-\alpha_0} = \lim_{\substack{t\to 0\\ t>0}} y_\alpha(t)\,,$$

if $\beta = \frac{\pi}{2}$. Then y_α is strictly positive (Lemma 6.2.4.8), continuous, twice differentiable on $[0,c]$ (on $]0,c]$ if $\beta = \frac{\pi}{2}$), and

$$y_\alpha(c) = \frac{1}{\varphi_\alpha(c)}\,.$$

Then on $[0,c]$ (or on $]0,c]$ if $\beta = \frac{\pi}{2}$)

$$y'_\alpha = \frac{x'_\alpha\varphi_\alpha - x_\alpha\varphi'_\alpha}{\varphi_\alpha^2}\,.$$

Hence

$$y'_\alpha(0) = \frac{x'_\alpha(0)(\alpha-\alpha_0)\cos\beta - x_\alpha(0)(\alpha-\alpha_0)\sin\beta}{(\alpha-\alpha_0)^2\cos^2\beta} = 0$$

if $\beta \neq \frac{\pi}{2}$. If $\beta = \frac{\pi}{2}$, then

$$x_\alpha(0) = x''_\alpha(0) = 0$$

and

$$\lim_{\substack{t\to 0\\ t>0}} y'_\alpha(t) = \lim_{\substack{t\to 0\\ t>0}} \frac{x'_\alpha(t)\mathrm{sh}(\alpha-\alpha_0)t - (\alpha-\alpha_0)x_\alpha(t)\mathrm{ch}(\alpha-\alpha_0)t}{\mathrm{sh}^2(\alpha-\alpha_0)t} =$$

$$= \lim_{\substack{t\to 0\\ t>0}} \frac{x''_\alpha(t)\mathrm{sh}(\alpha-\alpha_0)t - (\alpha-\alpha_0)^2 x_\alpha(t)\mathrm{sh}(\alpha-\alpha_0)t}{2(\alpha-\alpha_0)\mathrm{sh}(\alpha-\alpha_0)t\,\mathrm{ch}(\alpha-\alpha_0)t} =$$

$$= \lim_{\substack{t\to 0\\ t>0}} \frac{x''_\alpha(t) - (\alpha-\alpha_0)^2 x_\alpha(t)}{2(\alpha-\alpha_0)\mathrm{ch}(\alpha-\alpha_0)t} = 0$$

by de l'Hôpital's rule.

Now

$$x_\alpha = \varphi_\alpha y_\alpha\,,$$

$$x''_\alpha = \varphi_\alpha y''_\alpha + 2\varphi'_\alpha y'_\alpha + (\alpha-\alpha_0)^2\varphi_\alpha y_\alpha\,,$$

so that

$$0 = x''_\alpha + (q-\alpha^2)x_\alpha = \varphi_\alpha y''_\alpha + 2\varphi'_\alpha y'_\alpha + (q - 2\alpha_0\alpha + \alpha_0^2)\varphi_\alpha y_\alpha\,,$$

$$y''_\alpha + 2\frac{\varphi'_\alpha}{\varphi_\alpha} y'_\alpha + (q - 2\alpha_0\alpha + \alpha_0^2) y_\alpha = 0\,.$$

Since

$$q - 2\alpha_0\alpha + \alpha_0^2 \le \alpha_0^2 - 2\alpha_0\alpha + \alpha_0^2 = 2\alpha_0(\alpha_0 - \alpha) < 0\,,$$

it follows that

$$y''_\alpha(0) = (2\alpha_0\alpha - q(0) - \alpha_0^2) y_\alpha(0) > 0\,,$$

if $\beta \neq \frac{\pi}{2}$ (Lemma 6.2.4.8 b)). Assume that $\beta = \frac{\pi}{2}$. Then

$$\frac{\varphi'_\alpha}{\varphi_\alpha} y'_\alpha = \frac{x'_\alpha\varphi_\alpha\varphi'_\alpha - x_\alpha{\varphi'_\alpha}^2}{\varphi_\alpha^3} =$$

$$= (\alpha-\alpha_0)\mathrm{ch}(\alpha-\alpha_0)t\left(\frac{x'_\alpha \mathrm{sh}(\alpha-\alpha_0)t - (\alpha-\alpha_0)x_\alpha \mathrm{ch}(\alpha-\alpha_0)t}{\mathrm{sh}^3(\alpha-\alpha_0)t}\right),$$

so that

$$\lim_{\substack{t\to 0\\ t>0}} \frac{\varphi'_\alpha(t)}{\varphi_\alpha(t)} y'_\alpha(t) = (\alpha-\alpha_0)\lim_{\substack{t\to 0\\ t>0}} \frac{x''_\alpha(t)\mathrm{sh}(\alpha-\alpha_0)t - (\alpha-\alpha_0)^2 x_\alpha(t)\mathrm{sh}(\alpha-\alpha_0)t}{3(\alpha-\alpha_0)\mathrm{sh}^2(\alpha-\alpha_0)t\,\mathrm{ch}(\alpha-\alpha_0)t} =$$

$$= \frac{1}{3}\lim_{\substack{t\to 0\\ t>0}} \frac{(\alpha^2 - q(t) - (\alpha-\alpha_0)^2)x_\alpha(t)}{\mathrm{sh}(\alpha-\alpha_0)t} =$$

$$= \frac{1}{3(\alpha - \alpha_0)} (2\alpha_0\alpha - q(0) - \alpha_0^2)x'_\alpha(0)\,,$$

by de l'Hôpital's rule. Hence

$$\lim_{\substack{t\to 0\\ t>0}} y''_\alpha(t) = -\frac{2x'_\alpha(0)}{3(\alpha-\alpha_0)} (2\alpha_0\alpha - q(0) - \alpha_0^2) - (q(0) - 2\alpha_0\alpha + \alpha_0^2)\, \frac{x'_\alpha(0)}{\alpha - \alpha_0} =$$

$$= (2\alpha_0\alpha - q(0) - \alpha_0^2)\frac{x'_\alpha(0)}{3(\alpha-\alpha_0)} > 0$$

by Lemma 6.2.4.8 c).

We want to show that y'_α is strictly positive on $]0, c[$. Assume the contrary. By the above,

$$\lim_{\substack{t\to 0\\ t>0}} y'_\alpha(t) = 0\,, \quad \lim_{\substack{t\to 0\\ t>0}} y''_\alpha(t) > 0\,.$$

Hence there is a $t \in]0, c[$ such that $y'_\alpha(t) = 0$ and y'_α is strictly positive on $]0, t[$. Then

$$y''_\alpha(t) = (2\alpha_0\alpha - q - \alpha_0^2)y_\alpha(t) > 0\,,$$

so that y'_α is strictly increasing on a neighbourhood of t, which is a contradiction. Thus y'_α is strictly positive on $]0, c[$ and y_α is increasing.

We have

$$\frac{1}{\varphi_\alpha^2}(\varphi_\alpha^2 y'_\alpha)' = y''_\alpha + 2\frac{\varphi'_\alpha}{\varphi_\alpha} y'_\alpha = (2\alpha_0\alpha - q - \alpha_0^2)y_\alpha\,,$$

so that

$$\varphi_\alpha^2(c)y'_\alpha(c) = \int_0^c \left(\varphi_\alpha^2(t)y'_\alpha(t)\right)' dt = \int_0^c (2\alpha_0\alpha - q(t) - \alpha_0^2)\varphi_\alpha^2(t)y_\alpha(t)dt \leq$$

$$\leq (2\alpha_0\alpha + \alpha_0^2 - \alpha_0^2)\varphi_\alpha^2(c)y_\alpha(c)c\,,$$

$$y'_\alpha(c) \leq \frac{2\alpha_0\alpha c}{\varphi_\alpha(c)}\,,$$

$$\|x'_\alpha\| = x'_\alpha(c) = \varphi_\alpha(c)y'_\alpha(c) + \varphi'_\alpha(c)y_\alpha(c) \leq$$

$$\leq 2\alpha_0\alpha c + (\alpha - \alpha_0)\frac{\sin\beta\,\mathrm{ch}(\alpha-\alpha_0)c + (\alpha-\alpha_0)\cos\beta\,\mathrm{sh}(\alpha-\alpha_0)c}{\sin\beta\,\mathrm{sh}(\alpha-\alpha_0)c + (\alpha-\alpha_0)\cos\beta\,\mathrm{sh}(\alpha-\alpha_0)c}\,.$$

Since

$$\lim_{\alpha\to\infty} \frac{\sin\beta\,\mathrm{ch}(\alpha-\alpha_0)c+(\alpha-\alpha_0)\cos\beta\,\mathrm{sh}(\alpha-\alpha_0)c}{\sin\beta\,\mathrm{sh}(\alpha-\alpha_0)c+(\alpha-\alpha_0)\cos\beta\,\mathrm{ch}(\alpha-\alpha_0)c}=1\,,$$

we may take

$$\delta := 2\alpha_0 c+\sup_{\alpha>\alpha_0+1} \frac{\sin\beta\,\mathrm{ch}(\alpha-\alpha_0)c+(\alpha-\alpha_0)\cos\beta\,\mathrm{sh}(\alpha-\alpha_0)c}{\sin\beta\,\mathrm{sh}(\alpha-\alpha_0)c+(\alpha-\alpha_0)\cos\beta\,\mathrm{ch}(\alpha-\alpha_0)c}\,. \qquad \blacksquare$$

Lemma 6.2.4.11 (5) *Take* $c\in]0,\infty[$ *and* $p,q\in\mathcal{C}([0,c])$. *Let* φ *be a continuously differentiable function on* $[0,c]$. *Take* $\beta,\gamma,\beta',\gamma'\in\mathbb{R}$ *such that*

$$tg\,\beta' = tg\,\beta+\varphi(0)\,,\quad tg\,\gamma' = tg\,\gamma-\varphi(c)\,,$$

where we assign the value $+\infty$ *to the tangent at the poles. Let further* x *be a twice differentiable function on* $[0,c]$ *such that*

$$x''+px'+qx=0\,,$$

$$\begin{cases} x(0)\sin\beta-x'(0)\cos\beta=0\\ x(c)\sin\gamma+x'(c)\cos\gamma=1\,, \end{cases}$$

and define

$$y := xe^{\int_0^t\varphi(s)ds}\,.$$

Then y *is twice differentiable and*

$$y''+(p-2\varphi)y'+(q+\varphi^2-\varphi'-p\varphi)y=0\,,$$

$$\begin{cases} y(0)\sin\beta'-y'(0)\cos\beta'=0\\ y(c)\sin\gamma'+y'(c)\cos\gamma'=\frac{\cos\gamma'}{\cos\gamma}e^{\int_0^c\varphi(s)ds}\,, \end{cases}$$

where

$$\frac{\cos\gamma'}{\cos\gamma}:=1$$

if $tg\,\gamma=\infty$.

We have successively

$$x = ye^{-\int_0^t\varphi(s)ds}\,,$$

$$x' = (y' - \varphi y)e^{-\int_0^t \varphi(s)ds},$$

$$x'' = (y'' - 2\varphi y' + \varphi^2 y - \varphi' y)e^{-\int_0^t \varphi(s)ds}.$$

Thus

$$0 = (x'' + px' + qx)e^{\int_0^t \varphi(s)ds} =$$

$$= y'' - 2\varphi y' + \varphi^2 y - \varphi' y + p(y' - \varphi y) + qy =$$

$$= y'' + (p - 2\varphi)y' + (q + \varphi^2 - \varphi' - p\varphi)y.$$

If $tg\,\gamma \neq \infty$, then $tg\,\gamma' \neq \infty$ and it follows that

$$y(c)\sin\gamma' + y'(c)\cos\gamma' =$$

$$= (x(c)(\sin\gamma' + \varphi(c)\cos\gamma') + x'(c)\cos\gamma')e^{\int_0^c \varphi(s)ds} =$$

$$= (x(c)tg\gamma' + \varphi(c)) + x'(c))\cos\gamma' e^{\int_0^c \varphi(s)ds} =$$

$$= (x(c)tg\gamma + x'(c))\cos\gamma' e^{\int_0^c \varphi(s)ds} =$$

$$= \frac{\cos\gamma'}{\cos\gamma} e^{\int_0^c \varphi(s)ds}.$$

If $tg\,\gamma = \infty$, then $tg\,\gamma' = \infty$, so that

$$y(c)\sin\gamma' + y'(c)\cos\gamma' = y(c) = x(c)e^{\int_0^c \varphi(s)ds} = e^{\int_0^c \varphi(s)ds} =$$

$$= \frac{\cos\gamma'}{\cos\gamma} e^{\int_0^c \varphi(s)ds}.$$

A similar calculation can be carried out for β. ■

Proposition 6.2.4.12 (5) *Take* $c \in]0,\infty[$. *Let* p *(resp.* q*) be a continuously differentiable (resp. a continuous) real function on* $[0,c]$. *Take* $\beta,\gamma \in \mathbb{R}$, *such that*

$$tg\,\beta + \frac{p(0)}{2},\ tg\,\gamma - \frac{p(c)}{2} \in [0,\infty],$$

where the tangent takes the value $+\infty$ *at the poles.*

a) *For each* $\alpha > \|q - \frac{p^2}{4} - \frac{p'}{2}\|$ *there is a unique twice differentiable real-valued function* x_α *on* $[0, c]$ *such that*

$$x''_\alpha + px'_\alpha + (q - \alpha)x_\alpha = 0\,,$$

$$\begin{cases} x_\alpha(0)\sin\beta - x'_\alpha(0)\cos\beta = 0 \\ x_\alpha(c)\sin\gamma + x'_\alpha(c)\cos\gamma = 1 \end{cases}$$

$$\left(\textit{resp.} \quad \begin{cases} x_\alpha(0)\sin\beta - x'_\alpha(0)\cos\beta = 1 \\ x_\alpha(c)\sin\gamma + x'_\alpha(c)\cos\gamma = 0 \end{cases}\right).$$

b) *There is a* $\delta \in \mathbb{R}_+$ *such that*

$$\|x_\alpha\| \leq \delta\,, \quad \|x'_\alpha\| \leq \delta\sqrt{\alpha}$$

for every $\alpha > \|q - \frac{p^2}{4} - \frac{p'}{2}\| + 1$.

c) *If*

$$tg\,\beta + \frac{p(0)}{2}\,,\ tg\,\gamma - \frac{p(c)}{2} \in \mathbb{R}_+\,,$$

then there is a $\delta \in \mathbb{R}_+$ *such that*

$$\|x'_\alpha\| \leq \delta$$

for every $\alpha > \|q - \frac{p^2}{4} - \frac{p'}{2}\| + 1$.

a & b. Let $\beta', \gamma' \in [0, \frac{\pi}{2}]$ such that

$$tg\,\beta' = tg\,\beta + \frac{p(0)}{2}\,, \quad tg\,\gamma' = tg\,\gamma - \frac{p(c)}{2}\,.$$

By Theorem 6.2.2.5 d) and Corollary 3.2.1.7 a $\Rightarrow$ c, there is a unique twice differentiable function y_α on $[0, c]$ such that

$$y''_\alpha + \left(q - \frac{p^2}{4} - \frac{p'}{2} - \alpha\right) y_\alpha = 0\,,$$

$$y_\alpha(0)\sin\beta' - y'_\alpha(0)\cos\beta' = 0\,,$$

$$y_\alpha(c)\sin\gamma' + y'_\alpha(c)\cos\gamma' = \frac{\cos\gamma'}{\cos\gamma} e^{\frac{1}{2}\int_0^c p(s)ds}\,,$$

where

$$\frac{\cos\gamma'}{\cos\gamma} := 1$$

if $tg\,\gamma = \infty$. Put

$$x_\alpha := y_\alpha e^{-\frac{1}{2}\int_0^c p(s)ds}.$$

By Lemma 6.2.4.11, x_α is twice differentiable and

$$x_\alpha'' + px_\alpha' + (q-\alpha)x_\alpha = 0,$$

$$\begin{cases} x_\alpha(0)\sin\beta - x_\alpha'(0)\cos\beta = 0 \\ x_\alpha(c)\sin\gamma + x_\alpha'(c)\cos\gamma = 1. \end{cases}$$

The uniqueness of x_α follows from the uniqueness of y_α, using Lemma 6.2.4.11 again.

By Proposition 6.2.4.10, there is a δ' such that

$$\|y_\alpha\| \le \delta' \text{ and } \|y_\alpha'\| \le \delta'\sqrt{\alpha}$$

for every $\alpha > \|q - \frac{p^2}{4} - \frac{p'}{2}\| + 1$. Put

$$\delta := \left(\delta' + \frac{\|p\|}{2}\delta'\right) e^{\frac{1}{2}\int_0^c |p(s)|ds}.$$

Then

$$\|x_\alpha\| \le \|y_\alpha\| e^{\frac{1}{2}\int_0^c |p(s)|ds} \le \delta,$$

$$\|x_\alpha'\| = \left\|(y_\alpha' - \frac{p}{2}y_\alpha)e^{\frac{1}{2}\int_0^c |p(s)|ds}\right\| \le$$

$$\le \left(\|y_\alpha'\| + \frac{\|p\|}{2}\|y_\alpha\|\right) e^{\frac{1}{2}\int_0^c |p(s)|ds} \le \left(\delta'\sqrt{\alpha} + \frac{\|p\|}{2}\delta'\right) e^{\frac{1}{2}\int_0^c |p(s)|ds} \le \delta\sqrt{\alpha}.$$

The conclusion in parentheses can be derived from the other one by defining

$$\widetilde{p} : [0,c] \longrightarrow \mathbb{R}, \quad t \longmapsto -p(c-t),$$

$$\widetilde{q} : [0,c] \longrightarrow \mathbb{R}, \quad t \longmapsto q(c-t),$$

$$\widetilde{x}_\alpha : [0,c] \longrightarrow \mathbb{R}, \quad t \longmapsto x_\alpha(c-t),$$

$$\widetilde{\beta} := \gamma, \text{ and } \widetilde{\delta} := \beta.$$

Then

$$\begin{cases} tg\,\widetilde{\beta} + \frac{\widetilde{p}(0)}{2} = tg\,\gamma - \frac{p(c)}{2} \in [0,\infty] \\ tg\,\widetilde{\gamma} - \frac{\widetilde{p}(c)}{2} = tg\,\beta + \frac{p(0)}{2} \in [0,\infty], \end{cases}$$

$$\begin{cases} \widetilde{x}_\alpha(0)\sin\widetilde{\beta} - \widetilde{x}'_\alpha(0)\cos\widetilde{\beta} = x_\alpha(c)\sin\gamma + x'_\alpha(c)\cos\gamma = 1 \\ \widetilde{x}_\alpha(c)\sin\widetilde{\gamma} + \widetilde{x}'_\alpha(c)\cos\widetilde{\gamma} = x_\alpha(0)\sin\beta - x'_\alpha(0)\cos\gamma = 0 \end{cases}$$

and for any $t \in [0,c]$,

$$\widetilde{x}''_\alpha(t) + \widetilde{p}(t)\widetilde{x}'_\alpha(t) + (\widetilde{q}(t) - \alpha)\widetilde{x}_\alpha(t) =$$

$$= x''_\alpha(c-t) + p((c-t)x'_\alpha(c-t) + (q(c-t) - \alpha)x_\alpha(c-t) = 0.$$

c) follows from the above proof and Lemma 6.2.4.9.

Theorem 6.2.4.13 (5) *Take $c \in]0,\infty[$. Let p (resp. q) be a continuo differentiable (resp. a continuous) real function on $[0,c]$. Take $f_1, f_2 \in \mathcal{D}$ $\beta, \gamma \in \mathbb{R}$ such that*

$$tg\,\beta + \frac{p(0)}{2},\ tg\,\gamma - \frac{p(c)}{2} \in [0,\infty],$$

where the tangent takes the value $+\infty$ at the poles, and given $\alpha > \|q - \frac{p^2}{4} - \frac{p}{2}$ let y_α and z_α denote the twice differentiable real functions on $[0,c]$ such t

$$\begin{cases} y''_\alpha + py'_\alpha + (q-\alpha)y_\alpha = 0 \\ y_\alpha(0)\sin\beta - y'_\alpha(0)\cos\beta = 1 \\ y_\alpha(c)\sin\gamma + y'_\alpha(c)\cos\gamma = 0, \end{cases} \qquad \begin{cases} z''_\alpha + pz'_\alpha + (q-\alpha)z_\alpha = 0 \\ z_\alpha(0)\sin\beta - z'_\alpha(0)\cos\beta = 0 \\ z_\alpha(c)\sin\gamma + z'_\alpha(c)\cos\gamma = 1 \end{cases}$$

(Proposition 6.2.4.12 a)). Assume that $A \subset]\|q - \frac{p^2}{4} - \frac{p'}{2}\|, \infty[$ and that coefficients of u are $\mathcal{C}^\infty$-functions.

a) *There is a unique $y \in \mathcal{C}([a,b] \times [0,c])$ such that*

$$z : [0,c] \longrightarrow L^2(r\cdot\lambda), \quad s \longmapsto y(\cdot, s)$$

is twice differentiable, $(f(x)\langle z(s)|x\rangle)_{x\in B} \in \ell^2(B)$ for every $s \in [0,c]$,

$$\begin{cases} z(0)\sin\beta - z'(0)\cos\beta = f_1 \\ z(c)\sin\gamma + z'(c)\cos\gamma = f_2\,, \end{cases}$$

and

$$r(z'' + pz' + qz) - uz = 0$$

on $]0,c[$, *where* uz *is understood in the sense of* L^2*-distributions.*

Given $x \in B$, *define*

$$h_x : [a,b] \times [0,c] \longrightarrow \mathbb{K}\,,$$

$$(t,s) \longmapsto (\langle f_1|x\rangle y_{f(x)}(s) + \langle f_2|x\rangle z_{f(x)}(s))x(t)\,.$$

Then $(h_x)_{x\in B}$ *is summable in* $\mathcal{C}([a,b]\times[0,c])$ *and*

$$y = \sum_{x\in B} h_x\,.$$

b) *If* p *and* q *are* $\mathcal{C}^\infty$*- functions and* $\mathbb{K} = \mathbb{R}$, *then*

$$r\left(\frac{\partial^2 y}{\partial s^2} + p\frac{\partial y}{\partial s} + qy\right) - uy = 0$$

on $]a,b[\times]0,c[$ *in the sense of distributions.*

c) *If*

$$tg\beta + \frac{p(0)}{2}\,,\ tg\gamma - \frac{p(c)}{2} \in \mathbb{R}_+\,,$$

then the family $\left(\frac{\partial h_x}{\partial s}\right)_{x\in B}$ *is summable in* $\mathcal{C}([a,b]\times[0,c])$, $y(t,\cdot)$ *is differentiable for every* $t \in [a,b]$ *and*

$$\frac{\partial y}{\partial s} = \sum_{x\in B}\frac{\partial h_x}{\partial s} \in \mathcal{C}([a,b]\times[0,c])\,.$$

d) *If* $uf_1\,,\ uf_2 \in \mathcal{D}$, *then the families* $\left(\frac{\partial h_x}{\partial s}\right)_{x\in B}$, $\left(\frac{\partial^2 h_x}{\partial s^2}\right)_{x\in B}$ *are summable in* $\mathcal{C}([a,b]\times[0,c])$, $y(\cdot,s) \in \mathcal{D}$ *for every* $s \in [0,c]$, $y(t,\cdot)$ *is twice differentiable for every* $t \in [a,b]$ *and*

$$\frac{\partial y}{\partial s} = \sum_{x\in B}\frac{\partial h_x}{\partial s} \in \mathcal{C}([a,b]\times[0,c])\,,$$

$$\frac{\partial^2 y}{\partial s^2} = \sum_{x\in B}\frac{\partial^2 h_x}{\partial s^2} \in \mathcal{C}([a,b]\times[0,c])\,.$$

By Proposition 6.2.4.12 b), there is a $\delta \in \mathbb{R}_+$ such that

$$\|y_\alpha\| \leq \delta\,, \quad \|z_\alpha\| \leq \delta\,, \quad \|y'_\alpha\| \leq \delta\sqrt{\alpha}\,, \quad \|z'_\alpha\| \leq \delta\sqrt{\alpha}$$

for every $\alpha > \|q - \frac{p^2}{4} - \frac{p'}{2}\| + 1$. If

$$tg\beta + \frac{p(0)}{2}\,,\ tg\gamma - \frac{p(c)}{2} \in \mathbb{R}_+\,,$$

then, by Proposition 6.2.4.12 c), there is a $\delta \in \mathbb{R}_+$ such that

$$\|y'_\alpha\| \leq \delta\,, \quad \|z'_\alpha\| \leq \delta$$

for every $\alpha > \|q - \frac{p^2}{4} - \frac{p'}{2}\| + 1$. The assertions now follow from Theorem 6.2.4.4. ■

Lemma 6.2.4.14 (5) *Take* $c \in]0, \infty[$. *Let* q *be a strictly positive continuous real function on* $[0, c]$ *and* y, z *twice differentiable real functions on* $[0, c]$ *such that*

$$\begin{cases} y'' = qy \\ y(c) - y(0) = 0\,, \\ y'(c) - y'(0) = 1\,, \end{cases} \qquad \begin{cases} z'' = qz \\ z(c) - z(0) = 1\,, \\ z'(c) - z'(0) = 0\,. \end{cases}$$

Then

$$\|y\| \leq c + \frac{1}{c \inf\limits_{t\in[0,c]} q(t)}\,, \quad \|y'\| \leq 1\,, \quad \|z\| \leq 1\,, \quad \|z'\| \leq \frac{1}{c} + \|q\|c\,.$$

Step 1 y vanishes nowhere.

Take $t_0 \in [0, c]$ such that $y(t_0) = 0$. Then $y'(t_0) \neq 0$, since otherwise y would vanish identically. If $y'(t_0) > 0$ $(y'(t_0) < 0)$, then y is strictly increasing (strictly decreasing) on $[0, c]$, which contradicts the fact that $y(0) = y(c)$.

Step 2 y is strictly positive

Assume the contrary. By Step 1, y is strictly negative. Hence y' is decreasing and we obtain the contradiction

$$1 = y'(c) - y'(0) \leq 0\,.$$

Step 3 $\|y\| \leq c + \frac{1}{c \inf\limits_{t\in[0,c]} q(t)}\,, \quad \|y'\| \leq 1$

By Step 2, y is strictly positive and therefore y' is strictly increasing. Since $y(0) = y(c)$, it follows that y has a minimum on $[0, c[$ at a unique point t_0 and

$$y'(0) < 0 < y'(c).$$

Since

$$y'(c) - y'(0) = 1,$$

we see that $\|y'\| \leq 1$ and that

$$1 = y'(c) - y'(0) = \int_0^c y''(t)dt = \int_0^c q(t)y(t)dt \geq cy(t_0) \inf_{t\in[0,c]} q(t).$$

Hence

$$\|y\| = y(c) = y(t_0) + \int_{t_0}^c y'(t)dt \leq \frac{1}{c \inf\limits_{t\in[0,c]} q(t)} + c.$$

Step 4 $\exists\, t_0 \in]0, c[,\ z(t_0) = 0,\ z'(t_0) > 0$

If z does not vanish on $]0, c[$, then z' is strictly monotonic, which contradicts the fact that $z'(0) = z'(c)$. Hence there is a $t_0 \in]0, c[$, such that $z(t_0) = 0$. Then $z'(t_0) \neq 0$, since otherwise z would vanish identically. $z'(t_0) < 0$ implies z is strictly decreasing, which contradicts the fact that

$$z(c) - z(0) = 1.$$

Hence $z'(t_0) > 0$.

Step 5 $\|z\| \leq 1$

By Step 4, z is strictly increasing and vanishes at a single point of $]0, c[$. Hence

$$z(0) < 0 < z(c).$$

It follows from

$$z(c) - z(0) = 1$$

that $\|z\| \leq 1$.

Step 6 $\|z'\| \leq \frac{1}{c} + \|q\|c$

By Step 4, z' is increasing on $[t_0, c]$ and decreasing on $[0, t_0]$. By Step 5,

$$1 \geq z(c) = \int_{t_0}^{c} z'(t)dt \geq z'(t_0)c\,,$$

and

$$\|z'\| = z'(c) = z'(t_0) + \int_{t_0}^{c} q(t)z(t)dt \leq \frac{1}{c} + \|q\|c\,. \qquad \blacksquare$$

Theorem 6.2.4.15 (5) *Let q be a continuous real function on $\mathbf{T}$ and take $f_1, f_2 \in \mathcal{D}$. Given $\alpha > \sup_{t\in\mathbf{T}} q(t)$, let y_α and z_α be twice differentiable real functions on $[0,1]$ such that*

$$\begin{cases} y_\alpha'' + (q-\alpha)y_\alpha = 0 \\ y_\alpha(1) - y_\alpha(0) = 1 \\ y_\alpha'(1) - y_\alpha'(0) = 0\,, \end{cases} \qquad \begin{cases} z_\alpha'' + (q-\alpha)z_\alpha = 0 \\ z_\alpha(1) - z_\alpha(0) = 0 \\ z_\alpha'(1) - z_\alpha'(0) = 1 \end{cases}$$

(Proposition 3.2.2.11 and Corollary 3.2.1.7 a $\Rightarrow$ c) and assume that the coefficients of u are $\mathcal{C}^\infty$-functions and $A \subset]\|q\|, \infty[$.

a) *There is a unique $y \in \mathcal{C}([a,b] \times \mathbf{T})$ such that*

$$z : \mathbf{T} \longrightarrow L^2(r\cdot\lambda)\,, \quad s \longmapsto y(\cdot, s)$$

is twice differentiable, $(f(x)\langle z(s)|x\rangle)_{x\in B} \in \ell^2(B)$ for every $s \in [0,c]$,

$$z(0) = f_1\,, \quad z'(0) = f_2$$

and

$$r(z'' + qz) - uz = 0$$

on $\mathbf{T}$, where uz is understood in the sense of L^2-distributions.

We identify each $s \in [0,1]$ with $e^{2\pi i s} \in \mathbf{T}$ and define

$$h_x : [a,b] \times \mathbf{T} \longrightarrow \mathbb{R}\,,$$

$$(t,s) \longmapsto (\langle f_1|x\rangle y_{f(x)}(s) + \langle f_2|x\rangle z_{f(x)}(s))x(t)$$

for every $x \in B$. Then the families $(h_x)_{x\in B}$, $\left(\frac{\partial h_x}{\partial s}\right)_{x\in B}$ are summable in $\mathcal{C}([a,b] \times \mathbf{T})$, $y(t,\cdot)$ is differentiable for every $t \in [a,b]$ and

$$y = \sum_{x\in B} h_x\,,$$

$$\frac{\partial y}{\partial s} = \sum_{x\in B} \frac{\partial h_x}{\partial s} \in \mathcal{C}([a,b] \times \mathbf{T})\,.$$

b) *If q is a $\mathcal{C}^\infty$-function and $\mathbb{K} = \mathbb{R}$, then*

$$r\left(\frac{\partial^2 y}{\partial s^2} + qy\right) - uy = 0$$

on $]a,b[\times\mathbf{T}$ in the sense of distributions.

c) *If $uf_1, uf_2 \in \mathcal{D}$, then the family $\left(\frac{\partial^2 h_x}{\partial s^2}\right)_{x\in B}$ is summable in $\mathcal{C}([a,b]\times\mathbf{T})$, $y(\cdot,s) \in \mathcal{D}$ for every $s \in \mathbf{T}$, $y(t,\cdot)$ is twice differentiable for every $t \in [a,b]$ and*

$$\frac{\partial^2 y}{\partial s^2} = \sum_{x\in B} \frac{\partial^2 h_x}{\partial s^2} \in \mathcal{C}([a,b]\times\mathbf{T})\,.$$

Since A is discrete (Theorem 6.2.1.3 c),f)), there is an $\varepsilon > 0$ such that $A \subset]\|q\| + \varepsilon, \infty[$. By Lemma 6.2.4.14, there is a $\delta \in \mathbb{R}_+$ such that

$$\|y_\alpha\| \le \delta\,, \quad \|y'_\alpha\| \le \delta\,, \quad \|z'_\alpha\| \le \delta$$

for every $\alpha \in [\|q\| + \varepsilon\,, \infty[$. The assertion now follows by Theorem 6.2.4.4. ■

6.2.5 Selfadjoint Linear Partial Differential Equations

Throughout this subsection, we abide by the notation and hypotheses of Subsection 3.2.3. In addition, we assume T to be compact, u selfadjoint,

$$r := -ue_T$$

strictly positive, and

$$\int r d\lambda = 1 .$$

It follows that the Green function g of u is selfadjoint *(J.-M. Bony, Principe du maximum, inégalité de Harnack et unicité du problème de Cauchy pour les opérateurs elliptiques dégénérés, Ann. Inst. Fourier 19.1 (1969) 277-304, Théorème 6.1)*, e_T is a potential on T (Lemma 3.2.3.2), and

$$e_T = \int g(\cdot, t) r(t) d\lambda(t) .$$

Lemma 6.2.5.1 (5) *Every $x \in L^1(r\cdot\lambda)_+$ is e_T-integrable and*

$$\int (x \cdot e_T) r d\lambda = \int x r d\lambda .$$

First suppose that x is bounded. By Tonelli's Theorem

$$\int (x \cdot e_T) r d\lambda = \int \left(\int g(s,t) x(t) r(t) d\lambda(t) \right) r(s) d\lambda(s) =$$

$$= \int \left(\int g(t,s) x(t) r(t) d\lambda(t) \right) r(s) d\lambda(s) =$$

$$= \int \left(\int g(t,s) r(s) d\lambda(s) \right) x(t) r(t) d\lambda(t) =$$

$$= \int e_T(t) x(t) r(t) d\lambda(t) = \int x r d\lambda .$$

Given arbitrary x, we deduce that

$$\int x r d\lambda = \lim_{n \to \infty} \int (x \wedge n e_T) r d\lambda =$$

$$= \lim_{n\to\infty} \int ((x \wedge ne_T)\cdot e_T) r d\lambda = \int (x\cdot e_T) r d\lambda .$$

Hence $x\cdot e_T$ is $r\cdot\lambda$–integrable and therefore finite on a dense set. This implies that $x\cdot e_T$ is superharmonic, i.e. x is e_T–integrable. ■

Lemma 6.2.5.2 (5) *Let U be an open set of T, K a compact set of U and take $\varepsilon > 0$. There is a $\delta > 0$ such that for every $x \in L^1(r\cdot\lambda)_+$ vanishing on U, we have (Lemma 6.2.5.1)*

$$\int x r d\lambda < \delta \Longrightarrow \sup_{t\in K} (x\cdot e_T)(t) < \varepsilon .$$

Assume the contrary. Then there is a sequence $(x_n)_{n\in\mathbb{N}}$ in $L^1(r\cdot\lambda)_+$ such that the following holds for every $n \in \mathbb{N}$:

$$x_n \text{ vanishes on } U ,$$

$$\int x_n r d\lambda < \frac{1}{2^n} ,$$

$$\sup_{t\in K} (x_n\cdot e_T)(t) \geq \varepsilon .$$

Put

$$x := \sum_{n\in\mathbb{N}} x_n .$$

Then x is $r\cdot\lambda$–integrable and vanishes on U. By Lemma 6.2.5.1, x is e_T–integrable and harmonic on U. Since

$$x\cdot e_T = \sum_{n\in\mathbb{N}} x_n\cdot e_T ,$$

it follows by Dini's Theorem that $(x_n\cdot e_T)_{n\in\mathbb{N}}$ converges uniformly to 0 on K, which is a contradiction. ■

Lemma 6.2.5.3 (5) *Take $t \in T$ and $\varepsilon > 0$. There is an open neighbourhood U of t and a $\delta > 0$ such that for every Borel set A of T*

$$\sup\{(e_A\cdot e_T)(s) \mid s \in U\} < \varepsilon$$

whenever $\int_A r d\lambda < \delta$.

Given $n \in \mathbb{N}$, put

$$U_n := U_{\frac{1}{n}}(t) \text{ and } x_n := e_{U_n}.$$

Then $(x_n \cdot e_T)_{n\in\mathbb{N}}$ is a decreasing sequence of continuous real functions on T converging pointwise to 0. By Dini's Theorem, it converges uniformly to 0. Hence there is an $n \in \mathbb{N}$ such that

$$\sup\{(x_n \cdot e_T)(s) \mid s \in T\} < \frac{\varepsilon}{2}.$$

We set

$$U := U_{2n}.$$

By Lemma 6.2.5.2, there is a $\delta > 0$, such that for every Borel subset A of $T \backslash U_n$,

$$\sup\{(e_A \cdot e_T)(s) \mid s \in U\} < \frac{\varepsilon}{2}$$

whenever $\int_A r d\lambda < \delta$. Let A be a Borel set of T with

$$\int_A r d\lambda < \delta.$$

Then

$$e_A \cdot e_T = e_{A\cap U_n} \cdot e_T + e_{A\backslash U_n} \cdot e_T \leq x_n \cdot e_T + e_{A\backslash U_n} \cdot e_T.$$

Thus

$$\sup\{e_A \cdot e_T)(s) \mid s \in U\} \leq$$

$$\leq \sup\{(x_n \cdot e_T)(s) | s \in U\} + \sup\{(e_{A\backslash U} \cdot e_T)(s) \mid s \in U\} < \varepsilon. \quad \blacksquare$$

Lemma 6.2.5.4 (5) *For every $\varepsilon > 0$, there is a $\delta > 0$ such that for every Borel set A of T,*

$$\sup\{(e_A \cdot e_T)(t) \mid t \in T\} < \varepsilon,$$

whenever $\int_A r d\lambda < \delta$.

By Lemma 6.2.5.3, for every $t \in T$, there is an open neighbourhood U_t of t and a $\delta_t > 0$ such that for every Borel set A of T,

$$\sup\{(e_A \cdot e_T)(s) \mid s \in U_t\} < \varepsilon,$$

whenever $\int_A r d\lambda < \delta_t$. Since T is compact, there is a finite subset B of T such that

$$T = \bigcup_{t \in B} U_t .$$

Put

$$\delta := \inf_{t \in B} \delta_t .$$

Let A be a Borel set of T with

$$\int_A r d\lambda < \delta .$$

Take $s \in T$. There is a $t \in B$ such that $s \in U_t$. Then

$$(e_A \cdot e_T)(s) \leq \sup\{(e_A \cdot e_T)(s') \mid s' \in U_t\} < \varepsilon .$$

Hence

$$\sup\{(e_A \cdot e_T)(s) \mid s \in T\} < \varepsilon . \qquad \blacksquare$$

Theorem 6.2.5.5 (5) *Take* $\alpha \in]1, \infty]$.

a) *If* $x \in L^\alpha(r \cdot \lambda)$ *then* $x \cdot e_T \in L^\alpha(r \cdot \lambda)$ *and*

$$\|x \cdot e_T\|_\alpha \leq \|x\|_\alpha .$$

b) *The operator*

$$v : L^\alpha(r \cdot \lambda) \longrightarrow L^\alpha(r \cdot \lambda), \quad x \longmapsto x \cdot e_T$$

is compact.

a) follows from Lemma 6.2.5.1 for $\alpha \neq \infty$. For $\alpha = \infty$, the assertion is trivial.

b) Let $\mathcal{B}$ be the Banach space of bounded Borel functions on T (with the supremum norm). By Proposition 3.2.3.1, $\{x \cdot e_T \mid x \in \mathcal{B}^\#\}$ is a precompact set of $\mathcal{C}(T)$ and therefore a precompact set of $L^\infty(r \cdot \lambda)$. This proves the assertion for $\alpha = \infty$.

Assume now that $\alpha \neq \infty$. Take $\varepsilon > 0$. By Lemma 6.2.5.4, there is a $\delta > 0$ such that for every Borel set A of T,

$$\|e_A\|_1 \leq \delta \Longrightarrow \|e_A \cdot e_T\|_\infty < (\frac{\varepsilon}{2})^{\frac{\alpha}{\alpha - 1}} .$$

Let x be a Borel function of $L^\alpha(r\cdot\lambda)_+^\#$. Put

$$A := \{|x| > \delta^{-\frac{1}{\alpha}}\}.$$

Then

$$1 \geq \int |x|^\alpha r d\lambda \geq \int_A |x|^\alpha r d\lambda \geq \frac{1}{\delta}\int_A r d\lambda,$$

$$\|e_A\|_1 = \int_A r d\lambda \leq \delta,$$

$$\|e_A \cdot e_T\|_\infty < (\frac{\varepsilon}{2})^{\frac{\alpha}{\alpha-1}}.$$

By Hölder's Inequality,

$$v(xe_A) \leq (x^\alpha \cdot e_T)^{\frac{1}{\alpha}}(e_A \cdot e_T)^{\frac{\alpha-1}{\alpha}} \leq (x^\alpha \cdot e_T)^{\frac{1}{\alpha}}\frac{\varepsilon}{2},$$

$$v(xe_A)^\alpha \leq (\frac{\varepsilon}{2})^\alpha (x^\alpha \cdot e_T).$$

Hence by Lemma 6.2.5.1,

$$\|v(xe_A)\|_\alpha = |\int v(xe_A)^\alpha r d\lambda|^{\frac{1}{\alpha}} \leq$$

$$\leq \frac{\varepsilon}{2}(\int (x^\alpha \cdot e_T) r d\lambda)^{\frac{1}{\alpha}} = \frac{\varepsilon}{2}(\int x^\alpha r d\lambda)^{\frac{1}{\alpha}} = \frac{\varepsilon}{2}\|x\|_\alpha \leq \frac{\varepsilon}{2}.$$

Since $v(\delta^{-\frac{1}{\alpha}}\mathcal{B}^\#)$ is a precompact set of $L^\alpha(r\cdot\lambda)$, there is a finite subset B of $L^\alpha(r\cdot\lambda)$ such that

$$v(\delta^{-\frac{1}{\alpha}}\mathcal{B}^\#) \subset \bigcup_{y\in B} U_{\frac{\varepsilon}{2}}(y).$$

Since $x - xe_A \in \delta^{-\frac{1}{\alpha}}\mathcal{B}^\#$, there is a $y \in B$ with

$$v(x - xe_A) \in U_{\frac{\varepsilon}{2}}(y).$$

Thus

$$\|vx - y\|_\alpha \leq \|v(xe_A)\|_\alpha + \|v(x - xe_A) - y\|_\alpha \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,$$

$$vx \in U_\varepsilon(y)$$

and hence

$$v(L^\alpha(r\cdot\lambda)_+^\#) \subset \bigcup_{y\in B} U_\varepsilon(y).$$

Thus $v(L^\alpha(r\cdot\lambda)^\#)$ is a precompact set of $L^\alpha(r\cdot\lambda)$ and so v is compact. ■

Theorem 6.2.5.6 (5)

a) *The map*

$$v : L^1(r\cdot\lambda) \longrightarrow L^1(r\cdot\lambda), \quad x \longmapsto x\cdot e_T$$

is an operator with $\|v\| \leq 1$.

b) *For every* $x \in L^1(r\cdot\lambda)$,

$$u(vx) = -rx$$

and

$$(x \in \operatorname{Im} v) \Longleftrightarrow (ux \in L^1(r\cdot\lambda)) \Longrightarrow \left(v(\frac{1}{r}ux)) = -x\right).$$

c) *Given* $\alpha \in \mathbb{R}$, *put*

$$\mathcal{F}_\alpha := \{x \in L^1(r\cdot\lambda) \mid ux = -\alpha rx\}.$$

Then

$$\mathcal{F}_\alpha = \{x \in L^1(r\cdot\lambda) \mid vx = \frac{1}{\alpha}x\}$$

for every $\alpha \in \mathbb{R}\backslash\{0\}$, *and*

$$\{\alpha \in \mathbb{R} \mid \mathcal{F}_\alpha \neq \{0\}\} \subset]0, \infty[,$$

$$\sigma_p(v) \subset]0, \infty[.$$

a) follows from Lemma 6.2.5.1.

b) By a) and Lemma 3.2.3.4 a),

$$u(vx) = -rx.$$

Assume that $ux \in L^1(r\cdot\lambda)$. Then $\frac{1}{r}ux$ is e_T–integrable (Lemma 6.2.5.1). By Lemma 3.2.3.4 b) (and Lemma 3.2.3.2),

$$x = -v(\frac{1}{r}ux) \in \operatorname{Im} v.$$

Assume that $x \in \operatorname{Im} v$. Then there is a $y \in L^1(r\cdot\lambda)$ with

$$x = vy.$$

Then, by the above,

$$ux = uvy = -ry \in L^1(r\cdot\lambda)\,.$$

c) By b), if $\alpha \in \mathbb{R}\backslash\{0\}$, then

$$(x \in \mathcal{F}_\alpha) \Longleftrightarrow (ux = -\alpha rx) \Longleftrightarrow (vx = \frac{1}{\alpha}x)\,.$$

Since $\mathcal{H}(T) = \{0\}$ (Lemma 3.2.3.2), $\mathcal{F}_0 = \{0\}$. Take $\alpha > 0$ and $x \in \mathcal{F}_{-\alpha}$. Then

$$\alpha vx + x = 0\,,$$

so that

$$x = -\alpha vx = -\alpha(\alpha v_\alpha vx + v_\alpha x) = -\alpha v_\alpha(\alpha vx + x) = 0\,.$$

Hence

$$\{\alpha \in \mathbb{R} \mid \mathcal{F}_\alpha \neq \{0\}\} \subset]0,\infty[\,.$$

It now follows that

$$\sigma_p(v) \subset]0,\infty[\,.$$ ■

Theorem 6.2.5.7 (5) *The map*

$$v : L^2(r\cdot\lambda) \longrightarrow L^2(r\cdot\lambda)\,, \quad x \longmapsto x\cdot e_T$$

(Theorem 6.2.5.5 a)) is an injective, positive, compact operator.

Step 1 v is injective

This follows from Theorem 6.2.5.6 c).

Step 2 v is selfadjoint

Take $x, y \in L^2(r\cdot\lambda)$. Then, by Theorem 6.2.5.6 b),

$$\langle vx|y\rangle = \langle vx| - \frac{1}{r}uvy\rangle = -\int (vx)(uvy)d\lambda =$$

$$= -\int (uvx)(vy)d\lambda = \int rx(vy)d\lambda = \langle x|vy\rangle\,.$$

Step 3 v is compact

This was proved in Theorem 6.2.5.5 b).

Step 4 v is positive

This follows from Theorem 6.2.5.6 c). ■

Corollary 6.2.5.8 (5)

a) *There is an orthonormal basis* B *of* $L^2(r\cdot\lambda)$ *and an* $f \in \mathbb{R}^B$ *such that*

$$ux = -f(x)rx$$

for every $x \in B$.

b) *If* B *and* f *are as above, then* f *is strictly positive and* $\frac{1}{f} \in c_0(B)$.

c) *Given* $\alpha \in \mathbb{R}$, *define*

$$\mathcal{F}_\alpha := \{x \in L^2(r\cdot\lambda) \mid ux = -\alpha rx\}.$$

Then $\mathcal{F}_\alpha$ *is finite-dimensional and*

$$\{\alpha \in \mathbb{R} \mid \mathcal{F}_\alpha \neq \{0\}\} \subset]0, \infty[.$$

By Theorem 6.2.5.7,

$$v : L^2(r\cdot\lambda) \longrightarrow L^2(r\cdot\lambda), \quad x \longmapsto x\cdot e_T$$

is an injective, positive, compact operator. By Theorem 6.2.5.6 c),

$$\{x \in L^2(r\cdot\lambda) \mid ux = -\alpha rx\} = \{x \in L^2(r\cdot\lambda) \mid vx = \frac{1}{\alpha}x\}$$

for every $\alpha \in \mathbb{R}\backslash\{0\}$.

a) and b) follow from Corollary 6.1.3.18.

c) By Theorem 6.2.5.6 c),

$$\{\alpha \in \mathbb{R} \mid \mathcal{F}_\alpha \neq \{0\}\} \subset]0, \infty[.$$

Since v is compact, each $\mathcal{F}_\alpha$ is finite-dimensional. ■

6.2.6 Associated Parabolic and Hyperbolic Evolution Equations

This subsection retains the notation and all the hypotheses from the preceding subsection except that the hypothesis "r is strictly positive" is replaced by the weaker one "r is strictly positive, strictly negative, or it vanishes identically".
We put

$$\rho := \begin{cases} r \text{ if } r \text{ is strictly positive} \\ -r \text{ if } r \text{ is strictly negative} \\ \text{any strictly positive } C^\infty\text{–function on } T \text{ if } r \text{ vanishes identically.} \end{cases}$$

Given $\alpha \in \mathbb{R}$, put

$$\mathcal{F}_\alpha := \{x \in L^2(\lambda) \mid ux = -\alpha\rho x\}.$$

Finally define

$$A := \{\alpha \in \mathbb{R} \mid \mathcal{F}_\alpha \neq \{0\}\}.$$

Proposition 6.2.6.1 (5) *If $(x_n)_{n\in\mathbb{N}}$ is a sequence in $L^2(\rho\cdot\lambda)$ such that*

$$\frac{1}{\rho}ux_n = x_{n+1}$$

for every $n \in \mathbb{N}$, then each x_n $(n \in \mathbb{N})$ is a C^∞-function on T. In particular, any $x \in L^2(\rho\cdot\lambda)$ is a C^∞-function iff ux is a C^∞-function.

The assertion follows from Theorem 18 in *L.P. Rothschild and E.M. Stein, Hypoelliptic Differential Operators and Nilpotent Groups, Acta Math. 137, 1-2 (1976) 247-320.* ■

Proposition 6.2.6.2 (5)

a) *There are an orthonormal basis B of $L^2(\rho\cdot\lambda)$ and an $f \in \mathbb{R}^B$ such that*

$$ux = -f(x)\rho x$$

for every $x \in B$.

b) *If* B *and* f *are as above, then* $f(B)$ *is bounded below and has no point of accumulation, each* $f^{-1}(\alpha)$ *is finite* $(\alpha \in \mathbb{R})$ *and every element of* B *is a* $\mathcal{C}^\infty$*-function.*

c) A *is bounded below and each* $\mathcal{F}_\alpha$ *is finite-dimensional* $(\alpha \in A)$.

d) *If* B *and* f *are as in a) and if* $h \in \mathbb{R}^B$ *satisfies* $hf \in \ell^2(B)$, *then*

$$\sum_{x \in B} h(x)x \in \mathcal{C}(T),$$

$(h(x)ux)_{x \in B}$ *is summable in* $L^2(\rho \cdot \lambda)$ *and*

$$u\left(\sum_{x \in B} h(x)x\right) = -\rho \sum_{x \in B} h(x)f(x)x = \sum_{x \in B} h(x)ux\,.$$

Given $\alpha, \beta \in \mathbb{R}$, define

$$u_\beta := u - \beta\rho\,,$$

$$\mathcal{F}_{\beta,\alpha} := \{x \in L^2(\rho \cdot \lambda) \mid u_\beta x = -\alpha\rho x\}\,,$$

$$A_\beta := \{\alpha \in \mathbb{R} \mid \mathcal{F}_{\beta,\alpha} \neq \{0\}\}\,.$$

Take $\alpha, \beta \in \mathbb{R}$. Then for each $x \in L^2(\rho \cdot \lambda)$,

$$(x \in \mathcal{F}_{\beta,\alpha}) \Longleftrightarrow (u_\beta x = -\alpha\rho x) \Longleftrightarrow (ux = -(\alpha - \beta)\rho x) \Longleftrightarrow (x \in \mathcal{F}_{\alpha-\beta})\,.$$

It follows that

$$\mathcal{F}_{\beta,\alpha} = \mathcal{F}_{\alpha-\beta}\,,$$

$$(\alpha \in A_\beta) \Longleftrightarrow (\mathcal{F}_{\beta,\alpha} \neq \{0\}) \Longleftrightarrow (\mathcal{F}_{\alpha-\beta} \neq \{0\}) \Longleftrightarrow (\alpha - \beta \in A)\,,$$

$$A = A_\beta - \beta\,.$$

We have

$$u_2 e_T = -r - 2\rho = \begin{cases} -3\rho & if \quad r > 0 \\ -\rho & if \quad r < 0 \\ -2\rho & if \quad r = 0\,. \end{cases}$$

a) By Corollary 6.2.5.8 a), there is an orthonormal basis B of $L^2(\rho\cdot\lambda)$ and a $k \in \mathbb{R}^B$ such that

$$u_2 x = -k(x)\rho x\,,$$

$$ux = (2 - k(x))\rho x$$

for every $x \in B$.

b) Given $x \in B$,

$$u_2 x = -(f(x) + 2)\rho x\,.$$

By Corollary 6.2.5.8 b), $f + 2$ is strictly positive and $\frac{1}{f+2} \in c_0(B)$. It follows that $f(B)$ is bounded below, has no point of accumulation, and $\overset{-1}{f}(\alpha)$ is finite for each $\alpha \in \mathbb{R}$. By Proposition 6.2.6.1, each element of B is a $\mathcal{C}^\infty$-function.

c) follows from the above considerations and Corollary 6.2.5.8 c).

d) We write $\widetilde{T}$ for the harmonic space constructed on T with respect to u_2. Since $u_2 e_T$ is strictly negative, if $x \in L^2(\rho\cdot\lambda)$, then we may write $x\cdot e_T$ for multiplication with respect to $\widetilde{T}$. By Theorem 6.2.5.5 b), the map

$$v : L^2(\rho\cdot\lambda) \longrightarrow L^2(\rho\cdot\lambda)\,, \quad x \longmapsto x\cdot e_T$$

is an operator and by Theorem 6.2.5.6 b),

$$u_2(vx) = (u_2 e_T)x$$

for every $x \in L^2(\rho\cdot\lambda)$. Take $x \in B$. Then

$$u_2 x = -(f(x) + 2)\rho x = (f(x) + 2)\frac{\rho}{u_2 e_T} u_2(vx)$$

and so

$$-(f(x) + 2)\frac{\rho}{u_2 e_T} vx = x$$

(Theorem 6.2.5.6 c)).

Then

$$\sum_{x\in B} h(x)x = -\frac{\rho}{u_2 e_T}\sum_{x\in B} h(x)(f(x) + 2)vx =$$

$$= -\frac{\rho}{u_2 e_T} v\left(\sum_{x\in B} h(x)(f(x) + 2)x\right) \in \mathcal{C}(T)\,,$$

so that

$$u\left(\sum_{x\in B} h(x)x\right) = u_2\left(\sum_{x\in B} h(x)x\right) + 2\rho\sum_{x\in B} h(x)x =$$

$$= -\frac{\rho}{u_2 e_T} u_2 v\left(\sum_{x\in B} h(x)(f(x)+2)x\right) + 2\rho\sum_{x\in B} h(x)x =$$

$$= -\rho\sum_{x\in B} h(x)(f(x)+2)x + 2\rho\sum_{x\in B} h(x)x =$$

$$= -\rho\sum_{x\in B} h(x)f(x)x = \sum_{x\in B} h(x)ux\,. \qquad \blacksquare$$

Proposition 6.2.6.3 (5) *Let B be an orthonormal basis of $L^2(\rho\cdot\lambda)$ and take $f \in \mathbb{R}^B$ such that*

$$ux = -f(x)\rho x$$

for every $x \in B$. Take $n \in \mathbb{N}$ and let y be a $2n$-times continuously differentiable function on T. Then

$$(f^n(x)\langle y|x\rangle)_{x\in B} \in \ell^2(B)\,.$$

We prove the assertion by induction on n. Assume that the assertion holds for $n-1$. Then

$$f^n(x)\langle y|x\rangle = f^{n-1}(x)\langle y|f(x)x\rangle = -f^{n-1}(x)\langle y|\frac{1}{\rho}ux\rangle =$$

$$= -f^{n-1}(x)\langle\frac{1}{\rho}uy|x\rangle$$

for every $x \in B$. Since $\frac{1}{\rho}uy$ is $2(n-1)$- times continuously differentiable

$$\left(f^{n-1}(x)\langle\frac{1}{\rho}uy|x\rangle\right)_{x\in B} \in \ell^2(B)$$

by the inductive hypothesis. Hence

$$(f^n(x)\langle y|x\rangle)_{x\in B} \in \ell^2(B)\,. \qquad \blacksquare$$

Proposition 6.2.6.4 (5) *Let B be an orthonormal basis of $L^2(\rho\cdot\lambda)$ and take $f \in \mathbb{R}^B$ such that*

$$ux = -f(x)\rho x$$

for every $x \in B$. Take $h \in \ell^\infty(B)$ and let y be a $\mathcal{C}^\infty$-function on T. Then

$$\sum_{x\in B} h(x)\langle y|x\rangle x$$

is a $\mathcal{C}^\infty$-function on T.

By Proposition 6.2.6.3, if $n \in \mathbb{N}$, then

$$(f^n(x)h(x)\langle y|x\rangle)_{x\in B} \in \ell^2(B)\,,$$

and we define

$$y_n := (-1)^{n-1}\sum_{x\in B} f^{n-1}(x)h(x)\langle y|x\rangle x\,.$$

Then by Proposition 6.2.6.2 d),

$$\frac{1}{\rho}uy_n = (-1)^{n-1}\sum_{x\in B} f^{n-1}(x)h(x)\langle y|x\rangle\frac{1}{\rho}ux =$$

$$= (-1)^n\sum_{x\in B} f^n(x)h(x)\langle y|x\rangle x = y_{n+1}$$

for every $n \in \mathbb{N}$ and the assertion now follows from Proposition 6.2.6.1. ■

Theorem 6.2.6.5 (5) *Take $c \in]0,\infty]$ and let p, q be continuous real functions on $[0,c[$ with p strictly positive. Then for each $\mathcal{C}^\infty$-function h on T, there is a uniquely continuously differentiable map*

$$y : [0,c[\longrightarrow L^2(\rho\cdot\lambda)$$

such that $y(0) = h$, and $y(s)$ is a $\mathcal{C}^\infty$-function on T and

$$uy(s) = \rho\left(p(s)\frac{dy}{ds}(s) + q(s)y(s)\right)$$

whenever $s \in [0,c[$. If we define

$$z : T \times [0,c[\longrightarrow \mathbb{R}\,, \quad (t,s) \longmapsto (y(s))(t)\,,$$

then

$$uz = -\rho(p\frac{\partial z}{\partial s} + qz)$$

in the sense of distributions.

By Proposition 6.2.6.2 a),b), there is an orthonormal basis B of $L^2(\rho\cdot\lambda)$ such that every element of B is a $\mathcal{C}^\infty$–function on T and an $f \in \mathbb{R}^B$ such that $\overset{-1}{f}(]-\infty,0])$ is finite and

$$ux = -f(x)\rho x$$

for every $x \in B$. Define

$$P : [0,c[\longrightarrow \mathbb{R}\,, \quad s \longmapsto \int_0^s \frac{d\sigma}{p(\sigma)}\,,$$

$$Q : [0,c[\longrightarrow \mathbb{R}\,, \quad s \longmapsto \int_0^x \frac{q(\sigma)d\sigma}{p(\sigma)}\,,$$

and for each $x \in B$,

$$a_x : [0,c[\longrightarrow L^2(\rho\cdot\lambda)\,, \quad s \longmapsto \langle h|x\rangle e^{-f(x)P(s)-Q(s)}x\,.$$

By Proposition 6.2.6.3,

$$(f^n(x)\langle h|x\rangle)_{x\in B} \in \ell^2(B)$$

for every $n \in \mathbb{N}$. For each $n \in \mathbb{N}$, define

$$y_n : [0,c[\longrightarrow L^2(\rho\cdot\lambda)\,, \quad s \longmapsto (-1)^{n-1}\sum_{x\in B} f^{n-1}(x)a_x\,.$$

Then $y_1(0) = h$ and by Corollary 5.2.2.6, each y_n is continuous.

Take $s \in [0,c[$. Since

$$\frac{1}{\rho}ua_x(s) = -f(x)a_x$$

we see that

$$\frac{1}{\rho}uy_n(s) = y_{n+1}(s)$$

for every $n \in \mathbb{N}$. By Proposition 6.2.6.1, $y_1(s)$ is a $\mathcal{C}^\infty$–function on T. Given any $x \in B$,

$$\frac{d}{ds}\langle a_x(s)|x\rangle = -\frac{f(x)+q(s)}{p(s)}\langle a_x(s)|x\rangle\,.$$

By Proposition 5.5.1.20, y_1 is differentiable and

$$\frac{dy_1}{ds} = -\frac{1}{p}\sum_{x\in B}(f(x)+q)a_x = \frac{1}{p}y_2 - \frac{q}{p}y_1\,.$$

Hence

$$\rho(p\frac{dy_1}{ds} + qy_1) = uy_1 .$$

Now we prove the uniqueness. Assume that η has the properties described. Given $x \in B$, define

$$\eta_x : [0, c[\longrightarrow \mathbb{R}, \quad s \longmapsto \langle \eta(s)|x\rangle .$$

Take $s \in [0, c[$. Then

$$\eta(s) = \sum_{x\in B} \langle \eta(s)|x\rangle x = \sum_{x\in B} \eta_x(s)x ,$$

$$u\eta(s) = -\rho \sum_{x\in B} \eta_x(s) f(x)x$$

(Proposition 6.2.6.2 d)). Take $x \in B$. Then

$$f(x)\eta_x(s) = -\langle \frac{1}{\rho} u\eta(s)|x\rangle =$$

$$= -\langle p(s)\frac{d\eta}{ds}(s) + q(s)\eta(s)|x\rangle = -p(s)\eta_x'(s) - q(s)\eta_x(s) ,$$

$$p(s)\eta_x'(s) + (q(s) + f(x))\eta_x(s) = 0 .$$

Since

$$\eta_x(0) = \langle h|x\rangle ,$$

η_x is uniquely determined. It follows that η, too, is uniquely determined. ∎

Theorem 6.2.6.6 (5) *Take $c \in]0, \infty]$ and let p, q be continuous real functions on $[0, c[$. Then, given any C^∞-functions h_1, h_2 on T, there is a unique twice continuously differentiable map*

$$y : [0, c[\longrightarrow L^2(\rho\cdot\lambda)$$

such that

$$y(0) = h_1 , \quad \frac{dy}{ds}(0) = h_2$$

and for each $s \in [0, c[$, $y(s)$ is a C^∞-function on T and

$$uy(s) = \rho\left(\frac{d^2y}{ds^2}(s) + p(s)\frac{dy}{ds}(s) + q(s)y(s)\right).$$

Define

$$z : T \times [0, c[\longrightarrow \mathbb{R}, \quad (t, s) \longmapsto (y(s))(t).$$

Then

$$uz = \rho\left(\frac{\partial^2 z}{\partial s^2} + p\frac{\partial z}{\partial s} + qz\right)$$

in the sense of distributions.

By Proposition 6.2.6.2 a),b), there is an orthonormal basis B of $L^2(\rho\cdot\lambda)$ consisting of $\mathcal{C}^\infty$–functions on T and an $f \in \mathbb{R}^B$ such that $f^{-1}(]-\infty, 1])$ is finite and

$$ux = -f(x)\rho x$$

for every $x \in B$. By Proposition 6.2.6.3,

$$(f^n(x)\langle h_1|x\rangle)_{x\in B}, \quad (f^n(x)\langle h_2|x\rangle)_{x\in B} \in \ell^2(B)$$

for every $n \in \mathbb{N}$.

Given α, let y_α and z_α denote the twice differentiable real functions on $[0, c[$ which satisfy

$$y_\alpha'' + py_\alpha' + (q + \alpha)y_\alpha = 0,$$

$$z_\alpha'' + pz_\alpha' + (q + \alpha)z_\alpha = 0,$$

$$y_\alpha(0) = 1, \quad y_\alpha'(0) = 0, \quad z_\alpha(0) = 0, \quad z_\alpha'(0) = 1.$$

Given $x \in B$, define

$$a_x : [0, c[\longrightarrow L^2(\rho\cdot\lambda), \quad s \longmapsto (\langle h_1|x\rangle y_{f(x)}(s) + \langle h_2|x\rangle z_{f(x)}(s))x.$$

Take $c' \in]0, c[$ and define

$$\|\varphi\|_{(c')} := \sup_{s\in[0,c']} |\varphi(s)|$$

for every continuous real function φ on $[0, c[$. By Lemma 6.2.4.5, there is a $\beta \in \mathbb{R}_+$ such that

$$\|\langle q_x|x\rangle\|_{(c')} \leq \beta(|\langle g|x\rangle| + |\langle h|x\rangle|)\,,$$

$$\|\frac{d}{ds}\langle a_x|x\rangle\|_{(c')} \leq \beta(|\langle g|x\rangle| + |\langle h|x\rangle|)\sqrt{f(x)}$$

$$\|\frac{d^2}{ds^2}\langle a_x|x\rangle\|_{(c')} \leq \beta(|\langle g|x\rangle| + |\langle h|x\rangle|)(f(x) + \|p\|_{(c')}\sqrt{f(x)} + \|q\|_{(c')})$$

for every $x \in B$ with $f(x) \geq 1$. Hence $(f^n(x)a_x(s))_{x\in B}$ is summable whenever $s \in [0, c[$ and $n \in \mathbb{N} \cup \{0\}$. For each $n \in \mathbb{N}$ define

$$y_n : [0, c[\longrightarrow L^2(\rho\cdot\lambda)\,, \quad s \longmapsto (-1)^{n-1}\sum_{x\in B} f^{n-1}(x)a_x(s)\,.$$

By Corollary 5.2.2.6, y_n is continuous.

Take $s \in [0, c[$. Since

$$\frac{1}{\rho}ua_x(s) = -f(x)a_x$$

for every $x \in B$, it follows that

$$\frac{1}{\rho}uy_n(s) = y_{n+1}(s)$$

for every $n \in \mathbb{N}$ (Proposition 6.2.6.2 d)). By Proposition 6.2.6.1, $y_1(s)$ is a $\mathcal{C}^\infty$-function on T. If $x \in B$, then

$$\frac{d}{ds}\langle a_x(s)|x\rangle = \langle g|x\rangle y'_{f(x)}(s) + \langle h|x\rangle z'_{f(x)}(s)\,,$$

$$\frac{d^2}{ds^2}\langle a_x(s)|x\rangle = \langle g|x\rangle y''_{f(x)}(s) + \langle h|x\rangle z''_{f(x)}(s)$$

and so

$$\frac{d^2}{ds^2}\langle a_x(s)|x\rangle + p\frac{d}{ds}\langle a_x(s)|x\rangle + q\langle a_x(s)|x\rangle =$$

$$= -f(x)(\langle g|x\rangle y_{f(x)}(s) + \langle h|x\rangle z_{f(x)}(s)) = -f(x)\langle a_x(s)|x\rangle\,.$$

By Proposition 5.5.1.20, y_1 is twice continuously differentiable, $y_1(0) = h_1$, $\frac{dy_1}{ds}(0) = h_2$, and

$$\frac{d^2y_1}{ds^2} + p\frac{dy_1}{ds} + qy_1 = y_2\,.$$

Hence

$$\rho\left(\frac{d^2y_1}{ds^2}+p\frac{dy_1}{ds}+qy_1\right)=uy_1\,.$$

We now prove the uniqueness. Assume that η has the properties described. Given $x\in B$, define

$$\eta_x:[0,c[\longrightarrow \mathbb{R}\,,\quad s\longmapsto\langle\eta(s)|x\rangle\,.$$

Take $s\in[0,c[$. Then

$$\eta(s)=\sum_{x\in B}\langle\eta(s)|x\rangle x=\sum_{x\in B}\eta_x(s)x\,,$$

$$u\eta(s)=-\rho\sum_{x\in B}\eta_x(s)f(x)x$$

(Proposition 6.2.6.2 d)). Take $x\in B$. Then

$$f(x)\eta_x(s)=-\langle\frac{1}{\rho}u\eta(s)|x\rangle=$$

$$=-\langle\frac{d^2\eta}{ds^2}(s)+p(s)\frac{d\eta}{ds}(s)+q(s)\eta(s)|x\rangle=$$

$$=-\eta_x''(s)-p(s)\eta_x'(s)-q(s)\eta_x(s)\,,$$

$$\eta_x''+p\eta_x'+(q+f(x))\eta_x=0\,.$$

Since

$$\eta_x(0)=\langle h_1|x\rangle\,,\quad \eta_x'(0)=\langle h_2|x\rangle\,,$$

η_x is uniquely determined. It follows that η, too, is uniquely determined. ■

6.3 Von Neumann Algebras

This section is devoted to the study of von Neumann algebras. A von Neumann algebra on the Hilbert space H is a W^*-subalgebra of $\mathcal{L}(H)$. We first introduce a number of topologies on $\mathcal{L}(H)$ and establish several properties of these topologies, such as Kaplanski's Theorem. With the universal representation, this theorem shows that the bidual of a C^*-algebra has a natural W^*-algebra structure. This enables us to once again extend the functional calculus, this time to W^*-algebras. This extension is used to decompose self-normal operators with respect to bands of measures, in analogy with the manner employed with measures. The von Neumann algebras can be characterised by means of the abovementioned topologies, or by the order relation, or by commutants (von Neumann's Theorem). In the commutative case the last characterization leads a very elegant statement. The results allow us to characterize the irreducible representations (Segal's Theorem), to list all finite-dimensional C^*-algebras, and to prove that every reflexive C^*-algebra must be finite dimensional.

Finally, representations of W^*-algebras are examined. We prove that every W^*-algebra admits a faithful representation whose image is a von Neumann algebra.

In this section E is always a C^*-algebra and H is always a Hilbert space. We identify $\mathcal{L}(H)$ with $\mathcal{L}^1(H)'$ by means of the involutive isometry

$$\mathcal{L}(H) \longrightarrow \mathcal{L}^1(H)', \quad u \longmapsto \overset{1}{\widetilde{u}}$$

(Corollary 6.1.7.9). According to the notation introduced in Definition 1.2.6.1, $\mathcal{L}(H)_{\mathcal{L}^1(H)}$ and $\mathcal{L}(H)_{\mathcal{L}_f(H)}$ denote the set $\mathcal{L}(H)$ endowed with the topology of pointwise convergence in $\mathcal{L}^1(H)$ and $\mathcal{L}_f(H)$, respectively.

These topologies are usually called **the ultraweak operator topology** and **the weak operator topology** of $\mathcal{L}(H)$, respectively. This terminology is rather unfortunate, since the weak operator topology is weaker than the ultraweak operator topology. The topology of $\mathcal{L}(H)_H$ is called **the strong topology** of $\mathcal{L}(H)$. The topologies of $\mathcal{L}(H)_H$ and $\mathcal{L}(H)_{\mathcal{L}^1(H)}$ are finer than the topology of $\mathcal{L}(H)_{\mathcal{L}_f(H)}$ and

$$\mathcal{L}(H)^{\#}_{\mathcal{L}_f(H)} = \mathcal{L}(H)^{\#}_{\mathcal{L}^1(H)},$$

since $\mathcal{L}_f(H)$ is dense in $\mathcal{L}^1(H)$ (Theorem 6.1.3.12 d)). By the Alaoglu-Bourbaki Theorem, this space is compact. The usual norm topology of $\mathcal{L}(H)$ is finer than the topologies of $\mathcal{L}(H)_H$ and $\mathcal{L}(H)_{\mathcal{L}^1(H)}$.

6.3.1 The Strong Topology

Proposition 6.3.1.1 *If H is infinite-dimensional, then the maps*

$$\mathcal{L}(H)^{\#}_{\mathcal{L}^1(H)} \longrightarrow \mathcal{L}(H)^{\#}_H, \quad u \longmapsto u,$$

$$\mathcal{L}(H)_H \longrightarrow \mathcal{L}(H)_{\mathcal{L}^1(H)}, \quad u \longmapsto u,$$

$$\mathcal{L}(H)^{\#}_H \longrightarrow \mathcal{L}(H)^{\#}_H, \quad u \longmapsto u^*,$$

are not continuous.

Let $(x_n)_{n\in\mathbb{N}}$ be an orthonormal sequence in H.
For $p \in \mathbb{N}$, put

$$u_p : H \longrightarrow H, \quad x \longmapsto \sum_{n\in\mathbb{N}} \langle x|x_n\rangle x_{n+p}.$$

Then $(u_p)_{p\in\mathbb{N}}$ converges to 0 in $\mathcal{L}(H)^{\#}_{\mathcal{L}^1(H)}$ but does not converge to 0 in $\mathcal{L}(H)^{\#}_H$. Hence, the map

$$\mathcal{L}(H)^{\#}_{\mathcal{L}^1(H)} \longrightarrow \mathcal{L}(H)^{\#}_H, \quad u \longmapsto u$$

is not continuous.

Put

$$u := \sum_{n\in\mathbb{N}} \frac{1}{n^2} \langle\cdot|x_n\rangle x_n \in \mathcal{L}^1(H)$$

and for every $A \in \mathfrak{P}_f(H)$,

$$\mathcal{F}(A) := \{v \in \mathcal{L}(H) \mid x \in A \Longrightarrow vx = 0\}$$

and let $\mathfrak{F}$ denote the filter on $\mathcal{L}(H)$ generated by the filter base

$$\{\mathcal{F}(A) \mid A \in \mathfrak{P}_f(H)\}.$$

Then $\mathfrak{F}$ converges to 0 in $\mathcal{L}(H)_H$. Take $A \in \mathfrak{P}_f(H)$. There is a $p \in \mathbb{N}$, such that $x_p \notin A^{\perp\perp}$. Given $\alpha \in \mathbb{K}$, define

$$v_\alpha := \alpha\langle\cdot|(\pi_{A^\perp})x_p\rangle x_p\,.$$

Then $v_\alpha \in \mathcal{F}_A$ and

$$\tilde{v_\alpha}H)(u) = \alpha\sum_{n\in\mathbb{N}}\frac{1}{n^2}\langle v_\alpha x_n|x_n\rangle =$$

$$= \alpha\sum_{n\in\mathbb{N}}\frac{1}{n^2}\langle x_n|(\pi_{A^\perp})x_p\rangle\langle x_p|x_n\rangle = \frac{\alpha}{p^2}\|(\pi_{A^\perp})x_p\|^2$$

(Proposition 6.1.6.2, Theorem 5.5.1.15 c)). It follows that $\mathfrak{F}$ does not converge to 0 in $\mathcal{L}(H)_{\mathcal{L}^1(H)}$. Hence the map

$$\mathcal{L}(H)_H \longrightarrow \mathcal{L}(H)_{\mathcal{L}^1(H)}\,, \quad u \longmapsto u$$

is not continuous.

Let u_ℓ and u_r be the left and the right shift on ℓ^2, respectively. Then $u_\ell^* = u_r$ (Example 5.3.1.6), $(u_\ell^n)_{n\in\mathbb{N}}$ converges to 0 in $\mathcal{L}(H)_H^\#$ and $(u_r^n)_{n\in\mathbb{N}}$ does not converge in $\mathcal{L}(H)_H$. Hence the map

$$\mathcal{L}(H)_H^\# \longrightarrow \mathcal{L}(H)_H^\#\,, \quad u \longmapsto u^*$$

is not continuous. ■

Proposition 6.3.1.2 *The traces on $\mathcal{L}(H)_+^\#$ of the 0-neighbourhood filters with respect to the topologies $\mathcal{L}(H)_{\mathcal{L}^1(H)}$, $\mathcal{L}(H)_{\mathcal{L}_f(H)}$, and $\mathcal{L}(H)_H$ coincide.*

Take $x \in H$ and $\varepsilon > 0$. Put

$$\mathcal{U} := \{u \in \mathcal{L}(H)_+^\# \mid \|ux\| < \varepsilon\}\,.$$

$$\mathcal{V} := \{u \in \mathcal{L}(H)_+^\# \mid \widetilde{\langle\cdot|x\rangle x}(u)| < \varepsilon^2\}\,.$$

If $u \in \mathcal{V}$, then

$$0 \le u^2 \le u$$

(Corollary 4.2.7.2, Corollary 4.2.1.17 a $\Rightarrow$ b). Thus

$$\|ux\|^2 = \langle ux|ux\rangle = \langle u^2x|x\rangle \le \langle ux|x\rangle = \widetilde{\langle\cdot|x\rangle x}(u) < \varepsilon^2$$

(Corollary 5.3.3.7, Proposition 6.1.6.2). Hence, $u \in \mathcal{U}$ and $\mathcal{V} \subset \mathcal{U}$. Since x and ε are arbitrary, it follows that the traces on $\mathcal{L}(H)_+^\#$ of the 0-neighbourhood filter with respect to the topologies $\mathcal{L}(H)_H^\#$ and $\mathcal{L}(H)_{\mathcal{L}_f(H)}^\#$ coincide. The assertion now follows from

$$\mathcal{L}(H)_{\mathcal{L}_f(H)}^\# = \mathcal{L}(H)_{\mathcal{L}^1(H)}^\#\,.$$ ■

Proposition 6.3.1.3 (0) *The maps*

a) $\mathcal{L}(H)_H^{\#} \times \mathcal{L}(H)_H \longrightarrow \mathcal{L}(H)_H\,, \quad (u,v) \longmapsto uv\,,$

b) $\mathrm{No}\,\mathcal{L}(H)_H \longrightarrow \mathcal{L}(H)_H\,, \quad u \longmapsto u^*\,,$

c) $\mathcal{L}(H)_H \times \mathcal{L}(H)_H \to \mathcal{L}(H)_{\mathcal{L}_f(H)}\,, \quad (u,v) \mapsto v^*u\,,$

d) $\mathcal{L}(H)_H \to \mathcal{L}(H)_{\mathcal{L}_f(H)}\,, \quad u \mapsto u^*u\,.$

are continuous.

a) Take $(u_0, v_0) \in \mathcal{L}(H)^{\#} \times \mathcal{L}(H)$. Then

$$\|(uv - u_0v_0)x\| \leq \|u(v - v_0)x\| + \|(u - u_0)v_0x\| \leq$$

$$\leq \|(v - v_0)x\| + \|(u - u_0)v_0x\|$$

for every $(u,v) \in \mathcal{L}(H)^{\#} \times \mathcal{L}(H)$ and $x \in H$. The assertion now follows.

b) Take $u, v \in \mathrm{No}\,\mathcal{L}(H)$ and $x \in H$. Then

$$\|(u - v)x\|^2 = \langle (u - v)x \mid (u - v)x\rangle = \langle (u^* - v^*)(u - v)x \mid x\rangle =$$

$$= \langle (u^*u + v^*v)x|x\rangle - \langle u^*vx|x\rangle - \langle v^*ux|x\rangle\,,$$

so that

$$\|(u^* - v^*)x\|^2 - \|(u - v)x\|^2 =$$

$$= \langle u^*vx|x\rangle + \langle v^*ux|x\rangle - \langle uv^*x|x\rangle - \langle vu^*x|x\rangle =$$

$$= \langle vx|ux\rangle + \langle ux|vx\rangle - \langle x|vu^*x\rangle - \langle vu^*x|x\rangle\,.$$

Since

$$\lim_{v\to u}(\langle vx|ux\rangle + \langle ux|vx\rangle - \langle x|vu^*x\rangle - \langle vu^*x|x\rangle) =$$

$$= \langle ux|ux\rangle + \langle ux|ux\rangle - \langle x|uu^*x\rangle - \langle uu^*x|x\rangle = 0$$

we get that

$$\lim_{v\to u} \|(u^* - v^*)x\| = 0\,.$$

c) Take $u_0, v_0 \in \mathcal{L}(H)$, $x, y \in H$, and $\varepsilon > 0$. Put

$$\delta := \inf\left\{1, \frac{\varepsilon}{3(\|u_0 x\| + \|v_0 y\| + 1)}\right\}$$

$$U := \{u \in \mathcal{L}(H) \mid \|(u - u_0)x\| < \delta\},$$

$$V := \{v \in \mathcal{L}(H) \mid \|(v - v_0)x\| < \delta\}.$$

Then U and V are neigbourhoods of u_0 and v_0 in $\mathcal{L}(H)_H$, respectively. For $(u, v) \in U \times V$,

$$\left|\langle v^*u - v_0^*u_0, \overbrace{\langle \cdot | y\rangle x}^{\infty}\rangle\right| =$$

$$= |\langle (v^*u - v_0^*u_0)x | y\rangle| = |\langle ux | vy\rangle - \langle u_0 x | v_0 y\rangle| \leq$$

$$\leq |\langle (u - u_0)x \mid v_0 y\rangle| + |\langle u_0 x \mid (v - v_0)y\rangle| + |\langle (u - u_0)x \mid (v - v_0)y\rangle| \leq$$

$$\leq \|(u - u_0)x\| \, \|v_0 y\| + \|u_0 x\| \, \|(v - v_0)y\| + \|(u - u_0)x\| \, \|(v - v_0)y\| \leq$$

$$\leq \delta \|v_0 y\| + \|u_0 x\| \delta + \delta^2 < \varepsilon.$$

d) follows from c), since the diagonal map

$$\mathcal{L}(H)_H \longrightarrow \mathcal{L}(H)_H \times \mathcal{L}(H)_H, \quad x \longmapsto (x, x)$$

is continuous. ■

Proposition 6.3.1.4 (0) (Vigier, 1946) *If $\mathcal{F}$ is a nonempty downwards directed set of $\mathcal{L}(H)_+$ and $\mathfrak{F}$ is its lower section filter, then the following are equivalent:*

a) 0 *is the infimum of* $\mathcal{F}$.

b) *Given* $x \in H$,

$$\lim_{u,\mathfrak{F}} \langle ux | x\rangle = \inf_{u \in \mathcal{F}} \langle ux | x\rangle = 0.$$

c) $\mathfrak{F}$ *converges to* 0 *in* $\mathcal{L}(H)_H$.

We have

$$\widetilde{\overset{\infty}{\langle\cdot|x\rangle x}} \in \mathcal{L}(H)'_+$$

(Proposition 6.1.7.10 a $\Rightarrow$ e, Corollary 5.3.3.7) and

$$\langle ux|x\rangle = \widetilde{\overset{\infty}{\langle\cdot|x\rangle x}}(u)$$

for all $x \in H$ and $u \in \mathcal{L}(H)$ (Proposition 6.1.6.2).

a $\Rightarrow$ b follows from Theorem 4.4.1.8 b) (and Corollary 6.1.7.14 b)).

b $\Rightarrow$ c. Take $x \in H$. Then

$$\lim_{u,\mathfrak{F}} \widetilde{\langle\cdot|x\rangle x}(u) = \lim_{u,\mathfrak{F}} \langle ux|x\rangle = 0\,.$$

By Proposition 4.2.4.12,

$$0 = \lim_{u,\mathfrak{F}} \widetilde{\langle\cdot|x\rangle x}(u^2) = \lim_{u,\mathfrak{F}} \langle u^2x|x\rangle =$$

$$= \lim_{u,\mathfrak{F}} \langle ux|ux\rangle = \lim_{u,\mathfrak{F}} \|ux\|^2\,.$$

Since x is arbitrary, $\mathfrak{F}$ converges to 0 in $\mathcal{L}(H)_H$.

c $\Rightarrow$ a. Let v be the infimum of $\mathcal{F}$ (Theorem 4.4.1.8 c), Corollary 6.1.7.14 b)). Take $x \in H$. By a $\Rightarrow$ c,

$$\lim_{u,\mathfrak{F}} \|(u-v)x\| = 0\,.$$

We deduce that

$$\lim_{u,\mathfrak{F}} \langle (u-v)x|ux\rangle = \lim_{u,\mathfrak{F}} \langle (u-v)x|vx\rangle = 0\,.$$

Since

$$\|vx\|^2 = \langle vx|vx\rangle = \langle (v-u)x|vx\rangle + \langle ux|(v-u)x\rangle + \langle ux|ux\rangle$$

for every $v \in \mathcal{F}$, it follows that

$$vx = 0\,.$$

Since x is arbitrary, $v = 0$. ■

Proposition 6.3.1.5 (0) *Take* $u \in \mathcal{L}(H)^{\#}_+$ *and put*

$$F = \overline{\mathrm{Im}\, u}\,.$$

a) $0 < \alpha < \beta \Rightarrow u^\beta \leq u^\alpha$.

b) $\bigvee_{\alpha>0} u^\alpha = \lim_{\substack{\alpha\to 0\\ \alpha>0}} u^\alpha = \pi_F$ *in* $\mathcal{L}(H)_H$.

a) follows from Corollary 4.2.7.2 (and Corollary 4.2.1.17 a $\Rightarrow$ b).

b) By a) and Theorem 4.4.1.8 b) (and Corollary 6.1.7.14 b)), the supremum of $\{u^\alpha | \alpha > 0\}$ in $\mathcal{L}(H)$ exists and is equal to

$$\lim_{\substack{\alpha\to 0\\ \alpha>0}} u^\alpha$$

in $\mathcal{L}(H)_{\mathcal{L}^1(H)}$. Put

$$v := \bigvee_{\alpha>0} u^\alpha = \lim_{\substack{\alpha\to 0\\ \alpha>0}} u^\alpha .$$

By Proposition 6.3.1.4 a $\Rightarrow$ c,

$$v = \lim_{\substack{\alpha\to 0\\ \alpha>0}} u^\alpha$$

in $\mathcal{L}(H)_H$. By Proposition 6.3.1.3 a),

$$v^2 = \lim_{\substack{\alpha\to 0\\ \alpha>0}} u^{2\alpha} = v$$

in $\mathcal{L}(H)_H$. In particular, v is an orthogonal projection. We have

$$vx = \lim_{\substack{\alpha\to 0\\ \alpha>0}} u^\alpha x = 0$$

for every $x \in \operatorname{Ker} u$ and

$$0 = \|vx\|^2 = \langle vx|x\rangle = \lim_{\substack{\alpha\to 0\\ \alpha>0}} \langle u^\alpha x|x\rangle \geq \langle ux|x\rangle = \|u^{\frac{1}{2}}x\|^2$$

for every $x \in \operatorname{Ker} v$ (a) and Corollary 5.3.3.7). Hence

$$\operatorname{Ker} u = \operatorname{Ker} v$$

(Proposition 5.3.3.10 a)) and

$$\operatorname{Im} v = (\operatorname{Ker} v)^\perp = (\operatorname{Ker} u)^\perp = \overline{\operatorname{Im} u} = F$$

(Proposition 5.3.2.4). It follows that

$$v = \pi_F .$$

■

Proposition 6.3.1.6 (0) *Let A be a closed set of $\mathbb{R}$. Define*

$$\mathcal{F} := \{u \in \mathrm{Re}\,\mathcal{L}(H) \mid \sigma(u) \subset A\},$$

and let $f : A \to \mathbb{R}$ be a continuous function such that

$$\limsup_{\substack{\alpha\to\infty \\ \alpha\in A\setminus\{0\}}} |\frac{f(\alpha)}{\alpha}| < \infty$$

whenever A is unbounded. Then the map

$$\mathcal{F}_H \longrightarrow \mathcal{L}(H)_H\,, \quad u \longmapsto f(u)$$

is continuous.

Let $\mathcal{G}$ be the set of continuous functions $f : A \to \mathbb{R}$ for which the map

$$\mathcal{F}_H \longrightarrow \mathcal{L}(H)_H\,, \quad u \longmapsto f(u)$$

is continuous. Then $\mathcal{G}$ contains the maps

$$A \longrightarrow \mathbb{R}\,, \quad \alpha \longmapsto 1\,,$$

$$A \longrightarrow \mathbb{R}\,, \quad \alpha \longmapsto \alpha\,,$$

Step 1 $f, g \in \mathcal{G}$, g bounded $\Rightarrow fg \in \mathcal{G}$

Follows from Proposition 6.3.1.3 a).

Step 2 $\mathcal{G} \cap \mathcal{C}(A)$ is a closed set of $\mathcal{C}(A)$.

Take $f \in \overline{\mathcal{G} \cap \mathcal{C}(A)}$ and $\varepsilon > 0$. There is a $g \in \mathcal{G} \cap \mathcal{C}(A)$ such that

$$\|f - g\| < \varepsilon\,.$$

Given $u, v \in \mathcal{F}$ and $x \in H$,

$$\|(f(u) - f(v))x\| \leq$$

$$\leq \|(f(u) - g(u))x\| + \|(g(u) - g(v))x\| + \|(g(v) - f(v))x\| =$$

$$= \|((f - g)(u))x\| + \|(g(u) - g(v))x\| + \|(f - g)(v))x\| \leq$$

$$\leq 2\varepsilon\|x\| + \|(g(u) - g(v))x\| .$$

It follows that $f \in \mathcal{G} \cap \mathcal{C}(A)$.

Step 3 $f : A \to \mathbb{R}$, $\alpha \mapsto \frac{\alpha}{1+\alpha^2}$ belongs to $\mathcal{G}$.

Take $u, v \in \mathcal{F}$. Then

$$f(u) - f(v) = (1+u^2)^{-1}u - v(1+v^2)^{-1} =$$

$$= (1+u^2)^{-1}(u(1+v^2) - (1+u^2)v)(1+v^2)^{-1} =$$

$$= (1+u^2)^{-1}(u-v)(1+v^2)^{-1} + (1+u^2)^{-1}u(v-u)v(1+v^2)^{-1} .$$

Since

$$\|(1+u^2)^{-1}\| \leq 1 , \quad \|(1+u^2)^{-1}u\| \leq \frac{1}{2}$$

for every $u \in \mathcal{F}$, we get that

$$\|(f(u) - f(v))x\| \leq \|(u-v)x\| + \frac{1}{4}\|(u-v)x\| .$$

Hence,

$$\lim_{u \to v} f(u) = f(v) ,$$

and $f \in \mathcal{G}$.

Step 4 $f \in \mathcal{C}(A)$ & $\left\{ \begin{array}{l} 0 \in A \text{ and } A \text{ unbounded } \Rightarrow \\ \Rightarrow f(0) = \lim\limits_{A \ni \alpha \to \infty} f(\alpha) \end{array} \right\} \Rightarrow f \in \mathcal{G}$.

We set $\widetilde{A} := A$ if A is bounded, and denote by $\widetilde{A}$ the Alexandroff compactification of A whenever A is unbounded. Define

$$\widetilde{\mathcal{G}} := \{g \in \mathcal{C}(\widetilde{A}) \mid g|A \in \mathcal{G}\} .$$

By Step 1 and 2, $\mathcal{G}$ is a closed subalgebra of the real algebra $\mathcal{C}(\widetilde{A})$. By Step 3 and Theorem 1.3.5.14, $f \in \mathcal{G}$.

Step 5 The assertion

Let f be the function in the proposition. We may assume that

$$f(0) = 0 .$$

Given $n \in \mathbb{N}$, define

$$g_n : A \longrightarrow \mathbb{R}, \quad \alpha \longmapsto \frac{f(\alpha)\alpha^{n-1}}{1+\alpha^2}.$$

By Step 4, $g_1 \in \mathcal{G}$. By Step 1, we get successively that $g_2 \in \mathcal{G}$ and $g_3 \in \mathcal{G}$. Hence

$$f = g_1 + g_3 \in \mathcal{G}. \qquad \blacksquare$$

Proposition 6.3.1.7 (0) *Let u' be a linear form on $\mathcal{L}(H)$. Then the following are equivalent:*

a) $u' \in \mathcal{L}_f(H)$ *(where we invoke the identifications*

$$\mathcal{L}_f(H) \subset \mathcal{L}^1(H) \subset \mathcal{L}(H)').$$

b) u' *is continuous on* $\mathcal{L}(H)_H$.

a $\Rightarrow$ b is trivial.

b $\Rightarrow$ a. There is a finite family $(x_\iota)_{\iota \in I}$ in H such that

$$|u'(u)| \leq 1$$

for every $u \in \mathcal{L}(H)$ with

$$\sup_{\iota \in I} \|ux_\iota\| \leq 1.$$

It follows that

$$|u'(u)| \leq \left(\sum_{\iota \in I} \|ux_\iota\|^2 \right)^{\frac{1}{2}}$$

for every $u \in \mathcal{L}(H)$.

Given $u \in \mathcal{L}(H)$, put

$$\widetilde{u} := \bigoplus_{\iota \in I} u,$$

and let F denote the closure of

$$\{(ux_\iota)_{\iota \in I} \mid u \in \mathcal{L}(H)\}$$

in $\bigoplus_{\iota \in I} H$. Take $u, v \in \mathcal{L}(H)$ such that

$$(ux_\iota)_{\iota \in I} = (vx_\iota)_{\iota \in I}.$$

Then

$$u'(u) = u'(v)\,.$$

Hence, there is a linear form x' on F such that

$$x'((ux_\iota)_{\iota\in I}) = u'(u)$$

for every $u \in \mathcal{L}(H)$. Since x' is continuous, there is, by the Fréchet–Riesz Theorem, a family $(y_\iota)_{\iota\in I}$ in H, such that

$$x'((ux_\iota)_{\iota\in I}) = \sum_{\iota\in I}\langle ux_\iota|y_\iota\rangle$$

for every $u \in \mathcal{L}(H)$. It follows that

$$u'(u) = \sum_{\iota\in I}\langle ux_\iota|y_\iota\rangle = \langle u, \sum_{\iota\in I}\langle\cdot|y_\iota\rangle x_\iota\rangle$$

for every $u \in \mathcal{L}(H)$. Hence

$$u' = \sum_{\iota\in I}\langle\cdot|y_\iota\rangle x_\iota \in \mathcal{L}_f(H)\,. \qquad \blacksquare$$

Corollary 6.3.1.8 (0) *If $\mathcal{F}$ is a convex set of $\mathcal{L}(H)$, then its closure in $\mathcal{L}(H)_H$ and $\mathcal{L}(H)_{\mathcal{L}_f(H)}$ coincide.*

Let $\overline{\mathcal{F}}$ be the closure of $\mathcal{F}$ in $\mathcal{L}(H)_H$ and take $u \in \mathcal{L}(H)\backslash\overline{\mathcal{F}}$. There is a finite subset A of H such that

$$u + \mathcal{A} \subset \mathcal{L}(H)\backslash\overline{\mathcal{F}}\,,$$

where

$$\mathcal{A} := \{v \in \mathcal{L}(H) \mid x \in A \Longrightarrow \|vx\| < 1\}\,.$$

By Proposition 1.3.1.6, there is a linear form u' on $\mathcal{L}(H)$, which is bounded on $\mathcal{A}$, such that

$$\alpha := \sup_{v\in\overline{\mathcal{F}}} \operatorname{re} u'(v) < \operatorname{re} u'(u)\,.$$

u' is continuous in the topology of $\mathcal{L}(H)_H$. By Proposition 6.3.1.7, u' is continuous in the topology of $\mathcal{L}(H)_{\mathcal{L}_f(H)}$. Hence

$$\{v \in \mathcal{L}(H) \mid \operatorname{re} u'(v) \leq \alpha\}$$

is a closed set of $\mathcal{L}(H)_{\mathcal{L}_f(H)}$. Thus u does not belong to the closure of $\mathcal{F}$ in $\mathcal{L}(H)_{\mathcal{L}_f(H)}$ and this closure is contained in $\overline{\mathcal{F}}$. The reverse inclusion is trivial. $\blacksquare$

Proposition 6.3.1.9 (0) $\mathrm{Re}\,\mathcal{L}(H)$ *and* $\mathcal{L}(H)_+$ *are closed sets of* $\mathcal{L}(H)_{\mathcal{L}_f(H)}$.

Given $x, y \in H$, define

$$f_{x,y} : \mathcal{L}(H)_{\mathcal{L}_f(H)} \longrightarrow \mathbb{K}, \quad u \longmapsto \langle ux|y\rangle - \langle x|uy\rangle,$$

$$g_x : \mathcal{L}(H)_{\mathcal{L}_f(H)} \longrightarrow \mathbb{K}, \quad u \longmapsto \langle ux|x\rangle.$$

These functions are continuous. The assertion now follows from

$$\mathrm{Re}\,\mathcal{L}(H) = \bigcap_{x,y\in H} \overline{f}^{-1}_{x,y}(0)$$

and

$$\mathcal{L}(H)_+ = \bigcap_{x\in H} \overline{g}^{-1}_{x}(\mathbb{R}_+)$$

(Corollary 5.3.3.7). ■

Theorem 6.3.1.10 (0) (Kaplanski, 1951) *If* E *is a* C^**-subalgebra of* $\mathcal{L}(H)$, *then*

$$\overline{\mathrm{Re}\,E^{\#}} = \overline{E} \cap \mathrm{Re}\,\mathcal{L}(H)^{\#}, \quad \overline{E^{\#}_+} = \overline{E} \cap \mathcal{L}(H)^{\#}_+, \quad \overline{E^{\#}} = \overline{E} \cap \mathcal{L}(H)^{\#}$$

where closures are taken in $\mathcal{L}(H)_H$.

The inclusions from left to right are trivial, since the sets $\mathcal{L}(H)^{\#}$, $\mathrm{Re}\,\mathcal{L}(H)^{\#}$, and $\mathcal{L}(H)^{\#}_+$ are closed in $\mathcal{L}(H)_H$ (Proposition 6.3.1.9).

Take $v \in \overline{E} \cap \mathrm{Re}\,\mathcal{L}(H)^{\#}$ $\quad (v \in \overline{E} \cap \mathcal{L}(H)^{\#}_+)$. There is a filter $\mathfrak{F}$ on $\mathcal{L}(H)$ with $E \in \mathfrak{F}$ converging to v in $\mathcal{L}(H)_H$. Since the map

$$\mathcal{L}(H)_H \longrightarrow \mathcal{L}(H)_{\mathcal{L}_f(H)}, \quad u \longmapsto \mathrm{re}\,u$$

is continuous, $\mathrm{re}\,\mathfrak{F}$ converges to v in $\mathcal{L}(H)_{\mathcal{L}_f(H)}$. Hence v belongs to the closure of $\mathrm{Re}\,E$ in $\mathcal{L}(H)_{\mathcal{L}_f(H)}$. By Corollary 6.3.1.8, v belongs to the closure of $\mathrm{Re}\,E$ in $\mathcal{L}(H)_H$. Hence, there is a filter $\mathfrak{G}$ on $\mathcal{L}(H)$ with $\mathrm{Re}\,E \in \mathfrak{G}$ converging to v in $\mathcal{L}(H)_H$.

Define

$$f : \mathbb{R} \longrightarrow \mathbb{K}, \quad \alpha \longmapsto \sup\{-1, \inf\{\alpha, 1\}\} \quad (\text{resp. } \sup\{0, \inf\{\alpha, 1\}\}).$$

By Proposition 6.3.1.6, $f(\mathfrak{G})$ converges to $f(v)$ in $\mathcal{L}(H)_H$. Since

$$f(v) = v\,,$$

and since $\operatorname{Re} E^{\#}$ (resp. $E_+^{\#}$) belongs to $f(\mathfrak{G})$, it follows that

$$v \in \overline{\operatorname{Re} E^{\#}} \quad (v \in \overline{E_+^{\#}})\,.$$

Take $u \in \overline{E} \cap \mathcal{L}(H)^{\#}$ and define

$$H^{(2)} := \bigoplus_{i \in \mathbb{N}_2} H\,.$$

Let $E_{2,2}$ be the C^*-subalgebra of $\mathcal{L}(H^{(2)})$, comprising the 2×2-matrices with entries in E (Theorem 5.6.6.1 f)). Thus

$$\begin{bmatrix} 0 & u \\ u^* & 0 \end{bmatrix} \in \overline{E}_{2,2} \cap \operatorname{Re} \mathcal{L}(H^{(2)})\,.$$

By the above result, there is a filter $\mathfrak{F}$ on $\mathcal{L}(H^{(2)})$ containing $\operatorname{Re} E_{2,2}^{\#}$ and converging to

$$\begin{bmatrix} 0 & u \\ u^* & 0 \end{bmatrix}$$

in $\mathcal{L}(H^{(2)})_{H^{(2)}}$. Define

$$\varphi : \mathcal{L}(H^{(2)}) \longrightarrow \mathcal{L}(H)\,, \quad v \longmapsto v_{12}$$

(Theorem 5.6.6.1 f)). Then $\varphi(\mathfrak{F})$ is a filter on $\mathcal{L}(H)$ containing $E^{\#}$ (Theorem 5.6.6.1 a)) and converging to u in $\mathcal{L}(H)_H$. Hence

$$u \in \overline{E^{\#}}\,. \qquad \blacksquare$$

Corollary 6.3.1.11 *Let $\overline{E}$ be the closure in $\mathcal{L}(H)_H$ of the C^*-subalgebra E of $\mathcal{L}(H)$. Take $u \in \mathcal{L}^1(H)$. Then*

$$\|\overset{\infty}{\widetilde{u}}|E\| = \|\overset{\infty}{\widetilde{u}}|\overline{E}\|\,.$$

Since

$$\mathcal{L}(H)^{\#}_{\mathcal{L}_f(H)} = \mathcal{L}(H)^{\#}_{\mathcal{L}^1(H)}\,,$$

the restriction of $\overset{\infty}{\widetilde{u}}$ to $\mathcal{L}(H)^{\#}_{\mathcal{L}_f(H)}$ is continuous. Hence, the restriction of $\overset{\infty}{\widetilde{u}}$ to $\mathcal{L}(H)^{\#}_H$ is continuous. By Kaplanski's Theorem,

$$\|\overset{\infty}{\widetilde{u}}|E\| = \|\overset{\infty}{\widetilde{u}}|\overline{E}\|\,. \qquad \blacksquare$$

Corollary 6.3.1.12 (0) *The closures of a C^*-subalgebra E of $\mathcal{L}(H)$ in $\mathcal{L}(H)_H$, $\mathcal{L}(H)_{\mathcal{L}^1(H)}$, $\mathcal{L}(H)_{\mathcal{L}_f(H)}$ coincide and are the W^*-subalgebra of $\mathcal{L}(H)$ generated by E (Corollary 6.1.7.14 b)).*

Let F (resp. G) be the closure of E in $\mathcal{L}(H)_{\mathcal{L}^1(H)}$ (in $\mathcal{L}(H)_{\mathcal{L}_f(H)}$). By Corollary 6.3.1.8, G is the closure of E in $\mathcal{L}(H)_H$. Take

$$u \in G \cap \mathcal{L}(H)^{\#}.$$

By Theorem 6.3.1.10 and Corollary 6.3.1.8, u belongs to the closure of $E^{\#}$ in $\mathcal{L}(H)_{\mathcal{L}_f(H)}$. Since

$$\mathcal{L}(H)^{\#}_{\mathcal{L}_f(H)} = \mathcal{L}(H)^{\#}_{\mathcal{L}^1(H)},$$

$u \in F$.

By Corollary 4.4.2.9, F is a C^*-subalgebra of $\mathcal{L}(H)$ and by Corollary 4.4.4.12 a), it is the W^*-subalgebra of $\mathcal{L}(H)$ generated by E. ■

Corollary 6.3.1.13 *Take $A \in \{\mathbb{C}, \mathbb{H}\}$. Let H be a Hilbert right A-module and $\widetilde{H}$ the real Hilbert space obtained by endowing the underlying real vector space of H with the scalar product*

$$H \times H \longrightarrow \mathbb{R}, \quad (x, y) \longmapsto \mathrm{re}\langle x|y\rangle$$

(Proposition 5.6.2.5 a),e)). Then $\mathcal{L}_A(H)$ is a unital W^*-subalgebra of $\mathcal{L}(\widetilde{H})$.

By Proposition 5.6.2.5 b),c),e), $\mathcal{L}_A(H)$ is a strongly closed unital C^*-subalgebra of $\mathcal{L}(\widetilde{H})$. By Corollary 6.3.1.12, $\mathcal{L}_A(H)$ is a unital W^*- subalgebra of $\mathcal{L}(\widetilde{H})$. ■

Corollary 6.3.1.14 *Let $\mathbb{K} = \mathbb{C}$. If E is a unital C^*-subalgebra of $\mathcal{L}(H)$, then*

$$\{u \in \overline{E} \mid u \text{ is unitary } \} \subset \overline{\{u \in E \mid u \text{ is unitary}\}},$$

where the closures are taken in $\mathcal{L}(H)_H$.

By Corollary 6.3.1.12, $\overline{E}$ is a W^*-subalgebra of $\mathcal{L}(H)$.

Let u be a unitary element of $\overline{E}$. By Corollary 4.3.2.7 (and Theorem 4.4.1.8 c)), there ia a $v \in \mathrm{Re}\,\overline{E}$ such that

$$u = e^{iv}.$$

By Kaplanski's Theorem, there is a filter $\mathfrak{F}$ on $\mathcal{L}(H)$ containing $\operatorname{Re} E$ and converging to v in $\mathcal{L}(H)_H$. By Proposition 6.3.1.6,

$$u = e^{iv} = \lim_{w,\mathfrak{F}} e^{iw} \in \overline{\{w \in E \mid w \text{ is unitary}\}}$$

(Corollary 4.1.3.10). ■

Proposition 6.3.1.15 *Let u be a self-normal operator on H, $\mathfrak{B}$ the σ-algebra of Borel sets of $\sigma(u)$, $\mathcal{B}$ the Banach space of bounded Borel functions on $\sigma(u)$, and μ the spectral measure of u (Corollary 6.1.7.14 b), Corollary 4.4.1.11 a)). Given $\xi, \eta \in H$, define*

$$\mu_{\xi,\eta} : \mathfrak{B} \longrightarrow \mathbb{K}, \quad A \longmapsto \langle \mu(A)\xi | \eta \rangle .$$

a) *$\mu_{\xi,\eta}$ is a measure for all $\xi, \eta \in H$.*

b) *If $f \in \mathcal{B}$, then*

$$\langle f(u)\xi | \eta \rangle = \int f d\mu_{\xi,\eta} ,$$

$$\|f(u)\xi\|^2 = \int |f|^2 d\mu_{\xi,\xi}$$

for all $\xi, \eta \in H$.

c) *If $(f_n)_{n\in\mathbb{N}}$ is a bounded sequence in $\mathcal{B}$ converging pointwise to an $f \in \mathcal{B}$, then $(f_n(u))_{n\in\mathbb{N}}$ converges strongly to $f(u)$.*

d) *A closed vector subspace K of H reduces u iff $\pi_K \in \mu(\mathfrak{B})^c$. In particular, $\operatorname{Im}\mu(A)$ reduces u for every $A \in \mathfrak{B}$.*

a) $\mu_{\xi,\eta}$ is obviously additive. Let $(A_n)_{n\in\mathbb{N}}$ be a decreasing sequence in $\mathfrak{B}$ with empty intersection. Then $(\mu(A_n))_{n\in\mathbb{N}}$ is a drecreasing sequence in $\mathcal{L}(H)$. with infimum 0. By Theorem 5.3.3.14 b $\Rightarrow$ c,

$$\lim_{n\to\infty} \mu_{\xi,\eta}(A_n) = 0 .$$

Hence $\mu_{\xi,\eta}$ is a measure.

b) The first equality follows from Theorem 4.4.1.8 f) (and Corollary 6.1.7.14 b)). Then

$$\|f(u)\xi\|^2 = \langle f(u)\xi | f(u)\xi \rangle = \langle f(u)^* f(u)\xi | \xi \rangle =$$

$$= \langle |f|^2(u)\xi|\xi\rangle = \int |f|^2 d\mu_{\xi,\xi}\,.$$

c) follows from the second equality of b).

d) K reduces u iff $\pi_K \in \{u\}^c$ (Corollary 5.2.3.12 b)). The assertion now follows from

$$\mu(\mathfrak{B}) \subset \mu(\mathfrak{B})^c = \{u\}^c$$

(Corollary 4.4.1.11 d)). ■

Proposition 6.3.1.16 *If E is a C^*-subalgebra of $\mathcal{L}(H)$ acting non-degenerately on H, then any approximate unit of E converges strongly to 1_H.*

The assertion follows immediately from Proposition 5.3.2.21. ■

6.3.2 Bidual of a C^*-algebra

Theorem 6.3.2.1 (0) (Takeda, 1954) *Let E be a (unital) C^*-algebra, (H, φ) a faithful representation of E such that every continuous linear form on $\varphi(E)$ is continuous on $\varphi(E)_{\mathcal{L}^1(H)}$ (e.g. the complex universal representation of E), and $\overline{\varphi(E)}$ the strong closure of $\varphi(E)$.*

a) *The map*

$$E \longrightarrow \varphi(E), \quad x \longmapsto \varphi x$$

is an isometry of C^*-algebras.

b) *The Arens multiplications of E'' (Definition 2.2.7.13) coincide, E'' endowed with this multiplication is a W^*-algebra. j_E is an involutive (unital) algebra homomorphism, $\overline{\varphi(E)}$ is the W^*-subalgebra of $\mathcal{L}(H)$ generated by $\varphi(E)$, and there is a W^*-isometry $u : E'' \to \overline{\varphi(E)}$ such that*

$$u j_E x = \varphi x$$

for every $x \in E$.

c) *If E is Gelfand, then so is E'' and $\sigma(E'')$ is a compact hyperstonian space.*

d) *$E''^{\#}$, $\operatorname{Re} E''^{\#}$, and $E''^{\#}_+$ are the closures in $E''_{E'}$ of $j_E(E^{\#})$, $j_E(\operatorname{Re} E^{\#})$ and $j_E(E^{\#}_+)$, respectively. Moreover*

$$j_{E'}(E'_+) \subset E'''_+ .$$

If (H, φ) is the complex universal representation of E then by Proposition 5.4.2.6 b), φ is faithful, and by Corollary 6.1.6.3, every continuous linear form on $\varphi(E)$ is continuous on $\varphi(E)_{\mathcal{L}^1(H)}$.

a) follows from Theorem 4.2.6.6.

b) By Kaplanski's Theorem, $\varphi(E)^{\#}$ is dense in $\overline{\varphi(E)}^{\#}_H$, and by Corollary 6.3.1.12, $\overline{\varphi(E)}$ is the closure of $\varphi(E)$ in $\mathcal{L}(H)_{\mathcal{L}^1(H)}$ and the W^*-subalgebra of $\mathcal{L}(H)$ generated by $\varphi(E)$. By Theorem 4.4.4.18 h), the Arens multiplications on $\varphi(E)''$ coincide, $\varphi(E)''$ endowed with this multiplication is a W^*-algebra, and there is a W^*-isometry $v : \varphi(E)'' \to \overline{\varphi(E)}$ such that $v \circ j_{\varphi(E)}$ is the inclusion map $\varphi(E) \to \overline{\varphi(E)}$. Define

$$\psi : E \longrightarrow \varphi(E), \quad x \longmapsto \varphi x,$$

$$u := v \circ \psi'' .$$

By a), the Arens multiplications of E'' coincide, E'' endowed with this multiplication is a W^*–algebra, j_E is an involutive (unital) algebra homomorphism, and $u : E'' \to \overline{\varphi(E)}$ is a W^*–isometry such that

$$u \circ j_E = v \circ \psi'' \circ j_E = v \circ j_{\varphi(E)} \circ \psi$$

(Proposition 1.3.6.16). Hence

$$u j_E x = \varphi x$$

for every $x \in E$.

c) follows from Theorem 4.4.4.18 a) and Corollary 4.4.1.10.

d) By a) and b), we may indentify E and E'' with $\varphi(E)$ and $\overline{\varphi(E)}$. The first assertion follows from Kaplanski's theorem (Theorem 6.3.1.10), since on bounded sets of $\mathcal{L}(H)$ the strong topology is finer than the topology of $\mathcal{L}(H)_{\mathcal{L}^1(H)}$, which induces on E the topology of $E_{E'}$.

Take $x' \in E'_+$. Then

$$\langle j_E x, j_{E'} x' \rangle = \langle j_E x, x' \rangle = \langle x, x' \rangle \in \mathbb{R}_+$$

for every $x \in E_+$. By the above,

$$\langle x'', j_{E'} x' \rangle \in \mathbb{R}_+$$

for every $x'' \in E''_+$, i.e. $j_{E'} x' \in E'''_+$ and $j_{E'}(E'_+) \subset E'''_+$. ■

Remark. Take $(x', x'') \in E' \times E''$. The multiplication $x'x''$ defined in Chapter 4 using the above b) coincides with the multiplication defined in Chapter 2. Indeed, let us denote by $x' \top x''$ and $x' \perp x''$ these two multiplications, respectively. Then for every $y'' \in E''$,

$$\langle y'', x' \top x'' \rangle = \langle y'', (j_{E'} x') x'' \rangle = \langle x'' y'', j_{E'} x' \rangle =$$

$$= \langle x'' y'', x' \rangle = \langle x'' \dashv y'', x' \rangle = \langle y'', x' \perp x'' \rangle ,$$

so that $x' \top x'' = x' \perp x''$.

Theorem 6.3.2.2 (0) *Assume E is a W^*–algebra and let F be its predual. Denote by $\mathfrak{A}$ the set of downward directed sets of $\operatorname{Pr} E$ with infimum 0 and put (Theorem 6.3.2.1 b) and Theorem 4.4.1.8 i))*

$$\mathfrak{B} := \{A \in \mathfrak{A} \mid -A \text{ is well-ordered}\},$$
$$R := \{\bigwedge_{q \in A} j_E q \mid A \in \mathfrak{A}\} \subset \Pr E'',$$
$$S := \{\bigwedge_{q \in A} j_E q \mid A \in \mathfrak{B}\} \subset \Pr E''.$$

a) $j'_F : E'' \to E$ *is a unital* W^**-homomorphism (Theorem 6.3.2.1 b)),* j_F *is its pretranspose, and the canonical projection of the tridual of* F $(= E'')$ *(Proposition 1.3.6.19 b)) is an involutive unital algebra homomorphism.*

b) $(\operatorname{Im} j_F)^\circ = \operatorname{Ker} j'_F$ *is a closed ideal of* $E''_{E'}$.

c) *There is a* $p \in E''^c \cap \Pr E''$ *such that*

$$(\operatorname{Im} j_F)^\circ = pE''p.$$

d) $a'' \in F'' \Rightarrow pa'' = a''p$.

e) *The map*

$$F'' \longrightarrow F'', \quad a'' \longmapsto a'' - pa''$$

is a projection of F'' *onto* $\operatorname{Im} j_F$. $E''(1-p)$ *is a closed ideal of* $E''_{F''}$ *with predual* $(1-p)F'' = \operatorname{Im} j_F$ *and*

$$F'' \longrightarrow \operatorname{Im} j_F, \quad a'' \longmapsto a''(1-p)$$

the pretranspose of the inclusion map $E''(1-p) \to E''$. *The map*

$$\varphi : E''(1-p) \longrightarrow E, \quad x'' \longmapsto j'_F x''$$

is an isomorphism of W^**-algebras with pretranspose*

$$F \longrightarrow \operatorname{Im} j_F, \quad a \longmapsto j_F a.$$

f) p *is the supremum of* R *and of* S *in* $\Pr E''$ *and in* $E''^{\#}_{+}$.

g) $$\operatorname{Im} j_F = \left\{ x' \in E' \;\middle|\; \begin{array}{l} \text{the maps } E \to E',\ x \mapsto xx' (\text{resp. } x'x) \\ \text{are order continuous.} \end{array} \right\}$$

i.e. $\operatorname{Im} j_F = \overbrace{E'}^{\cdot}$, *where* E' *is considered an* E*-module (Definition 4.3.1.7).*

h) *The following are equivalent:*

h_1) *E is reflexive.*

h_2) $p = 0$.

h_3) *j_E is a W^*-homomorphism.*

h_4) *The canonical projection of the tridual of F is a W^*-homomorphism.*

i) (Takesaki, 1959) *The following are equivalent for every $x' \in E'_+$:*

i_1) $x'p = x'$.

i_2) *For every $q \in \Pr E \backslash \{0\}$, there is an $r \in \Pr E$ with*

$$0 < r \leq q, \quad x'(r) = 0.$$

j) (Takesaki, 1959) *Assume $\mathbb{K} = \mathbb{C}$ ($\mathbb{K} = \mathbb{R}$). The following are equivalent for every $x' \in E'$ ($x' \in \operatorname{Re} E'$):*

j_1) $x'p = 0$.

j_2) $x' \in \operatorname{Im} j_F$.

j_3) *x' is order continuous.*

j_4) *$x'|G$ is order continuous for every maximal Gelfand C^*-subalgebra G of E.*

j_5) *For every $A \in \mathfrak{B}$,*

$$\lim_{q,\mathfrak{F}} \langle q, x' \rangle = 0,$$

where $\mathfrak{F}$ denotes the lower section filter of A.

k) *If $x' \in \operatorname{Re} E'$ is order constinuous then x'^+, x'^-, and $|x'|$ are also order continuous.*

a) By Corollary 4.4.2.10, F is an involutive E–submodule of E'. By Theorem 2.2.7.15 f_4) (and Theorem 4.4.1.8 h)), j'_F and the canonical projection of the tridual of F are algebra homomorphisms. By Proposition 2.3.2.22 d),e), j_F and j'_F are involutive, so that the canonical projection of the tridual of F is also involutive. It is easy to see that j'_F is unital. Hence, by Theorem 6.3.2.1 b), the canonicaml projection of the tridual of F is also unital. Since the map

$$j'_F : E''_{F''} \longrightarrow E_F$$

is continuous, it follows that j'_F is a W^*– homomorphism (Proposition 4.4.4.6). It is obvious that j_F is the pretranspose of j'_F (Corollary 4.4.4.8 c)).

b) follows from a) since

$$(\operatorname{Im} j_F)^\circ = \operatorname{Ker} j_F' .$$

c) follows from b) and Corollary 4.4.4.12 d).

d) follows from c).

e) By c), for $a \in F$,

$$p(j_F a) = 0 ,$$

so that

$$j_F a = j_F a - p(j_F a) .$$

Let $a'' \in F''$ such that

$$pa'' = 0 .$$

By c), for every $x'' \in (\operatorname{Im} j_F)^\circ$,

$$\langle a'', x'' \rangle = \langle a'', x''p \rangle = \langle pa'', x'' \rangle = 0 .$$

Hence

$$a'' \in {}^\circ((\operatorname{Im} j_F)^\circ) = \operatorname{Im} j_F$$

(Proposition 1.3.5.7). It follows that the map

$$F'' \longrightarrow F'', \quad a'' \longmapsto a'' - pa''$$

is a projection of F'' onto $\operatorname{Im} j_F$.

The second assertion follows from c) and Corollary 4.4.4.12 c),d). By a), φ is a W^*-homomorphism with the given pretranspose and by b) and c), φ is injective. From

$$x = j_F' j_E x$$

for every $x \in E$ (Proposition 1.3.6.19 a)) it follows that φ is surjective.

f) Step 1 $r \in R \Rightarrow r \leq p$

By c), we have to show $r \in (\operatorname{Im} j_F)^\circ$. Take $a \in F$. We must prove

$$\langle r, j_F a \rangle = 0 .$$

In the real case we have

$$\langle r, j_F a - j_F a^* \rangle = \langle r, j_F a^* - j_F a \rangle$$

and therefore

$$\langle r, j_F a - j_F a^* \rangle = 0\,,$$

so that we may assume $a \in \operatorname{Re} F$ if $\mathbb{K} = \mathbb{R}$. Let $A \in \mathfrak{A}$ with

$$r = \bigwedge_{q \in A} j_E q$$

and let $\mathfrak{F}$ be the lower section filter of A. By Corollary 4.4.4.3 a $\Leftrightarrow$ b,

$$\langle r, j_F a \rangle = \lim_{q,\mathfrak{F}} \langle j_E q, j_F a \rangle = \lim_{q,\mathfrak{F}} \langle q, j_F a \rangle =$$

$$= \lim_{q,\mathfrak{F}} \langle q, a \rangle = 0\,.$$

Step 2 $p = \bigvee_{s \in S} s$, where the supremum is taken in $\Pr E''$

By Theorem 4.4.1.8 i), $\bigvee_{s \in S} s$ is well–defined. Put

$$p_0 := p - \bigvee_{s \in S} s$$

and assume $p_0 \neq 0$. By Step 1, p_0 is positive. By Corollary 4.4.1.5, there is an $a'' \in F''_+$ with

$$\langle p_0, a'' \rangle > 0\,.$$

By Corollary 4.3.1.5 d $\Rightarrow$ a,

$$p_0 a'' p_0 > 0\,.$$

Take $A \in \mathfrak{B}$ and put

$$s := \bigwedge_{q \in A} j_E q\,.$$

Then

$$s p_0 = 0$$

(Corollary 4.2.7.6 a $\Rightarrow$ c), so that

$$0 = \langle s, p_0 a'' p_0 \rangle = \lim_{q,\mathfrak{F}} \langle j_E q, p_0 a'' p_0 \rangle ,$$

where $\mathfrak{F}$ denote the lower section filter of A. By Theorem 4.4.4.2 c $\Rightarrow$ a, $p_0 a'' p_0 \in \operatorname{Im} j_F$. Hence

$$0 = \langle p, p_0 a'' p_0 \rangle = \langle p_0 p p_0, a'' \rangle = \langle p_0, a'' \rangle ,$$

which is a contradiction. It follows

$$p_0 = 0 , \quad p = \bigvee_{s \in S} s .$$

Step 3 p is the supremum of S in $E_+''^{\#}$

Take $S_0 \in \mathfrak{P}_f(S)$. By Proposition 4.2.7.21 c), the supremum of S_0 in $\Pr E''$ is also the supremum of S_0 in $E_+'^{\#}$. By Theorem 4.4.1.8 i), p is the supremum of S in $E_+''^{\#}$.

Step 4 p is the supremum of R in $E_+''^{\#}$

The assertion follows from Step 1 and Step 3.

g) Take $x' \in E'$ such that the maps

$$E \longrightarrow E' , \quad x \longmapsto xx' \quad (\text{resp. } x'x)$$

are order continuous. Let $A \in \mathfrak{A}$ and put

$$r := \bigwedge_{q \in A} j_E q \in \Pr E'' .$$

By Corollary 4.4.3.10,

$$rx' = \lim_{q,\mathfrak{F}} (j_E q) x' = \lim_{q,\mathfrak{F}} q x' = 0 ,$$
$$x'r = \lim_{q,\mathfrak{F}} x' (j_E q) = \lim_{q,\mathfrak{F}} x' q = 0 ,$$

where $\mathfrak{F}$ denotes the lower section filter of A.

Put

$$G := \{ x'' \in E'' \mid x''x' = x''^* x' = x'x'' = x'x''^* = 0 \} .$$

By Proposition 4.3.4.14, G is a hereditary C^*-subalgebra of E'', which is obviously closed in $E''_{E'}$. By the above, $R \subset G$. By f) and Proposition 4.4.4.11 c), $p \in G$. By c),

$$(\operatorname{Im} j_F)^\circ = pEp \subset G$$

so that

$$x' \in {}^\circ G \subset {}^\circ((\operatorname{Im} j_F)^\circ) = \operatorname{Im} j_F$$

(Proposition 1.3.5.7).

The reverse inclusion follows from Corollary 4.4.3.10.

$h_1 \Rightarrow h_2$. By Proposition 1.3.8.4, F is reflexive. Hence

$$(\operatorname{Im} j_F)^\circ = E'^\circ = \{0\}.$$

By c), $p = 0$.

$h_2 \Rightarrow h_3$. By c),

$$(\operatorname{Im} j_F)^\circ = \{0\}$$

so that

$$\operatorname{Im} j_F = {}^\circ((\operatorname{Im} j_F)^\circ) = F''$$

(Proposition 1.3.5.7). Take $a'' \in F''$. By the above, there is an $a \in F$ with $j_F a = a''$. Then

$$\langle a'', j_E x\rangle = \langle j_F a, j_E x\rangle = \langle j_F a, x\rangle = \langle a, x\rangle$$

for every $x \in E$ and the map

$$j_E : E_F \longrightarrow E''_{E'}$$

is continuous, i.e. j_E is a W^*–homomorphism (Proposition 4.4.4.6).

$h_3 \Rightarrow h_4$ follows from a).

$h_4 \Rightarrow h_1$. By Corollary 4.4.4.8 b) and Proposition 1.3.6.19 b), $\operatorname{Im} j_E$ is a W^*–subalgebra of E'', i.e. it is closed in $E''_{E'}$. But $\operatorname{Im} j_E$ is dense in $E''_{E'}$ (Corollary 1.3.6.5), so that j_E is surjective. Hence E is reflexive.

$i_1 \Rightarrow i_2$. We may assume $x'(q) > 0$. There is a $y' \in \operatorname{Im} j_F$ with

$$y'(q) > x'(q)$$

(Corollary 4.4.1.5). Put

$$P := \{r \in \operatorname{Pr} E \mid r \leq q\,,\ y'(r) \leq x'(r)\}.$$

Take a totally ordered subset Q of P and define

$$r_0 := \bigvee_{r \in Q} r .$$

Then

$$y'(r_0) = \sup_{r \in Q} y'(r) \leq \sup_{r \in Q} x'(r) \leq x'(r_0)$$

(Theorem 4.4.1.8 c),i)), so that $r_0 \in P$. Hence P is inductive. By Zorn's Lemma, there is a maximal element r of P. Since $q \notin P$, it follows $q - r > 0$. By the maximality of r,

$$y'(s) > x'(s)$$

for every $s \in \Pr E$, $0 < s \leq q - r$. Since $(q-r)E(q-r)$ is a W^*-algebra (Proposition 4.4.2.2), it follows

$$0 \leq (q-r)x'(q-r) \leq (q-r)y'(q-r)$$

(Theorem 4.4.1.8 f)). By Corollary 4.4.2.13,

$$(q-r)x'(q-r) \in \operatorname{Im} j_F .$$

By e),

$$(q-r)x'(q-r) = (1-p)(q-r)x'(q-r) = 0 ,$$

so that

$$x'(q-r) = \langle 1, (q-r)x'(q-r) \rangle = 0 .$$

$i_2 \Rightarrow i_1$. By e), $x'(1-p) \in \operatorname{Im} j_F$. Let q be the carrier of $x'(1-p)$ (Corollary 4.4.2.17). Then

$$\langle r, x'(1-p) \rangle = 0$$

for every $r \in \Pr E$, $r \leq q$. By i_2), $x'(1-p) = 0$ so that $x' = x'p$.

$j_1 \Rightarrow j_2$ follows from e).

$j_2 \Rightarrow j_3$ follows from Theorem 4.4.1.8 c).

$j_3 \Rightarrow j_4$ follows from Corollary 4.4.2.7.

$j_4 \Rightarrow j_5$. Since A is commutative (Corollary 4.2.7.6 a $\Rightarrow$ b), there is a maximal Gelfand C^*-subalgebra G of E containing A (Corollary 4.1.2.3). Hence

$$\lim_{q,\mathfrak{F}} \langle q, x' \rangle = 0 .$$

$j_5 \Rightarrow j_1$. We may assume $x' \in \operatorname{Re} E$. By e), $x'(1-p) \in \operatorname{Im} j_F$, so that, by $j_2 \Rightarrow j_5$,

$$\lim_{q \in \mathfrak{F}} \langle q, x'(1-p) \rangle = 0$$

for every $A \in \mathfrak{B}$, where $\mathfrak{F}$ denotes the lower section filter of A. The same assertion follows therefore for $x'p$.

Let $q \in \operatorname{Pr} E$ and let $\mathfrak{P}$ be the set of well-ordered subsets P of $\operatorname{Pr} E$ such that

$$r \leq q\,, \quad \langle r, |x'|p \rangle = 0$$

for every $r \in P$. We order $\mathfrak{P}$ by

$$P \leq Q$$

iff $P \subset Q$ and

$$(r, s) \in P \times Q\,,\ s \leq r \Longrightarrow s \in P\,.$$

By Zorn's Lemma, $\mathfrak{P}$ has a maximal element P. By $i_1 \Rightarrow i_2$,

$$\bigvee_{r \in P} r = q\,.$$

For every $r \in P$,

$$0 \leq \langle r, x'^{+}p \rangle \leq \langle r, |x'|p \rangle = 0\,,$$

$$0 \leq \langle r, x'^{-}p \rangle \leq \langle r, |x'|p \rangle = 0\,,$$

so that

$$\langle r, x'p \rangle = 0\,.$$

Let $\mathfrak{F}$ be the upper section filter of P. Then

$$\{q - r | r \in P\} \in \mathfrak{B}$$

and so

$$0 = \lim_{r, \mathfrak{F}} \langle q - r | x'p \rangle = \langle q, x'p \rangle\,.$$

Since q is arbitrary, it follows $x'p = 0$ (Theorem 4.4.1.8 f)).

k) follows from $j_3 \Rightarrow j_2$ and Theorem 4.4.3.9. ■

Remark. E is reflexive iff it is finite-dimensional (Theorem 6.3.6.15).

Corollary 6.3.2.3 (0) *If E, F are C^*-algebras and $u : E \to F$ is an involutive algebra homomorphism, then u'' is a W^*-homomorphism and $\operatorname{Im} u'$ is closed. If F has a predual G, then $j'_G \circ u''$ is the unique W^*-homomorphism $E'' \to F$ such that*

$$j'_G \circ u'' \circ j_E = u .$$

By Theorem 6.3.2.1 b), E'' and F'' are W^*-algebras and by Proposition 4.4.4.6, u'' is a W^*-homomorphism. By Theorem 4.2.6.6, $\operatorname{Im} u''$ is closed, so that by Proposition 1.4.2.11, $\operatorname{Im} u'$ is also closed. By Theorem 6.3.2.2 a), j'_G is a W^*-homomorphism, so that $j'_G \circ u''$ is also a W^*-homomorphism. By Proposition 1.3.6.16 and 1.3.6.19 a),

$$j'_G \circ u'' \circ j_E = j'_G \circ j_{G'} \circ u = u .$$

The uniqueness follows from the fact that $\operatorname{Im} j_E$ is dense in $E''_{E'}$ (Corollary 1.3.6.5). ■

Corollary 6.3.2.4 a) *If F is a C^*-subalgebra of the C^*-algebra E and $i : F \to E$ is the inclusion map, then F^{00} is the W^*-subalgebra of E'' generated by $j_E(F)$, $\operatorname{Im} i'' = F^{00}$, and the map*

$$F'' \longrightarrow F^{00}, \quad y'' \longmapsto i''y''$$

is an isometry of W^*-algebras. In particular, if F is also Gelfand, then F^{00} is also Gelfand.

b) *If G is a W^*-subalgebra of E'', then ${}^{00}G = \overset{-1}{j_E}(G)$ is a C^*-subalgebra of E and $({}^{00}G)^{00}$ is the W^*-subalgebra of E'' generated by $G \cap j_E(E)$.*

a) By Corollary 6.3.2.3, i'' is a W^*-homomorphism. By Proposition 1.3.6.14, F^{00} is the closure of $j_E(F)$ in $E''_{E'}$ and by Proposition 1.3.6.17, $\operatorname{Im} i'' = F^{00}$ and the map

$$F'' \longrightarrow F^{00}, \quad y'' \longmapsto i''y''$$

is an isometry of W^*-algebra. By Corollary 4.4.4.12 a), F^{00} is the W^*-subalgebra of E'' generated by $j_E(F)$. If F is Gelfand, then by Theorem 2.2.7.15 g), F'' is Gelfand. Thus, F^{00} is Gelfand.

b) Since

$$({}^0G)^0 = G$$

(Proposition 1.3.5.4), we have

$$^{00}G = \overset{-1}{j_E}(G)\,.$$

By Theorem 6.3.2.1 b), $\overset{-1}{j_E}(G)$ is a C^*–subalgebra of E. It follows that

$$G \cap j_E(E) = j_E(\overset{-1}{j_E}(G)) = j_E(^{00}G)\,.$$

and so, by a), $(^{00}G)^{00}$ is the W^*–subalgebra of E'' generated gy $G \cap j_E(E)$. ■

Corollary 6.3.2.5 *If E'' is simple (Theorem 6.3.2.1 b)), then E is simple.*

Let F be a closed ideal of E and $q : E \to E/F$ the quotient map. Then q'' is a W^*–homomorphism (Corollary 6.3.2.3, Theorem 4.2.6.5) and

$$F^{00} = (\operatorname{Ker} q)^{00} = \operatorname{Ker} q''$$

(Theorem 4.2.6.6). Hence F^{00} is a closed ideal of E'' and therefore

$$F^{00} = \{0\} \text{ or } E''\,,$$

$$F = \{0\} \text{ or } E\,.$$

Thus E is simple. ■

Example 6.3.2.6 (0) *Let T be a locally compact space. Put $E := \mathcal{C}_0(T)$, and let $\mathcal{B}$ be the C^*-algebra of bounded Borel functions on T. We identify E' canonically with the involutive Banach space of bounded Radon measures on T and for each $x \in \mathcal{B}$, we define*

$$\widetilde{x} : E' \longrightarrow \mathbb{K}\,, \quad x' \longmapsto \int x dx'\,.$$

a) *$\widetilde{x} \in E''$, for every $x \in \mathcal{B}$.*

b) *$\widetilde{x}x' = x \cdot x'$, for every $x \in \mathcal{B}$ and $x' \in E'$.*

c) *The map*

$$\mathcal{B} \longrightarrow E''\,, \quad x \longmapsto \widetilde{x}$$

is an injective involutive algebra homomorphism (Theorem 6.3.2.1 b)) such that $\bigwedge\limits_{n\in\mathbb{N}} \widetilde{x}_n = 0$ for any decreasing sequence $(x_n)_{n\in\mathbb{N}}$ in $\mathcal{B}$ whose infimum is 0.

d) *If* $(x_\iota)_{\iota\in I}$ *is an upward directed family of bounded lower semicontinuous real functions on* T *whose supremum* x *is bounded, then*

$$\widetilde{x} = \bigvee_{\iota\in I} \widetilde{x}_\iota .$$

e) *If* A *is a closed set of* T *and*

$$G := \{x \in E \mid xe_A = 0\},$$

then

$$\{x'' \in E'' \mid x''\widetilde{e}_A = 0\}$$

is the order faithful vector subspace of E'' *generated by* $j_E(G)$.

f) E'' *is the order faithful vector subspace of* E'' *generated by* $j_E(E)$.

a) is obvious.

b) Given $y \in E$,

$$\langle y, \widetilde{x}x'\rangle = \langle \widetilde{x}, yx'\rangle = \langle \widetilde{x}, y\cdot x'\rangle =$$

$$= \int x d(y\cdot x') = \int xy dx' = \int y d(x\cdot x') = \langle y, x\cdot x'\rangle .$$

Hence

$$\widetilde{x}x' = x\cdot x' .$$

c) The map is obviously injective and linear. Take $x \in \mathcal{B}$. Then

$$\widetilde{x^*}(x') = \int x^* dx' = \overline{\int x dx'^*} = \overline{\widetilde{x}(x'^*)} = \widetilde{x}^*(x')$$

for every $x' \in E'$ and thus

$$\widetilde{x^*} = \widetilde{x}^* .$$

Hence the map is involutive.

Take $x, y \in \mathcal{B}$. Then by b) (and Theorem 6.3.2.1 b),c)), for every $x' \in E'$,

$$\langle x', \widetilde{xy}\rangle = \int xy dx' = \int x d(y\cdot x') = \int x d(\widetilde{y}x') =$$

$$= \langle \widetilde{y}x', \widetilde{x}\rangle = \langle x', \widetilde{x}\widetilde{y}\rangle .$$

Hence

$$\widetilde{xy} = \widetilde{x}\widetilde{y}$$

and therefore the map is an algebra homomorphism.

The final claim follows from integration theory.

d) follows from integration theory.

e) Let G_0 be an order faithful vector subspace of E'' containing $j_E(G)$. By c) and d), G_0 contains $\{\widetilde{x} \mid x \in \mathcal{B},\ xe_A = 0\}$. Let $\mathfrak{B}$ be the set of Borel sets of T which are disjoint from A. Let U be a clopen set of $\sigma(E'')$ (Theorem 6.3.2.1 b),c)) such that

$$e_U \widetilde{e}_A = 0$$

and take $x', y' \in E'_+$ with

$$\operatorname{Supp} x' \subset U\,, \quad U \cap \operatorname{Supp} y' = \emptyset\,,$$

where we have identified E' with $(E'')^\pi$ (Corollary 4.4.4.3 a $\Leftrightarrow$ b). Then

$$x' \wedge y' = 0\,.$$

Thus there is a $B_{y'} \in \mathfrak{B}$ such that

$$x'(T \backslash B_{y'}) = y'(B_{y'}) = 0$$

(Hahn's Theorem). By c), $\widetilde{e}_{B_{y'}} \in \operatorname{Pr} E''$, and so $\{\widetilde{e}_{B_{y'}} \neq 0\}$ is a clopen set of $\sigma(E'')$. Hence

$$\operatorname{Supp} x' \subset \{\widetilde{e}_{B_{y'}} \neq 0\}\,, \quad \{\widetilde{e_{B_{y'}}} \neq 0\} \cap \operatorname{Supp} y' = \emptyset\,.$$

Let $V_{x'}$ denote the clopen set of $\sigma(E'')$ such that

$$e_{V_{x'}} = \bigwedge_{y'} \widetilde{e}_{B_{y'}}\,,$$

where y' runs through the set of elements of E'_+ with

$$U \cap \operatorname{Supp} y' = \emptyset\,.$$

Then $e_{V_{x'}} \in G$,

$$\operatorname{Supp} x' \subset V_{x'}\,,$$

and by Example 1.7.2.14 d),

$$V_{x'} \subset U\,.$$

Again, by Example 1.7.2.14 d),

$$e_U = \bigvee_{x'} e_{V_{x'}}\,,$$

where x' runs through the set of elements of E'_+ with

$$\operatorname{Supp} x' \subset U\,.$$

Hence $e_U \in G$. Since U is arbitrary, it follows that

$$\{x'' \in E'' \mid x''\widetilde{e}_A = 0\} \subset G\,.$$

Thus

$$\{x'' \in E'' \mid x''\widetilde{e}_A = 0\}$$

is the order faithful vector subspace of E'' generated by $j_E(G)$.

f) follows from e) by putting $A := \emptyset$. ■

Example 6.3.2.7 (0) *Let T be a compact space, and put $E := \mathcal{C}(T)$. We canonically identify T and $\sigma(E)$ (Example 2.4.3.1) and define*

$$\varphi : \sigma(E'') \longrightarrow \sigma(E) = T\,, \quad x''' \longmapsto j'_E x''' = x''' \circ j_E$$

(Theorem 6.3.2.1 a), Proposition 2.4.1.17 a)),

$$S := \{x''' \in \sigma(E'') \mid \{x'''\} \text{ is open}\}\,.$$

For $t \in T$ we write t' for the Dirac measure on T at t and define

$$t'' : E' \longrightarrow \mathbb{K}\,, \quad x' \longmapsto x'(\{t\})\,,$$

$$t''' := j_{E'} t'\,.$$

a) $t''x' = t''(x')t'$ *for every* $(t, x') \in T \times E'$.

b) $t'x'' = x''(t')t'$ *and* $t''x'' = x''(t')t''$ *for every* $(t, x'') \in T \times E''$.

c) $t''' \in S$ *and* $\varphi(t''') = t$, *for every* $t \in T$.

d) $\overline{\varphi^{-1}(t)} \backslash \{t'''\}$ *is meager for every* $t \in T$.

e) *The map*

$$S \longrightarrow T\,, \quad s \longmapsto \varphi(s)$$

is bijective.

f) *Given* $x' \in E'$, *let* x'_a *denote its atomic part (as Radon measure). Take* $f \in \ell^\infty(T)$, *and define*

$$x'' : E' \longrightarrow \mathbb{K}\,, \quad x' \longmapsto \int f dx'_a\,.$$

Then $x'' \in E''$, $\|x''\| = \|f\|_\infty$, $\widehat{x''} = f \circ \varphi$ *on* S, *and* $\widehat{x''} = 0$ *on* $\sigma(E'')\backslash\overline{S}$. *if* f *is continuous then* $\widehat{x''} = e_{\overline{S}}(f \circ \varphi)$.

a) Take $x \in E$. Then

$$\langle x, t''x'\rangle = \langle xx', t''\rangle = (xx')(\{t\}) = x(t)x'(\{t\}) = t''(x')\langle x, t'\rangle\,,$$

so that

$$t''x' = t''(x')t'\,.$$

b) Take $x \in E$. Then

$$\langle x, t'x''\rangle = \langle xt', x''\rangle = \langle x(t)t', x''\rangle =$$

$$= x(t)\langle t', x''\rangle = x''(t')\langle x, t'\rangle\,,$$

so that

$$t'x'' = x''(t')t'\,.$$

By a),

$$\langle x', t''x''\rangle = \langle x't'', x''\rangle = t''(x')\langle t', x''\rangle = x''(t')\langle x', t''\rangle$$

for every $x' \in E'$. Hence

$$t''x'' = x''(t')t''\,.$$

c) By b),

$$\langle x''y'', t'''\rangle = \langle x''y'', t'\rangle = \langle y'', t'x''\rangle = x''(t')\langle y'', t'\rangle =$$

$$= \langle x'', t'''\rangle\langle y'', t'''\rangle$$

for all $x'', y'' \in E''$. Thus $t''' \in \sigma(E'')$. Then

$$\langle t'', t''' \rangle = \langle t''', t' \rangle = t'(\{t\}) = 1 .$$

Take $s \in \sigma(E'')$, $s \neq t'''$. There is an $x'' \in E''$ such that

$$t'''(x'') = 1 , \quad s(x'') = 0$$

(Theorem 2.4.1.13). By b),

$$0 = s(t'')s(x'') = s(t''x'') = x''(t')s(t'') = s(t'') .$$

Hence

$$\{t'''\} = \{t'' \neq 0\} ,$$

$$t''' \in S .$$

Take $x \in E$. Then

$$\langle x, \delta_{\varphi(t''')} \rangle = \langle x, j'_E t''' \rangle \langle j_E x, t''' \rangle = \langle j_E x, t' \rangle = \langle x, t' \rangle .$$

Hence

$$\varphi(t''') = t .$$

d) Assume that $\overset{-1}{\varphi}(t) \backslash \{t'''\}$ is not meager. Since $\sigma(E'')$ is hyperstonian (Theorem 6.3.2.1 c)), there is an $x''' \in \varphi(E'')^{\pi}_{+}$ such that

$$\operatorname{Supp} x''' \subset \overset{-1}{\varphi}(t) \backslash \{t'''\} , \quad \text{and } \|x'''\| = 1 .$$

By Corollary 4.4.4.3 b $\Rightarrow$ a, there is an $x' \in E'$ such that

$$x''' = j_{E'} x' .$$

Then

$$\langle x, x' \rangle = \langle x, j'_E j_{E'} x' \rangle = \langle x, j'_E x''' \rangle = \langle j_E x, x''' \rangle = \int (x \circ \varphi) dx''' =$$

$$= x(t) = \langle x, t' \rangle$$

for every $x \in E$ (Proposition 1.3.6.19 a)), so that

$$x' = t'$$

$$x''' = j_{E'}t' = t''',$$

which is a contradiction.

e) By c), the map is surjective, and by d), it is injective.

f) It is easy to see that $x'' \in E''$, and $\|x''\| = \|f\|_\infty$. By c)

$$\widehat{x''}(t''') = \langle x'', t''' \rangle = \langle x'', t' \rangle = f(t) = f \circ \varphi(t''')$$

for every $t \in T$. Thus, by e), $\widehat{x''} = f \circ \varphi$ on S.

Assume that $\widehat{x''}$ does not vanish identically on $\sigma(E'') \backslash \overline{S}$. Since $\sigma(E'')$ is hyperstonian (Theorem 6.3.2.1 c)), there is an $x''' \in (E'')^\pi$ such that

$$\int \widehat{x''} dx''' \neq 0, \quad \text{and } \overline{S} \cap \operatorname{Supp} x''' = \emptyset.$$

By Corollary 4.4.4.3 b $\Rightarrow$ a, there is an $x' \in E'$ such that

$$x''' = j_{E'}x'.$$

Since

$$\overline{S} \cap \operatorname{Supp} j_{E'}x' = \emptyset,$$

x' is atomless, and we get the contradiction

$$0 = \langle x'', x' \rangle = \langle x'', j_{E'}x' \rangle = \langle x'', x''' \rangle = \int \widehat{x''} dx''' \neq 0.$$

The last assertion follows by continuity. ■

Example 6.3.2.8 (0) *Let T be a locally compact space, $\mathcal{C} := \mathcal{C}_0(T)$, $\mathfrak{U}$ the (order complete) lattice of clopen sets of $\sigma(\mathcal{C}'')$ (Theorem 6.3.2.1 b),c)), and $\mathfrak{N}$ the (order complete) lattice of bands of $\mathcal{C}'$. We identify $\operatorname{Im} j_{\mathcal{C}'}$ with $(\mathcal{C}'')^\pi$ (Corollary 4.4.4.3 a $\Leftrightarrow$ b) and for each $U \in \mathfrak{U}$ we define*

$$\widetilde{U} := \{\mu \in \mathcal{C}' \mid \operatorname{Supp} \mu' \subset U\}.$$

Then $\widetilde{U} \in \mathfrak{N}$ for every $U \in \mathfrak{U}$ and the map

$$\mathfrak{U} \longrightarrow \mathfrak{N}, \quad U \longmapsto \widetilde{U}$$

is an isomorphism of lattices. If we identify $\mathcal{C}(\sigma(\mathcal{C}''))$ with $\mathcal{C}''$ via the Gelfand transform, then for every $U \in \mathfrak{U}$ and every $\mu \in \mathcal{C}'$ the measure $e_U\mu$ is the projection of μ on the band $\widetilde{U}$.

By Theorem 6.3.2.1 b),c), $\mathcal{C}''$ is a Gelfand W^*-algebra and $\sigma(\mathcal{C}'')$ is a hyperstonian compact space. The first assertion follows from Example 1.7.2.16, while the second one follows from the relation

$$j_{\mathcal{C}'}(e_U\mu) = e_U(j_{\mathcal{C}'}\mu)$$

(Corollary 4.4.2.10). ■

Proposition 6.3.2.9 *Let T be a locally compact space. Let $\mathfrak{M}$ be the set of all sets $\mathcal{M}$ of positive bounded Radon measures on T such that $\|\mu\| = 1$ for every $\mu \in \mathcal{M}$ and $\mu \wedge \nu = 0$ for all distinct $\mu, \nu \in \mathcal{M}$. Let further $\mathcal{M}$ be a maximal element of $\mathfrak{M}$, where $\mathfrak{M}$ is ordered by inclusion. Put*

$$L := \{x \in \prod_{\mu\in\mathcal{M}} L^1(\mu) \mid \sum_{\mu\in\mathcal{M}} \|x_\mu\| < \infty\}$$

and denote by M the C^*-direct product of the family of C^*-algebras $(L^\infty(\mu))_{\mu\in\mathcal{M}}$.

a) *L is an involutive vector subspace of $\prod_{\mu\in\mathcal{M}} L^1(\mu)$ and L is an involutive Banach space with respect to the norm*

$$L \longrightarrow \mathbb{R}_+, \quad x \longmapsto \sum_{\mu\in\mathcal{M}} \|x_\mu\|.$$

b) *If we identify $\mathcal{C}_0(T)'$ in the usual way with the involutive Banach space of bounded Radon measures on T, then the map*

$$L \longrightarrow \mathcal{C}_0(T)', \quad x \longmapsto \sum_{\mu\in\mathcal{M}} x_\mu \cdot \mu$$

is an isometry of involutive Banach spaces. We shall identify L and $\mathcal{C}_0(T)'$ using this map.

c) *Given $y \in M$, define*

$$\widetilde{y} : L \longrightarrow \mathbb{K}, \quad x \longmapsto \sum_{\mu\in\mathcal{M}} \int y_\mu d(x_\mu \cdot \mu).$$

Then $\widetilde{y} \in \mathcal{C}_0(T)''$ for every $y \in M$ and the map

$$M \longrightarrow \mathcal{C}_0(T)'', \quad y \longmapsto \widetilde{y}$$

is an isometry of C^*-algebras (Theorem 6.3.2.1 b)).

a) is easy to verify.

b) The map is obviously linear and involutive. Take $x \in L$ and $\varepsilon > 0$. There is a finite subset $\mathcal{N}$ of $\mathcal{M}$, such that

$$\sum_{\mu \in \mathcal{M} \setminus \mathcal{N}} \|x_\mu\| < \frac{\varepsilon}{3} .$$

Let $\mathcal{B}$ be the Banach space of bounded Borel functions on T. Given $\mu \in \mathcal{N}$, there is a $y_\mu \in \mathcal{B}^\#$ such that

$$\|x_\mu\| = \int y_\mu d(x_\mu \cdot \mu) .$$

By Hahn's Theorem, there is a $y \in \mathcal{B}^\#$ such that

$$y = y_\mu \qquad \mu\text{–a.e.}$$

for every $\mu \in \mathcal{N}$. Since $\mathcal{C}_0(T)$ is dense in $\mathcal{C}_0(T)''_{\mathcal{C}_0(T)'}$, there is a $z \in \mathcal{C}(T)$ such that

$$\left| \int (y_\mu - z) d(x_\mu \cdot \mu) \right| < \frac{\varepsilon}{3(1 + \operatorname{Card} \mathcal{N})}$$

for every $\mu \in \mathcal{N}$. We may assume that $z \in \mathcal{C}(T)^\#$, replacing z by

$$T \longrightarrow \mathbb{K} , \quad t \longmapsto \frac{z(t)}{\sup\{|z(t)|, 1\}} .$$

Then

$$\| \sum_{\mu \in \mathcal{M}} x_\mu \cdot \mu \| \geq \left| \langle z, \sum_{\mu \in \mathcal{M}} x_\mu \cdot \mu \rangle \right| \geq$$

$$\geq \sum_{\mu \in \mathcal{N}} \langle y_\mu, x_\mu \cdot \mu \rangle - \sum_{\mu \in \mathcal{N}} |\langle z - y_\mu, x_\mu \cdot \mu \rangle| - \sum_{\mu \in \mathcal{M} \setminus \mathcal{N}} |\langle z, x_\mu \cdot \mu \rangle| \geq$$

$$\geq \sum_{\mu \in \mathcal{N}} \|x_\mu\| - \frac{\varepsilon \operatorname{Card} \mathcal{N}}{3(1 + \operatorname{Card} \mathcal{N})} - \sum_{\mu \in \mathcal{M} \setminus \mathcal{N}} \|x_\mu\| > \sum_{\mu \in \mathcal{M}} \|x_\mu\| - \varepsilon .$$

Hence

$$\| \sum_{\mu \in \mathcal{M}} x_\mu \cdot \mu \| \geq \sum_{\mu \in \mathcal{M}} \|x_\mu\| .$$

The converse inequality being trivial, the map in b) preserves norms.

We now show that the map is surjective. Take $\nu \in \mathcal{C}_0(T)'_+$. By the Radon–Nikodym Theorem, for every $\mu \in \mathcal{M}$, there is an $x_\mu \in L^1(\mu)_+$ such that

$$\nu \wedge \mu = x_\mu \cdot \mu .$$

Then

$$\|\nu \wedge \mu\| = \|x_\mu\|_1$$

for every $\mu \in \mathcal{M}$. Since $\mathcal{M}$ is maximal,

$$\nu = \sum_{\mu \in \mathcal{M}} \nu \wedge \mu = \sum_{\mu \in \mathcal{M}} x_\mu \cdot \mu .$$

hence the map of b) is surjective.

c) It is easy to see that $\widetilde{y} \in \mathcal{C}_0(T)''$ for every $y \in M$ and that the map

$$M \longrightarrow \mathcal{C}_0(T)'', \quad y \longmapsto \widetilde{y}$$

is linear, involutive, and injective. We show that this map is surjective. Take $x'' \in \mathcal{C}_0(T)''$. For every $\mu \in \mathcal{M}$, the map

$$L^1(\mu) \longrightarrow \mathbb{K}, \quad x \longmapsto \langle x \cdot \mu, x'' \rangle$$

is linear and continuous with norm at most $\|x''\|$. Hence there is a $y_\mu \in L^\infty(\mu)$ such that

$$\langle x \cdot \mu, x'' \rangle = \int y_\mu x d\mu$$

for every $x \in L^1(\mu)$. We have $\|y_\mu\| \leq \|x''\|$ for every $\mu \in \mathcal{M}$, so

$$y := (y_\mu)_{\mu \in \mathcal{M}} \in M .$$

For every $x \in L$,

$$\langle x, x'' \rangle = \sum_{\mu \in \mathcal{M}} \langle x_\mu \cdot \mu, x'' \rangle = \sum_{\mu \in \mathcal{M}} \int y_\mu d(x_\mu \cdot \mu) = \langle x, \widetilde{y} \rangle$$

and therefore $\widetilde{y} = x''$. This proves the surjectivity of the map. By Theorem 4.2.6.6, the map is an isometry of C^*–algebras. ■

Corollary 6.3.2.10 *Assume* $\mathbb{K} = \mathbb{R}$. *Let* T *be a locally compact space and* $\mathcal{F}$ *an order faithful set of* $\mathcal{C}_0(T)''$ *containing* $\mathcal{C}_0(T)$ *(identified with a subset of* $\mathcal{C}_0(T)''$*). Then* $\mathcal{F} = \mathcal{C}_0(T)''$.

If we identify the C^*–algebra $\mathcal{B}$ of real bounded Borel functions on T with a C^*–subalgebra of $\mathcal{C}_0(T)''$ in the usual way, then $\mathcal{B} \subset \mathcal{F}$.

Let $\mathfrak{M}$ be the set of all sets $\mathcal{M}$ of positive, bounded Radon measures on T with

$$\|\mu\| = 1\,,$$

for every $\mu \in \mathcal{M}$ and

$$\mu \wedge \nu = 0$$

for distinct $\mu, \nu \in \mathcal{M}$. If we order $\mathfrak{M}$ by the inclusion relation, then by Zorn's Lemma, $\mathfrak{M}$ has a maximal element $\mathcal{M}$.

Take $x'' \in \mathcal{C}_0(T)''_+$. By Proposition 6.3.2.9 (using the notation of this proposition), there is an $y \in M$ such that

$$\widetilde{y} = x''\,.$$

For every finite subset $\mathcal{N}$ of $\mathcal{M}$, the set

$$\{z \in \mathcal{B} \mid \mu \in \mathcal{N} \Longrightarrow z \in y_\mu\}$$

is downward directed. Its infimum $z_\mathcal{N}$ in $\mathcal{C}_0(T)''$ belongs to $\mathcal{F}$. We have

$$(z_\mathcal{N})_\mu = y_\mu$$

for every $\mu \in \mathcal{N}$ and

$$(z_\mathcal{N})_\mu = 0$$

for every $\mu \in \mathcal{M}\backslash\mathcal{N}$. The set $\{z_\mathcal{N} \mid \mathcal{N} \in \mathfrak{P}_f(\mathcal{M})\}$ is upward directed and its supremum in $\mathcal{C}_0(T)''$ is x''. Hence $x'' \in \mathcal{F}$ and

$$\mathcal{F} = \mathcal{C}_0(T)''\,. \qquad \blacksquare$$

Proposition 6.3.2.11 (0) *Let E be a C^*–algebra and take $x' \in E'$ and $y' \in E'_+$. Then the following are equivalent:*

a) *$y' = |x'|$, where x' is considered as an element of the predual of the W^*–algebra E'' (Theorem 6.3.2.1 b)).*

b) *$\|x'\| = \|y'\|$ and*

$$|x'(x)|^2 \leq \|x'\| y'(xx^*)$$

for every $x \in E$.

a $\Rightarrow$ b. Let (x'', y') be the polar representation of x' (Theorem 4.4.3.5 a)). Then $\|x'\| = \|y'\|$ and

$$|x'(x)|^2 = |\langle x, x''y'\rangle|^2 = |\langle xx'', y'\rangle|^2 \leq y'(xx^*)y'((x'')^*x'') \leq \|x'\|y'(xx^*)$$

for every $x \in E$ (Proposition 2.3.4.6 c)).

b $\Rightarrow$ a. Let (H, φ) be the complex universal representation of E. By Proposition 5.4.2.6 d), there is a $\xi \in H$, such that $\|\xi\| = \|y'\|^{\frac{1}{2}}$ and

$$y'(x) = \langle(\varphi x)\xi|\xi\rangle$$

for every $x \in E$. We identify E with $\varphi(E)$ (Proposition 5.4.2.6 b)) and E'' with the closure of E in $\mathcal{L}(H)_{\mathcal{L}^1(H)}$ (Theorem 6.3.2.1 a)), which is the same as its strong closure (Corollary 6.3.1.12). By b),

$$|x'(x^*)|^2 \leq \|x'\|y'(x^*x) = \|x'\|\langle x^*x\xi|\xi\rangle = \|x'\|\,\|x\xi\|^2$$

for every $x \in E$. Take $x'' \in (E'')^{\#}$. By Kaplanski's Theorem , there is a filter $\mathfrak{F}$ on $\mathcal{L}(H)$ converging strongly to x'' such that $E^{\#} \in \mathfrak{F}$. Hence $\mathfrak{F}$ converges to x in $\mathcal{L}(H)_{\mathcal{L}^1(H)}$ and we get

$$|x'((x'')^*)|^2 \leq \|x'\|\,\|x''\xi\|^2.$$

Let $(x'', |x'|)$ be the polar representation of x'. Then

$$|\langle x, |x'|\rangle|^2 = |\langle x^*, |x'|\rangle|^2 = |\langle x^*, (x'')^*x''|x'|\rangle|^2 =$$

$$= |\langle x^*(x'')^*, x'\rangle|^2 \leq \|x'\|\,\|x''x\xi\|^2 \leq \|x'\|\,\|x\xi\|^2$$

for every $x \in E$. Define

$$u : E \longrightarrow H, \quad x \longmapsto x\xi.$$

Let $q : E \to E/\mathrm{Ker}\,u$ be the quotient map and $v : E/\mathrm{Ker}\,u \to \mathrm{Im}\,u$ the associated algebraic isomorphism of u. By the above inequality

$$\mathrm{Ker}\,u \subset \mathrm{Ker}\,|x'|.$$

Let w be the factorization of $|x'|$ through $E/\mathrm{Ker}\,u$. Take $\eta \in \mathrm{Im}\,u$. There is an $x \in E$ with

$$\eta = ux.$$

We get

$$\eta = vqx\,, \quad qx = v^{-1}\eta\,,$$

$$\langle x, |x'|\rangle = wqx = wv^{-1}\eta\,,$$

$$|wv^{-1}\eta| = |\langle x, |x'|\rangle| \leq \|x'\|^{\frac{1}{2}}\|ux\| = \|x'\|^{\frac{1}{2}}\|\eta\|\,.$$

Hence, $w \circ v^{-1}$ is continuous with

$$\|w \circ v^{-1}\| \leq \|x'\|^{\frac{1}{2}}\,.$$

We extend $w \circ v^{-1}$ to a linear form z' on H with the same norm. By the Fréchet–Riesz Theorem , there is a $\zeta \in H$, such that $\|\zeta\| = \|z'\|$ and

$$z'(\omega) = \langle\omega|\zeta\rangle$$

for every $\omega \in H$. We get

$$\langle x, |x'|\rangle = wv^{-1}ux = z'(ux) = \langle x\xi|\zeta\rangle$$

for every $x \in E$. If $\mathfrak{G}$ is an approximate unit of E, then, by Proposition 5.3.2.21 (and Proposition 2.3.4.9),

$$\langle\xi|\zeta\rangle = \lim_{x,\mathfrak{G}}\langle x^*x\xi|\zeta\rangle = \lim_{x,\mathfrak{G}}\langle x^*x, |x'|\rangle = \|x'\| \geq \|\xi\|\,\|\zeta\|$$

(Proposition 2.3.4.10 a)). By Proposition 5.1.1.11 b $\Rightarrow$ c,

$$\|\zeta\|\xi = \|\xi\|\zeta\,.$$

Thus

$$\langle x, |x'|\rangle = \frac{\|\zeta\|}{\|\xi\|}\langle x\xi|\xi\rangle = \frac{\|\zeta\|}{\|\xi\|}y'(x)$$

for every $x \in E$, and hence

$$|x'| = \frac{\|\zeta\|}{\|\xi\|}y'\,.$$

It follows

$$\|y'\| = \|x'\| = \|\,|x'|\,\| = \frac{\|\zeta\|}{\|\xi\|}\|y'\|\,,$$

$$\frac{\|\zeta\|}{\|\xi\|} = 1\,,$$

$$|x'| = y'\,. \qquad \blacksquare$$

Proposition 6.3.2.12 (0) *Let E be a C^*-algebra and take $n \in \mathbb{N}$. Given $p \in \mathbb{N}$ and $a \in E_{n,n}^{(p)}$, define*

$$\widetilde{a} : E_{n,n}^{(p-1)} \longrightarrow \mathbb{K}, \quad b \longmapsto \sum_{i,j=1}^{n} \langle b_{ij}, a_{ji} \rangle ,$$

endow $E_{n,n}^{(p)}$ with the norm

$$E_{n,n}^{(p)} \longrightarrow \mathbb{R}_+ , \quad a \longmapsto \|\widetilde{a}\|$$

defined inductively for all $p \in \mathbb{N}$, and identify $E_{n,n}^{(p)}$ with $(E_{n,n})^{(p)}$ (Proposition 2.3.6.7 and Theorem 5.6.6.1 a)). We have for every $p \in \mathbb{N}$:

a) *The above identification map $E_{n,n}^{(2p)} \to (E_{n,n})^{(2p)}$ is an isomorphism of C^*-algebras (Theorems 5.6.6.1 f) and 6.3.2.1 b)).*

b) *If $a \in E_{n,n}^{(2p-1)}$ and $x \in E_{n,n}^{(2p)}$, then*

$$(ax)_{ij} = \sum_{k=1}^{n} a_{ik} x_{kj} , \quad (xa)_{ij} = \sum_{k=1}^{n} x_{ik} a_{kj}$$

for all $i, j \in \mathbb{N}_n$.

We pove the assertions by induction on p.

a) The only thing to be proved is that the products in $E_{n,n}^{(2p)}$ and $(E_{n,n})^{(2p)}$ are identified too (Proposition 2.3.6.7 b)). The assertion is obvious for the elements of $j_{E_{n,n}^{(2p-2)}} E_{n,n}^{(2p-2)}$. This set being dense in $(E_{n,n}^{(2p)})_{E_{n,n}^{(2p-1)}}$ (Corollary 1.3.6.5) the assertion holds for any elements of $E_{n,n}^{(2p)}$ by Theorem 2.2.7.15 d).

b) Take $y \in E_{n,n}^{(p-2)}$. Then, by Proposition 5.6.6.14,

$$\langle y, ax \rangle = \langle ya, x \rangle = \sum_{j,k=1}^{n} \langle (ya)_{jk} , x_{kj} \rangle =$$

$$= \sum_{j,k=1}^{n} \left\langle \sum_{i=1}^{n} y_{ji} a_{ik}, x_{kj} \right\rangle = \sum_{i,j,k=1}^{n} \langle y_{ji} a_{ik}, x_{kj} \rangle =$$

$$= \sum_{i,j,k=1}^{n} \langle y_{ji}, a_{ik} x_{kj} \rangle = \sum_{i,j=1}^{n} \left\langle y_{ji}, \sum_{k=1}^{n} a_{ik} x_{kj} \right\rangle =$$

$$= \left\langle y, \left(\sum_{k=1}^{n} a_{ik} x_{kj} \right)_{i,j \in \mathbb{N}_n} \right\rangle ,$$

so that

$$(ax)_{ij} = \sum_{k=1}^{n} a_{ik} x_{kj} .$$

The relation

$$(xa)_{ij} = \sum_{k=1}^{n} x_{ik} a_{kj}$$

is proved similarly. ■

Corollary 6.3.2.13 (0) *Let H be a Hilbert space and $n \in \mathbb{N}$. We set*

$$\widetilde{a} : \mathcal{K}(H)_{n,n} \longrightarrow \mathbb{K}, \quad b \longmapsto \sum_{i,j=1}^{n} \langle b_{ij}, \overset{0}{\widetilde{a}}_{ji} \rangle$$

for every $a \in \mathcal{L}^1(H)_{n,n}$ and

$$\widetilde{a} : \mathcal{L}^1(H)_{n,n} \longrightarrow \mathbb{K}, \quad b \longmapsto \sum_{i,j=1}^{n} \langle b_{ij}, \overset{1}{\widetilde{a}}_{ji} \rangle$$

for every $a \in \mathcal{L}(H)_{n,n}$ (Definition 6.1.6.1). Then (with the identification of Proposition 6.3.2.12) the maps

$$\mathcal{L}^1(H)_{n,n} \longrightarrow (\mathcal{K}(H)_{n,n})', \quad a \longmapsto \widetilde{a},$$

$$\mathcal{L}(H)_{n,n} \longrightarrow (\mathcal{L}^1(H)_{n,n})', \quad a \longmapsto \widetilde{a}$$

are isometries of $\mathcal{K}(H)$-modules. Moreover, the W^*-algebras $\mathcal{L}(H)_{n,n}$ and $(\mathcal{K}(H)_{n,n})''$ (Theorem 5.6.6.5 c) and 6.3.2.1 b)) may be identified. ■

Example 6.3.2.14 *Let E be the C^*-direct sum of the family $(E_t)_{t\in T}$ of C^*-algebras and G the C^*-direct product of the family $(E_t'')_{t\in T}$ (Example 4.1.1.6, Theorem 6.3.2.1 b)). Define*

$$F := \{x' \in \prod_{t\in T} E_t' \mid \sum_{t\in T} \|x_t'\| < \infty\}$$

and endow F with the norm

$$F \longrightarrow \mathbb{R}_+, \quad x' \longmapsto \sum_{t\in T} \|x_t'\| .$$

Given $x' \in F$ and $y \in G$, define

$$\widetilde{x'} : E \longrightarrow \mathbb{K}\,, \quad x \longmapsto \sum_{t\in T}\langle x_t, x'_t\rangle$$

$$\widetilde{y} : F \longrightarrow \mathbb{K}\,, \quad x' \longmapsto \sum_{t\in T}\langle y_t, x'_t\rangle\,.$$

Then $\widetilde{x'} \in E'$ *for every* $x' \in F$, $\widetilde{y} \in F'$ *for every* $y \in G$, *the maps*

$$u : F \longrightarrow E'\,, \quad x' \longmapsto \widetilde{x'}\,,$$

$$v : G \longrightarrow F'\,, \quad y' \longmapsto \widetilde{y}$$

are isometries of involutive Banach spaces, G *is a* W^**-algebra with* F *as predual, and* $v^{-1}\circ u'$ *is a* W^**-isometry (i.e.* G *is the bidual of* E*).*

The first assertions are easy to see (Proposition 1.2.2.13, Proposition 2.3.2.22 b)). It follows that $v^{-1}\circ u'$ is an isometry of involutive Banach spaces (Proposition 2.3.2.22 d)). Now we prove that $v^{-1}\circ u'$ is an algebra homomorphism.

Given $t \in T$, let p_t and q_t denote the natural projections $E \to E_t$ and $G \to E''_t$, respectively, and i_t the natural injection $E'_t \to F$.

Step 1 $\quad t \in T \Rightarrow u \circ i_t = p'_t$

Take $x'_t \in E'_t$. Given $x \in E$,

$$\langle x, ui_t x'_t\rangle = \langle p_t x, x'_t\rangle = \langle x, p'_t x'_t\rangle\,,$$

so that

$$ui_t x'_t = p'_t x'_t\,.$$

Step 2 $\quad t \in T \Rightarrow q_t \circ v^{-1} \circ u' = p''_t$

Take $x'' \in E''$. By Step 1, if $x'_t \in E'_t$, then

$$\langle x'_t, q_t v^{-1} u' x''\rangle = \langle i_t x'_t, u' x''\rangle = \langle ui_t x'_t, x''\rangle =$$

$$= \langle p'_t x'_t, x''\rangle = \langle x'_t, p''_t x''\rangle\,,$$

so that

$$q_t v^{-1} u' x'' = p''_t x''\,.$$

Step 3 $v^{-1} \circ u'$ is an algebra homomorphism

Take $x'', y'' \in E''$ and $t \in T$. p_t and q_t are involutive algebra homomorphisms. By Corollary 6.3.2.3, p_t'' is an involutive algebra homomorphism too. By Step 2,

$$q_t((v^{-1}u'x'')(v^{-1}u'y'')) =$$

$$= (q_t v^{-1}u'x'')(q_t v^{-1}u'y'') = (p_t''x'')(p_t''y'') = p_t''(x''y'') = q_t v^{-1}u'(x''y'')\,.$$

Since t is arbitrary,

$$(v^{-1}u'x'')(v^{-1}u'y'') = v^{-1}u'(x''y'')\,.$$

Hence $v^{-1} \circ u'$ is an algebra homomorphism and therefore an isomorphism of C^*–algebras.

Since E'' is a W^*–algebra (Theorem 6.3.2.1 b)), G is also a W^*–algebra. By the above and Corollary 4.4.4.4, F is the predual of G. Using again the above result and Proposition 4.4.4.6, we deduce that $v^{-1} \circ u'$ is a W^*–isometry. ■

Example 6.3.2.15 *Let $(H_t)_{t\in T}$ be a family of Hilbert spaces, E the C^*–direct sum of the family $(\mathcal{K}(H_t))_{t\in T}$, and G the C^*–direct product of the family $(\mathcal{L}(H_t))_{t\in T}$ (Example 4.1.1.6). Define*

$$F := \{u \in \prod_{t\in T} \mathcal{L}^1(H_t) \mid \sum_{t\in T} \|u_t\|_1 < \infty\}\,,$$

and endow F with the norm

$$F \longrightarrow \mathbb{R}_+\,, \quad u \longmapsto \sum_{t\in T} \|u_t\|_1\,.$$

For each $v \in F$ and $w \in G$, define

$$\widetilde{v} : E \longrightarrow \mathbb{K}\,, \quad u \longmapsto \sum_{t\in T} \overset{\circ}{\widetilde{v}}_t(u_t)\,,$$

$$\widetilde{w} : F \longrightarrow \mathbb{K}\,, \quad v \longmapsto \sum_{t\in T} \overset{1}{\widetilde{w}}_t(v_t)$$

(Definition 6.1.6.1). Then $\widetilde{v} \in E'$ for every $v \in F$, $\widetilde{w} \in F'$ for every $w \in G$, the maps

$$\varphi : F \longrightarrow E', \quad v \longmapsto \widetilde{v},$$

$$\psi : G \longrightarrow F', \quad w \longmapsto \widetilde{w}$$

are isometries of involutive Banach spaces, G is a W^*-algebra with predual F, and $\psi^{-1} \circ \varphi'$ is a W^*-isometry (i.e. G is the bidual of E).

The assertion follows immediately from Example 6.3.2.14 and Theorem 6.1.6.6. ■

Proposition 6.3.2.16 (0) *Let E be a real C^*-algebra. For $(x', y') \in \overset{\circ}{\widehat{E'}}$ and $(x'', y'') \in \overset{\circ}{\widehat{E''}}$ put*

$$\widetilde{(x', y')} : \overset{\circ}{E} \longrightarrow \mathbb{C}, \quad (x, y) \longmapsto x'(x) - y'(y) + i(x'(y) + y'(x)),$$

$$\widetilde{(x'', y'')} : (\overset{\circ}{E})' \longrightarrow \mathbb{C}, \quad (x', y') \longmapsto x''(x') - y''(y') + i(x''(y') + y''(x')),$$

where we identified $(\overset{\circ}{E})'$ with $\overset{\circ}{\widehat{E'}}$ using the map

$$\overset{\circ}{\widehat{E'}} \longrightarrow (\overset{\circ}{E})', \quad (x', y') \longmapsto \widetilde{(x', y')}$$

(Corollary 4.3.6.2 b)). Then $\widetilde{(x'', y'')} \in (\overset{\circ}{E})''$ for every $(x'', y'') \in \overset{\circ}{\widehat{E''}}$ and the map

$$\overset{\circ}{\widehat{E''}} \longrightarrow (\overset{\circ}{E})'', \quad (x'', y'') \longmapsto \widetilde{(x'', y'')}$$

is a W^*-isometry (Theorem 6.3.2.1 b), Theorem 4.4.2.21).

By Corollary 4.3.6.2 g_1),g_2), $\widetilde{(x'', y'')} \in (\overset{\circ}{E})''$ for every $(x'', y'') \in \overset{\circ}{\widehat{E''}}$ and the map

$$\overset{\circ}{\widehat{E''}} \longrightarrow (\overset{\circ}{E})'', \quad (x'', y'') \longmapsto \widetilde{(x'', y'')}$$

is an isomorphism of involutive complex vector spaces. Take

$$(x_1'', y_1''), (x_2'', y_2'') \in \overset{\circ}{\widehat{E''}}.$$

By Corollary 4.3.6.2 g_3), for every $(x', y') \in \overset{\circ}{\widetilde{E'}}$,

$$\left\langle \widetilde{(x', y')},\ \widetilde{(x_1'', y_1'')}\ \widetilde{(x_2'', y_2'')} \right\rangle = \left\langle \widetilde{(x', y')}\ \widetilde{(x_1'', y_1'')},\ \widetilde{(x_2'', y_2'')} \right\rangle =$$

$$= \left\langle \widetilde{(x'x_1'' - y'y_1'', x'y_1'' + y'x_1'')},\ \widetilde{(x_2'', y_2'')} \right\rangle =$$

$$= \langle x'x_1'' - y'y_1'', x_2'' \rangle - \langle x'y_1'' + y'x_1'', y_2'' \rangle +$$

$$+ i(\langle x'x_1'' - y'y_1'', y_2'' \rangle + \langle x'y_1'' + y'x_1'', x_2'' \rangle) =$$

$$= \langle x', x_1''x_2'' - y_1''y_2'' \rangle - \langle y', y_1''x_2'' + x_1''y_2'' \rangle +$$

$$+ i(\langle x', x_1''y_2'' + y_1''x_2'' \rangle + \langle y', x_1''x_2'' - y_1''y_2'' \rangle) =$$

$$= \left\langle \widetilde{(x', y')},\ \widetilde{(x_1''x_2'' - y_1''y_2'', x_1''y_2'' + y_1''x_2'')} \right\rangle =$$

$$= \left\langle \widetilde{(x', y')},\ \widetilde{(x_1'', y_1'')(x_2'', y_2'')} \right\rangle .$$

Hence

$$\widetilde{(x_1'', y_1'')}\ \widetilde{(x_2'', y_2'')} = \widetilde{(x_1'', y_1'')(x_2'', y_2'')}$$

and the map

$$\overset{\circ}{\widetilde{E''}} \longrightarrow (\overset{\circ}{E})'', \quad (x'', y'') \longmapsto \widetilde{(x'', y'')}$$

is an isomorphism of C^*–algebras. By Proposition 4.4.4.6, this map is even a W^*–isometry. ■

Proposition 6.3.2.17 *Put*

$$A := \{x \in E_+ \mid \|x\| < 1\}.$$

a) *A is upward directed and its upper section filter $\mathfrak{F}$ is an approximate unit of E.*

b) *For every* $x'' \in E''_+$, $\left(x''^{\frac{1}{2}}(j_E x)x''^{\frac{1}{2}}\right)_{x\in A}$ *is upward directed and* x'' *is its supremum.*

c) *For every* $x'' \in E''$,

$$x'' = \lim_{x,\mathfrak{F}}(j_E x)x'' = \lim_{x,\mathfrak{F}} x''(j_E x)$$

in $E''_{E'}$.

a) was proved in Proposition 4.2.8.1 and Theorem 4.2.8.2.

b) By a) and Corollary 4.2.2.3, $\left(x''^{\frac{1}{2}}(j_E x)x''^{\frac{1}{2}}\right)_{x\in A}$ is upward directed. Let y'' be its supremum in E'' (Theorem 4.4.1.8 c), Theorem 6.3.2.1 b)). We want to prove $x'' = y''$.

Step 1 $\quad x' \in E'_+ \Rightarrow \langle x'', x'\rangle = \langle y'', x'\rangle$

By Proposition 2.3.6.4,

$$x''^{\frac{1}{2}}x'x''^{\frac{1}{2}} \in E'_+$$

so that by a), Proposition 2.3.4.10 a), and Theorem 4.4.1.8 c) (and Theorem 6.3.2.1 b)),

$$\langle x'', x'\rangle = \langle 1_{E''}, x''^{\frac{1}{2}}x'x''^{\frac{1}{2}}\rangle = \|x''^{\frac{1}{2}}x'x''^{\frac{1}{2}}\| = \lim_{x,\mathfrak{F}}\langle x, x''^{\frac{1}{2}}x'x''^{\frac{1}{2}}\rangle =$$

$$= \lim_{x,\mathfrak{F}}\langle j_E x, x''^{\frac{1}{2}}x'x''^{\frac{1}{2}}\rangle = \lim_{x,\mathfrak{F}}\langle x''^{\frac{1}{2}}(j_E x)x''^{\frac{1}{2}}, x'\rangle = \langle y'', x'\rangle$$

Step 2 $\quad x' \in \operatorname{Re} E' \Rightarrow \langle x'', x'\rangle = \langle y'', x\rangle$

The assertion follows from Step 1 and Corollary 4.1.2.7 d).

Step 3 $\quad x'' = y''$

In the complex case the assertion follows immediately from Step 2. So assume $\mathbb{K} = \mathbb{R}$ and let $x' \in E'$. Put

$$y' := \frac{1}{2}(x' + x'^*), \quad z' := \frac{1}{2}(x' - x'^*).$$

By Step 2,

$$\langle x'', y'\rangle = \langle y'', y'\rangle.$$

Since

$$\langle x'', z'\rangle = 0 = \langle y'', z'\rangle,$$

it follows

$$\langle x'', x' \rangle = \langle y'', x' \rangle,$$

$$x'' = y''.$$

c) Case 1 $x'' \in E''_+$, x'' invertible

Take $x' \in E'$. By b),

$$\langle x'', x' \rangle = \langle x''^2, x' x''^{-1} \rangle = \lim_{x,\mathfrak{F}} \langle x''(j_E x)x'', x' x''^{-1} \rangle = \lim_{x,\mathfrak{F}} \langle (j_E x)x'', x' \rangle,$$

$$\langle x'', x \rangle = \langle x''^2, x''^{-1} x' \rangle = \lim_{x,\mathfrak{F}} \langle x''(j_F x)x'', x''^{-1} x' \rangle = \lim_{x,\mathfrak{F}} \langle x''(j_F x), x' \rangle.$$

Case 2 $x'' \in \operatorname{Re} E''$

The assertion follows from Case 1 since x'' is the difference of two invertible positive elements of E''.

Case 3 $\mathbb{K} = \mathbb{C}$

Follows immediately from Case 2.

Case 4 $\mathbb{K} = \mathbb{R}$

Follows from Case 3 and Proposition 6.3.2.16. ∎

Proposition 6.3.2.18 *Denote by ρ the canonical projection of the tridual of E, i.e.*

$$\rho := j_{E'} \circ j'_E.$$

a) $\operatorname{Im} \rho$ *is an involutive E''–submodule of E''',* $\operatorname{Ker} \rho = (\operatorname{Im} j_E)^\circ$ *is an involutive E–submodule of E''', and ρ is a homomorphism of involutive E–modules.*

b) $(x''', y''') \in E''' \times \operatorname{Im} \rho$, $0 \leq x''' \leq y''' \Rightarrow x''' \in \operatorname{Im} \rho$.

c) $(x''', y''') \in E''' \times \operatorname{Ker} \rho$, $0 \leq x''' \leq y''' \Rightarrow x''' \in \operatorname{Ker} \rho$.

d) *If* $\operatorname{Im} j_E$ *is an ideal of E'' then* $\operatorname{Ker} \rho$ *is an involutive E''–submodule of E''' and ρ is a homomorphism of involutive E''–modules.*

e) $j_{E'}(E'_+) = E'''_+ \cap \operatorname{Im} \rho$.

f) $j_{E'}(\tau(E)) = \tau(E'') \cap \operatorname{Im}\rho$.

g) *The map*

$$\tau(E'') \cap \operatorname{Im}\rho \longrightarrow \tau(E), \quad x''' \longmapsto j'_E x'''$$

is bijective and continuous with respect to the topologies of τ.

h) $j_{E'}(\tau(E))$ *is a dense set of* $\tau(E'')$.

i) $j_{E'}(\tau_0(E)) = \tau_0(E'') \cap \operatorname{Im}\rho$.

a) By Corollary 4.4.2.10, $\operatorname{Im}\rho$ is an involutive E''–submodule of E'''. By Proposition 2.3.2.22 b),d),e), $j_E, j_{E'}$, and j'_E are involutive. Hence ρ is involutive. By Proposition 2.2.7.5, j_E and $j_{E'}$ are homomorphisms of E–modules so that by Proposition 2.2.7.6, j'_E is also a homomorphism of E–modules. Hence ρ is a homomorphism of E–modules. By Proposition 1.3.6.19 c), $\operatorname{Ker}\rho = (\operatorname{Im} j_E)^\circ$.

b) follows from Corollary 4.4.2.13.

c) By a), for every $x \in E_+$,

$$0 \leq \langle j_E x, x''' \rangle \leq \langle j_E x, y''' \rangle = 0$$

so that

$$\langle j_E x, x''' \rangle = 0\,.$$

It follows $x''' \in (\operatorname{Im} j_E)^\circ = \operatorname{Ker}\rho$.

d) Take $(x'', x''') \in E'' \times \operatorname{Ker}\rho$. By a), for every $x \in E$,

$$\langle j_E x, x''x''' \rangle = \langle (j_E x)x'', x''' \rangle = 0\,,$$

$$\langle j_E x, x'''x'' \rangle = \langle x''(j_E x), x''' \rangle = 0\,,$$

so that

$$x''x''', x'''x'' \in (\operatorname{Im} j_E)^\circ = \operatorname{Ker}\rho\,.$$

Hence (by a)), $\operatorname{Ker}\rho$ is an involutive E''–submodule of E'''.

Take $(x'', x''') \in E'' \times E'''$. By a) and the above,

$$x''(\rho x'''), (\rho x''')x'' \in \operatorname{Im}\rho\,, \quad x''(x''' - \rho x'''), (x''' - \rho x''')x'' \in \operatorname{Ker}\rho\,.$$

Since

$$\rho(x''x'''), \rho(x'''x'') \in \operatorname{Im}\rho\,, \quad x''x''' - \rho(x''x'''), x'''x'' - \rho(x'''x'') \in \operatorname{Ker}\rho\,,$$

it follows

$$x''(\rho x''') - \rho(x''x''')\,,\ (\rho x''')x'' - \rho(x'''x'') \in \operatorname{Im}\rho \cap \operatorname{Ker}\rho$$

i.e.

$$\begin{cases} x''(\rho x''') - \rho(x''x''') = 0 \\ (\rho x''')x'' - \rho(x'''x'') = 0\,, \end{cases}$$

$$\rho(x''x''') = x''(\rho x''')\,, \quad \rho(x'''x'') = (\rho x''')x''\,.$$

Hence (by a)), ρ is a homomorphism of involutive E''–modules.

e) Take $x''' \in E'''_+ \cap \operatorname{Im}\rho$. Then

$$\langle x, j'_E x'''\rangle = \langle j_E x, x'''\rangle \in \mathbb{R}_+$$

for every $x \in E_+$ (Theorem 6.3.2.1 b), Corollary 4.2.2.11), so that

$$j'_E x''' \in E'_+\,,$$

$$x''' = \rho x''' = j_{E'} j'_E x''' \in j_{E'}(E'_+)\,.$$

Hence

$$E'''_+ \cap \operatorname{Im}\rho \subset j_{E'}(E'_+)\,.$$

The reverse inclusion follows from Theorem 6.3.2.1 d).

f) follows from e).

g) follows from f).

h) follows from Corollary 4.4.1.9.

i) Take $x' \in \tau_0(E)$. By f),

$$j_{E'}x' \in \tau(E'') \cap \operatorname{Im}\rho\,.$$

Let $y''', z''' \in \tau(E'')$ and $\alpha, \beta \in]0,1[$ such that

$$\alpha + \beta = 1\,, \quad \alpha y''' + \beta z''' = j_{E'}x'\,.$$

By b), $y''', z''' \in \operatorname{Im}\rho$. Put

$$y' := j'_E y'''\,, \quad z' := j'_E z'''\,.$$

By g), $y', z' \in \tau(E)$. From

$$x' = j'_E j_{E'} x' = \alpha y' + \beta z'$$

(Proposition 1.3.6.19 a)) it follows

$$y' = z' ,$$

$$y''' = \rho y''' = j_{E'} j'_E y''' = j_{E'} y' = j_{E'} z' = j_{E'} j'_E z''' = \rho z''' = z''' .$$

Hence $j_{E'} x' \in \tau_0(E'')$ and

$$j_{E'}(\tau_0(E)) \subset \tau_0(E'') \cap \operatorname{Im} \rho .$$

Take $x''' \in \tau_0(E'') \cap \operatorname{Im} \rho$. Put

$$x' := j'_E x''' .$$

By g), $x' \in \tau(E)$. Let $y', z' \in \tau(E)$ and $\alpha, \beta \in]0,1[$ such that

$$\alpha + \beta = 1 , \quad \alpha y' + \beta z' = x' .$$

Then

$$x''' = \rho x''' = j_{E'} j'_E x''' = j_{E'} x' = \alpha j_{E'} y' + \beta j_{E'} z' .$$

By f), $j_{E'} y', j_{E'} z' \in \tau(E'')$. It follows

$$j_{E'} y' = j_{E'} z' ,$$

$$y' = z' .$$

Hence $x' \in \tau_0(E)$,

$$x''' \in j_{E'}(\tau_0(E)) ,$$

$$\tau_0(E'') \cap \operatorname{Im} \rho \subset j_{E'}(\tau_0(E)) . \qquad \blacksquare$$

Remark. Take $E := c$. Then $E' \approx \ell^1 \times \mathbb{K}$, $E'' \approx \ell^\infty \times \mathbb{K}$, $E''' \approx (\ell^\infty)' \times \mathbb{K}$. Let $\mathfrak{F}$, $\mathfrak{G}$ be two distinct free ultrafilters on $\mathbb{N}$, $A \in \mathfrak{F} \backslash \mathfrak{G}$,

$$x'' := (e_A, 0) \in E'' ,$$

and

$$x''' : E'' \longrightarrow \mathbb{K} , \quad (y'', \alpha) \longmapsto \lim_{\mathfrak{F}} y'' - \lim_{\mathfrak{G}} y'' .$$

Then $x''' \in \operatorname{Ker} \rho$ and $x'' x''' \notin \operatorname{Ker} \rho$. Hence in this case d) does not hold.

Proposition 6.3.2.19 *Denote by ρ the cononical projection of the tridual of E, i.e.*

$$\rho := j_{E'} \circ j'_E$$

and assume that for every $x'' \in E''_+$ there is an upward directed family $(x_\iota)_{\iota \in I}$ in E_+ such that

$$x'' = \bigvee_{\iota \in I} (j_E x_\iota)$$

(by Proposition 6.3.2.17 b), this condition is fulfilled whenever $\operatorname{Im} j_E$ is an ideal of E'').

a) $x''' \in E'''_+ \Rightarrow 0 \leq \rho x''' \leq x'''$.

b) *If $x''' \in \operatorname{Re} E'''$ then*

$$\|x'''\| = \|\rho x'''\| + \|x''' - \rho x'''\|,$$

$$(\rho x''')^+ = \rho x'''^+, \quad (\rho x''')^- = \rho x'''^-,$$

$$(x''' - \rho x''')^+ = x'''^+ - \rho x'''^+, \quad (x''' - \rho x''')^- = x'''^- - \rho x'''^-.$$

c) *For every $x''' \in \operatorname{Re} E'''$, x''' is order continuous iff $x''' \in \operatorname{Im} \rho$.*

d) $\tau_0(E'') \subset \operatorname{Im} \rho \cup \operatorname{Ker} \rho$.

a) Take $x'' \in E''_+$ and let $(x_\iota)_{\iota \in I}$ be an upward directed family in E_+ such that

$$x'' = \bigvee_{\iota \in I} (j_E x_\iota).$$

By Corollary 4.4.4.3 a $\Rightarrow$ b,

$$\langle x'', \rho x''' \rangle = \langle x'', j_{E'} j'_E x''' \rangle = \langle x'', j'_E x''' \rangle =$$

$$= \sup_{\iota \in I} \langle j_E x_\iota, j'_E x''' \rangle = \sup_{\iota \in I} \langle x_\iota, j'_E x''' \rangle = \sup_{\iota \in I} \langle j_E x_\iota, x''' \rangle.$$

Since

$$0 \leq \langle j_E x_\iota, x''' \rangle \leq \langle x'', x''' \rangle$$

for every $\iota \in I$, it follows

$$0 \leq \langle x'', \rho x''' \rangle \leq \langle x'', x''' \rangle ,$$

i.e.

$$0 \leq \rho x''' \leq x''' .$$

b) If $x''' \in E'''_+$, then by a) and Corollary 2.3.4.14,

$$\|x'''\| = \|\rho x'''\| + \|x''' - \rho x'''\| .$$

Now assume $x''' \in \operatorname{Re} E'''$. By Proposition 6.3.2.18 a), ρ is involutive so that $\rho x''' \in \operatorname{Re} E'''$. By a),

$$\rho x'''^{+} , \rho x'''^{-} , x'''^{+} - \rho x'''^{+} , x'''^{-} - \rho x'''^{-} \in E'''_+ .$$

Since

$$x''' = x'''^{+} - x'''^{-} , \quad \rho x''' = \rho x'''^{+} - \rho x'''^{-} ,$$

$$x''' - \rho x''' = (x'''^{+} - \rho x'''^{+}) - (x'''^{-} - \rho x'''^{-}) ,$$

it follows from the above,

$$\|x'''\| \leq \|\rho x'''\| + \|x''' - \rho x'''\| \leq$$

$$\leq \|\rho x'''^{+}\| + \|\rho x'''^{-}\| + \|x'''^{+} - \rho x'''^{+}\| + \|x'''^{-} - \rho x'''^{-}\| =$$

$$= \|x'''^{+}\| + \|x'''^{-}\| = \|x'''\| .$$

Hence

$$\|x'''\| = \|\rho x'''\| + \|x''' - \rho x'''\| ,$$

$$\|\rho x'''\| = \|\rho x'''^{+}\| + \|\rho x'''^{-}\| ,$$

$$\|x''' - \rho x'''\| = \|x'''^{+} - \rho x'''^{+}\| + \|x'''^{-} - \rho x'''^{-}\| ,$$

so that

$$(\rho x''')^+ = \rho x'''^+, \quad (\rho x''')^- = \rho x'''^-,$$

$$(x''' - \rho x''')^+ = x'''^+ - \rho x'''^+, \quad (x''' - \rho x''')^- = x'''^- - \rho x'''^-.$$

c) If $x''' \in \operatorname{Im}\rho$ then by Theorem 4.4.1.8 c), x''' is order continuous. Assume now that x''' is order continuous. By the above, $\rho x'''$ is order continuous, so that $x''' - \rho x'''$ is order continuous. Since ρ is a projection,

$$x''' - \rho x''' \in \operatorname{Ker}\rho = (\operatorname{Im} j_E)^\circ.$$

Let $x'' \in E''_+$ and let $(x_\iota)_{\iota\in I}$ be an upward directed family in E_+ such that

$$x'' = \bigvee_{\iota\in I}(j_E x_\iota).$$

Then

$$\langle x'', x''' - \rho x'''\rangle = 0$$

and therefore

$$x''' - \rho x''' = 0,$$

$$x''' = \rho x''' \in \operatorname{Im}\rho.$$

d) Take $x'' \in \tau_0(E'')$. By a) and b),

$$\rho x''', x''' - \rho x''' \in E'''_+,$$

$$\|\rho x'''\| + \|x''' - \rho x'''\| = \|x'''\| = 1.$$

By Proposition 2.3.5.4 a $\Rightarrow$ b,

$$\rho x''' = \|\rho x'''\| x''', \quad x''' - \rho x''' = \|x''' - \rho x'''\| x'''.$$

If $\rho x''' \neq 0$ and $x''' - \rho x''' \neq 0$ then

$$\frac{1}{\|\rho x'''\|}\rho x''', \frac{1}{\|x''' - \rho x'''\|}(x''' - \rho x''') \in \tau(E'').$$

We get

$$x''' = \rho x''' + (x''' - \rho x''') =$$

$$= \|\rho x'''\| \frac{1}{\|\rho x'''\|} \rho x''' + \|x''' - \rho x'''\| \frac{1}{\|x''' - \rho x'''\|} (x''' - \rho x''') ,$$

which contradicts the hypothesis $x''' \in \tau_0(E'')$. Hence

$$\rho x''' = 0 \quad \text{or} \quad x''' - \rho x''' = 0$$

and

$$x''' \in \operatorname{Im} \rho \cup \operatorname{Ker} \rho . \qquad \blacksquare$$

Remark. In the above proposition we cannot simply drop the initial hypothesis as the example $E := c$ shows. Indeed, in this case

$$E' \approx \ell^1 \times \mathbb{K} , \; E'' \approx \ell^\infty \times \mathbb{K} , \; E''' \approx (\ell^\infty)' \times \mathbb{K} .$$

Let $\mathfrak{F}$ be a free ultrafilter on $\mathbb{N}$ and define

$$x''' : E'' \longrightarrow \mathbb{K} , \quad (x'', \alpha) \longmapsto \lim_{n, \mathfrak{F}} x''(n) .$$

Then $x''' \in E'''_+$,

$$\langle x, j'_E x''' \rangle = \langle j_E x, x''' \rangle = \lim_{n \to \infty} x(n)$$

for every $x \in E$,

$$\rho x''' = j_{E'} j'_E x''' = \delta ,$$

where

$$\delta : E'' \longrightarrow \mathbb{K} , \quad (x'', \alpha) \longmapsto \alpha .$$

Hence

$$\rho x''' \not\leqslant x''' .$$

Theorem 6.3.2.20 (8) *Let E be a C^*-algebra and S, T sets. We put*

$$H_0 := \bigoplus_{t \in T} E = \ell^2(T, E) , \quad K_0 := \bigoplus_{s \in S} E = \ell^2(S, E) ,$$

$$H := \overset{W}{\bigoplus_{t \in T}} E'' , \quad K := \overset{W}{\bigoplus_{s \in S}} E'' ,$$

$$\mathcal{K} := \mathcal{K}_E(H_0, K_0)\,, \quad \mathcal{L} := \mathcal{L}_{E''}(H, K)$$

and

$$f_{t_0} := (\delta_{tt_0}1)_{t\in T} \in H\,, \quad g_{s_0} := (\delta_{ss_0}1)_{s\in S} \in K$$

for every $t_0 \in T$ *and* $s_0 \in S$. *Moreover, for every* $x \in E^{(S\times T)}$ *and* $x' \in (E')^{(T\times S)}$ *define*

$$\widetilde{x} : H_0 \longrightarrow K_0\,, \quad \xi \longmapsto \left(\sum_{t\in T} x_{st}\xi_t\right)_{s\in S}\,,$$

$$\widehat{x} : H \longrightarrow K\,, \quad \xi \longmapsto \left(\sum_{t\in T} (j_E x_{st})\xi_t\right)_{s\in S}$$

$$\widetilde{x'} : \{\widetilde{x} | x \in E^{(S\times T)}\} \longrightarrow \mathbb{K}\,, \quad \widetilde{x} \longmapsto \sum_{(s,t),\in S\times T} \langle x_{st}, x'_{ts}\rangle$$

(Proposition 5.6.1.8 h), Proposition 5.6.4.16 b)), and

$$\widehat{x'} := \sum_{(t,s)\in T\times S} \widetilde{(x'_{ts}, f_t, g_s)}\,.$$

a) *The map*

$$\{\widetilde{x} | x \in E^{(S\times T)}\} \longrightarrow \mathcal{L}\,, \quad \widetilde{x} \longmapsto \widehat{x}$$

(Proposition 5.6.1.8 h), Proposition 5.6.4.16 b)) is linear and preserves the norms.

b) *The map*

$$\left\{\widetilde{x'} | x' \in (E')^{(T\times S)}\right\} \longrightarrow \ddot{H}\,, \quad \widetilde{x'} \longmapsto \widehat{x'}$$

is well-defined, linear, and preserves the norms; we denote by

$$\varphi : \mathcal{K}' \longrightarrow \ddot{H}$$

its continuous extension (Proposition 5.6.5.19).

c) *Let* $y' \in E'$, $(s,t) \in S \times T$, $x'', y'' \in E''$, *and*

$$x' := (\delta_{ss'}\delta_{tt'}x''y'y''^*)_{(t',s')\in T\times S} \in (E')^{(T\times S)}\,.$$

Then

$$\widehat{x'} = \widetilde{(y', f_t x'', g_s y'')}\,.$$

d) *φ is an isomorphism of Banach spaces.*

e) $x' \in \mathcal{K}' \Longrightarrow (\varphi x')^* = \varphi x'^*$.

f) *If we identify $\mathcal{K}'$ and $\ddot{H}$ by φ, then $\mathcal{L}$ is identified (by φ') to the bidual of $\mathcal{K}$ (Theorem 5.6.3.5 a)),*

$$j_{\mathcal{K}}\widetilde{x} = \widehat{x}$$

for every $x \in E^{(S\times T)}$ (Propositions 5.6.5.6 d) and 5.6.5.9 b)), and

$$j_{\mathcal{K}}(\eta\langle\cdot|\xi\rangle) = (j_E\eta_s)_{s\in S}\langle\cdot|(j_E\xi_t)_{t\in T}\rangle$$

for every $(\xi, \eta) \in H_0 \times K_0$.

g) *Assume T is a one-point set and put*

$$\vec{\eta} : E \longrightarrow K_0, \quad x \longmapsto \eta x$$

for every $\eta \in K_0$. Then

$$K_0 \longrightarrow \mathcal{K}\,,\ \eta \longmapsto \vec{\eta}\langle\cdot|1\rangle\,,$$

$$K \longrightarrow \mathcal{L}\,,\ \eta'' \longmapsto \eta''\langle\cdot|1\rangle\,,$$

are isometries of Banach spaces, so that (by f)) K may be identified with the bidual of K_0 , in which case

$$j_{K_0}\eta = (j_E\eta)_{s\in T}$$

for every $\eta \in K_0$.

h) *Put*

$$\mathfrak{M} := \{A \times B \mid A \in \mathfrak{P}_f(S)\,, \quad B \in \mathfrak{P}_f(T)\}\,,$$

order $\mathfrak{M}$ by inclusion, and denote by $\mathfrak{F}$ its upper section filter. Then for every $u' \in \ddot{H}$ there is a unique $x' \in (E')^{T\times S}$ such that

$$u' = \lim_{A\times B,\mathfrak{F}} \widetilde{x'e^{T\times S}_{B\times A}}$$

is the topology of pointwise convergence on $\mathcal{L}$.

a) Take $x \in E^{(S\times T)}$. There is a suitable $n \in \mathbb{N}$ such that x may be considered an element of $E_{n,n}$. Then

$$j_{E_{n,n}} x \in (E_{n,n})'' = E''_{n,n}$$

(Proposition 6.3.2.12 a)). Our notation identifies $\widetilde{x}$ and $\widehat{x}$ with the corresponding operators associated to the matrices x and $j_{E_{n,n}} x$, respectively. The assertion follows from the fact that $j_{E_{n,n}}$ is linear and preserves the norms.

b) The map is well-defined since the map $x' \mapsto \widetilde{x'}$ is linear and injective. Take $x' \in (E')^{(T\times S)}$. There is a $u \in \mathcal{L}^{\#}$ such that

$$\langle u, \widehat{x'} \rangle = \|\widehat{x'}\|$$

(Corollary 1.3.3.8 a), Theorem 5.6.3.5 a)). Put

$$x''_{st} := \langle u f_t | g_s \rangle \in E''$$

for all $(s,t) \in S \times T$. Then

$$\langle u, \widehat{x'} \rangle = \sum_{(s,t)\in S\times T} \left\langle u, \overbrace{(x'_{ts, f_t, g_s})}^{\sim} \right\rangle =$$

$$= \sum_{(s,t)\in S\times T} \langle \langle u f_t | g_s \rangle, x'_{ts} \rangle = \sum_{(s,t)\in S\times T} \langle x''_{st}, x'_{ts} \rangle .$$

Let $\varepsilon > 0$. By Theorem 6.3.2.1 d) and Proposition 6.3.2.12 a), there is an $x \in E^{(S\times T)}$ such that $\|\widetilde{x}\| \leq 1$ and

$$\sum_{(s,t)\in S\times T} \langle x''_{st}, x'_{ts} \rangle - \varepsilon < \sum_{(s,t)\in S\times T} \langle x_{st}, x'_{ts} \rangle = \langle \widetilde{x}, \widetilde{x'} \rangle .$$

It follows

$$\|\widehat{x'}\| - \varepsilon \leq \langle \widetilde{x}, \widetilde{x'} \rangle \leq \|\widetilde{x'}\|$$

and so

$$\|\widehat{x'}\| \leq \|\widetilde{x'}\| .$$

Take $y \in E^{(S\times T)}$ For every $(s,t) \in S \times T$,

$$\langle \widehat{y} f_t | g_s \rangle = \sum_{t' \in T} (j_E y_{st'}) \delta_{tt'} f_{t'} = j_E y_{st}$$

and so (by the above),

$$\langle \widetilde{y}, \widetilde{x'} \rangle = \sum_{(s,t)\in S\times T} \langle y_{st}, x'_{ts} \rangle = \sum_{(s,t)\in S\times T} \langle j_E y_{st}, x'_{ts} \rangle =$$

$$= \langle \widehat{y}, \widetilde{x'} \rangle .$$

By a),

$$|\langle \widetilde{y}, \widetilde{x'} \rangle| = |\langle \widehat{y}, \widehat{x'} \rangle| \leq \|\widehat{y}\| \, \|\widehat{x'}\| = \|\widetilde{y}\| \, \|\widehat{x'}\| ,$$

$$\|\widetilde{x'}\| \leq \|\widehat{x'}\| ,$$

$$\|\widetilde{x'}\| = \|\widehat{x'}\| .$$

c) By Proposition 5.6.3.1 d),

$$\widetilde{(y', f_t x'', g_s y'')} = \widetilde{(x'' y' y''^*, f_t, g_s)} =$$

$$= \sum_{(t',s')\in T\times S} \widetilde{(\delta_{ss'}\delta_{tt'} x'' y' y''^*, f_{t'}, g_{s'})} = \widehat{x'} .$$

d) Take $u \in \mathcal{L}$ with $u = 0$ on $\operatorname{Im}\varphi$. We deduce successively for $x' \in E'$, $s \in S$, and $t \in T$,

$$\langle \langle u f_t | g_s \rangle, x' \rangle = \langle u, \widetilde{(x', f_{t'}, g_s)} \rangle = 0 ,$$

$$\langle u f_t | g_0 \rangle = 0 ,$$

$$u f_t = 0 ,$$

$$u = 0$$

(Propositions 5.6.4.6 c) and 5.6.3.4 c)). By b), φ is an isomorphism of Banach spaces.

e) Take $x' \in (E')^{(T\times S)}$. By Proposition 5.6.3.1 b) and Proposition 5.6.5.18 b),

$$\widehat{x'}^* = \sum_{(t,s)\in T\times S} \widetilde{((x'_{ts})^*, g_s, f_t)} = \widehat{x'^*} ,$$

$$\widetilde{x'^*} = \widetilde{x'}^* .$$

It follows

$$\varphi(\widetilde{x'}^*) = \varphi(\widetilde{x'^*}) = \widehat{x'^*} = \widehat{x'}^* = (\varphi\widetilde{x'})^* .$$

By continuity,

$$\varphi x'^* = (\varphi x')^*$$

for every $x' \in \mathcal{K}'$ (Proposition 5.6.5.19).

f) Let $y' \in E'$, $(s,t) \in S \times T$, $x'', y'' \in E''$, and

$$x' := (\delta_{ss'}\delta_{tt'}x''y'y''^*)_{(t',s')\in T\times S} \in (E')^{(T\times S)} .$$

By c),

$$\langle \widehat{x}, \widehat{x'} \rangle = \langle \widehat{x}, \widetilde{(y', f_t x'', g_s y'')} \rangle = \langle \langle \widehat{x} f_t x'' | g_s y'' \rangle, y' \rangle =$$

$$= \langle y''^* \langle \widehat{x} f_t | g_s \rangle x'', y' \rangle = \langle \langle \widehat{x} f_t | g_s \rangle, x''y'y''^* \rangle =$$

$$= \langle j_E x_{st}, x''y'y''^* \rangle = \langle x_{st}, x''y'y''^* \rangle = \langle \widetilde{x}, \widetilde{x'} \rangle =$$

$$= \langle \widetilde{x}, \widehat{x'} \rangle = \langle j_{\mathcal{K}} \widetilde{x}, \widehat{x'} \rangle .$$

By Proposition 5.6.5.19,

$$\widehat{x} = j_{\mathcal{K}} \widetilde{x} .$$

Put

$$\mathfrak{M} := \{ A \times B \mid A \in \mathfrak{P}_f(S) , B \in \mathfrak{P}_f(T) \} ,$$

order $\mathfrak{M}$ by inclusion, and denote by $\mathfrak{F}$ the upper section filter of $\mathfrak{M}$. By the above, Proposition 5.6.5.9 a), and Proposition 5.6.5.2 m),

$$j_{\mathcal{K}}(\eta\langle\cdot|\xi\rangle) = \lim_{A\times B,\mathfrak{F}} j_{\mathcal{K}}(\eta e_A^S \langle\cdot|\xi e_B^T\rangle) =$$

$$= \lim_{A\times B,\mathfrak{F}} ((j_E\eta_s)e_A^S(s))_{s\in S}\langle\cdot|((j_E\xi_t)e_B^T(t))_{t\in T}\rangle =$$

$$= (j_E\eta_s)_{s\in S}\langle\cdot|(j_E\xi_t)_{t\in T}\rangle .$$

g) By Proposition 5.6.5.6 d),e), the maps

$$\varphi : K_0 \longrightarrow \mathcal{K}, \quad \eta \longmapsto \vec{\eta},$$

$$\psi : K \longrightarrow \mathcal{L}, \quad \eta'' \longmapsto \eta''\langle\cdot|1\rangle,$$

are isometries of Banach spaces with

$$\psi^{-1} : \mathcal{L} \longrightarrow K, \quad u'' \longmapsto u''1.$$

By f),

$$j_{K_0}\eta = \psi^{-1} j_{\mathcal{K}} \varphi\eta = \psi^{-1} j_{\mathcal{K}} \vec{\eta} = \psi^{-1}((j_E\eta_s)_{s\in S}\langle\cdot|1\rangle) =$$

$$= (j_E\eta_s)_{s\in S},$$

for every $\eta \in E^{(S)}$, since in this case $\vec{\eta} = \widetilde{\eta}$. The general assertion follows by continuity (Proposition 5.6.4.1 e)).

h) The uniqueness of x' follows from

$$x'_{ts} : E \longrightarrow \mathbb{K}, \quad x \longmapsto \left\langle \widetilde{(\delta_{ss'}\delta_{tt'}x)_{(s',t')\in S\times T}}, u' \right\rangle$$

for every $(t,s) \in T \times S$. Put

$$\pi_A : K \longrightarrow K, \quad \eta \longmapsto \eta e_A^S,$$

$$\pi_B : H \longrightarrow H, \quad \xi \longmapsto \xi e_B^T$$

for every $A \subset S$ and $B \subset T$. By Proposition 5.6.4.20 i),

$$\pi_A \in \Pr\mathcal{L}_E(K), \quad \pi_B \in \Pr\mathcal{L}_E(H)$$

for every $A \subset S$ and $B \subset T$ and

$$u' = \lim_{A\times B,\mathfrak{F}} \pi_B u' \pi_A = \lim_{A\times B,\mathfrak{F}} \widetilde{x' e_{B\times A}^{T\times S}}$$

in the topology of pointwise convergence on $\mathcal{L}$, with the above defined $x' \in (E')^{T\times S}$. ■

6.3.3 Extension of the Functional Calculus

Throughout this subsection, E will denote a W^*–algebra.

Theorem 6.3.3.1 (0) *Denote by F the predual of E and take $x \in \operatorname{Sn} E$. Let $j : \sigma(x) \to \mathbb{K}$ be the inclusion map. Put $\mathcal{C} := \mathcal{C}(\sigma(x))$ and consider*

$$\varphi : \mathcal{C} \longrightarrow E, \quad f \longmapsto f(x).$$

a) *There is a unique unital W^*–homomorphism $u : \mathcal{C}'' \to E$ (Theorem 6.3.2.1 b)) such that $uj_{\mathcal{C}}j = x$. We define $f(x) := uf$ for $f \in \mathcal{C}''$.*

b) $u = j'_F \circ \varphi''$, $\quad u \circ j_{\mathcal{C}} = \varphi$.

c) *If f is a bounded Borel function on $\sigma(x)$ and*

$$\widetilde{f} : \mathcal{C}' \longrightarrow \mathbb{K}, \quad \mu \longmapsto \int f d\mu,$$

then

$$u(\widetilde{f}) = f(x),$$

where $f(x)$ was defined in Corollary 4.3.2.5 b).

d) $\operatorname{Im} u$ *is the unital W^*–subalgebra of E generated by x.*

e) $\sigma(f(x)) \subset \widehat{f}(\sigma(\mathcal{C}''))$ *for every $f \in \mathcal{C}''$.*

f) $g \circ \widehat{f} \in \mathcal{C}(\sigma(\mathcal{C}'')) = \mathcal{C}''$ *and* $g(f(x)) = (g \circ \widehat{f})(x)$, *whenever $f \in \mathcal{C}''$ and $g \in \mathcal{C}(\widehat{f}(\sigma(\mathcal{C}'')))$.*

g) *Take $f \in \mathcal{C}''$, $\mathcal{D} := \mathcal{C}(\widehat{f}(\sigma(\mathcal{C}'')))$, and*

$$\psi : \mathcal{D} \longrightarrow \mathcal{C}'' = \mathcal{C}(\sigma(\mathcal{C}'')), \quad h \longmapsto h \circ \widehat{f}.$$

If $g \in \mathcal{D}''$, then $j'_{\mathcal{C}'}\psi''g \in \mathcal{C}''$ and

$$g(f(x)) = (j'_{\mathcal{C}'}\psi''g)(x);$$

for every $\alpha \in \widehat{f}(\sigma(\mathcal{C}''))$ we have that

$$j'_{\mathcal{C}'}\psi''g = g(\delta_\alpha)$$

on the interior of $\overset{-1}{\widehat{f}}(\alpha)$, where δ_α denotes the Dirac measure on $\widehat{f}(\sigma(\mathcal{C}''))$ at α.

h) *If the predual of E is separable, then for every $f \in \mathcal{C}''$ there is a bounded Borel function g on $\sigma(x)$ such that*

$$f(x) = g(x);$$

in particular

$$\{h(x) \mid h \text{ bounded Borel function on } \sigma(x)\}$$

is the unital W^*-subalgebra of E generated by x.

i) *If μ denotes the spectral measure of x (4.4.1.11 a)), then $x' \circ \mu$ is a measure for every $x' \in E^\pi$ and*

$$\langle f(x), x' \rangle = \langle f, x' \circ \mu \rangle$$

for every $f \in \mathcal{C}''$. Moreover, if $\mathfrak{A}$ (resp. $\mathfrak{B}$) denotes the σ-algebra of Borel sets of $\sigma(\mathcal{C}'')$ (resp. $\sigma(f(x))$) and if we set

$$\widetilde{A} : \mathcal{C}' \longrightarrow \mathbb{K}, \quad \lambda \longmapsto (j_{\mathcal{C}'}\lambda)(A)$$

for every $A \in \mathfrak{A}$, then

$$\mathfrak{B} \longrightarrow P_r E, \quad B \longmapsto \widetilde{\widehat{f}^{-1}(B)}\ (x)$$

is the spectral measure of $f(x)$.

a & b. First we prove the uniqueness. $u \circ j_{\mathcal{C}}$ is an involutive algebra homomorphism (Theorem 6.3.2.2 a)) such that

$$u \circ j_{\mathcal{C}}(j) = x.$$

Hence

$$u \circ j_{\mathcal{C}} = \varphi.$$

Since the map

$$u : \mathcal{C}''_{\mathcal{C}'} \longrightarrow E_F$$

is continuous and since $\operatorname{Im} j_{\mathcal{C}}$ is dense in $\mathcal{C}''_{\mathcal{C}'}$ (Corollary 1.3.6.5), the map u is uniquely determined.

Now we prove the existence. Put

$$u := j'_F \circ \varphi'' .$$

By Corollary 6.3.2.3 and Theorem 6.3.2.2 a), u is a unital W^*–homomorphism. By Proposition 1.3.6.16 and Proposition 1.3.6.19 a),

$$u \circ j_{\mathcal{C}} = j'_F \circ \varphi'' \circ j_{\mathcal{C}} = j'_F \circ j_E \circ \varphi = \varphi .$$

In particular,

$$u j_{\mathcal{C}} j = \varphi j = x .$$

c) Let $\mathcal{B}$ be the set of bounded Borel functions on $\sigma(x)$ and put

$$\mathcal{B}_0 := \{ f \in \mathcal{B} \mid u(\widetilde{f}) = f(x) \} .$$

By b) and Corollary 4.3.2.5 b)), $\mathcal{C} \subset \mathcal{B}_0$. Let $(f_n)_{n \in \mathbb{N}}$ be an increasing sequence in $\mathcal{B}_0$ with the supremum f in $\mathcal{B}$. Since

$$\widetilde{f} = \bigvee_{n \in \mathbb{N}} \widetilde{f}_n$$

we get

$$u(\widetilde{f}) = \bigvee_{n \in \mathbb{N}} u(\widetilde{f}_n) = \bigvee_{n \in \mathbb{N}} f_n(x) = f(x)$$

by Proposition 4.4.4.8 d) and Corollary 4.3.2.5 a). Hence, $f \in \mathcal{B}_0$. It follows that $\mathcal{B}_0 = \mathcal{B}$.

d) By a) and Corollary 4.4.4.8 b), $\operatorname{Im} u$ is a unital W^*–subalgebra of E containing x. Let G be a unital W^*–subalgebra of E containing x. By Corollary 4.4.4.9 a $\Rightarrow$ b, G is a W^*–algebra. By the uniqueness of u, we get that $\operatorname{Im} u \subset G$. Hence $\operatorname{Im} u$ is the unital W^*–subalgebra of E generated by x.

e) By a) and Proposition 2.4.1.17, there is an injective continuous map $\varphi_0 : \sigma(\operatorname{Im} u) \longrightarrow \sigma(\mathcal{C}'')$ such that

$$\widehat{f} \circ \varphi_0 = \widehat{f(x)} .$$

By Corollary 2.4.1.7 a),

$$\sigma(f(x)) = \widehat{f(x)}(\sigma(\operatorname{Im} u)) = \widehat{f} \circ \varphi_0(\sigma(\operatorname{Im} u)) \subset \widehat{f}(\sigma(\mathcal{C}'')) .$$

f) By Example 2.4.3.1, we may canonically identify $\mathcal{C}(\sigma(\mathcal{C}''))$ with $\mathcal{C}''$. First suppose that there is a $P \in \mathbb{K}[s,t]$ such that

$$g(\alpha) = P(\alpha, \overline{\alpha})$$

for every $\alpha \in \widehat{f}(\sigma(\mathcal{C}''))$. By b), Theorem 6.3.2.1 b), and Corollary 6.3.2.3 (and Corollary 4.1.3.8),

$$g(f(x)) = P(f(x), f(x)^*) = P(j'_F\varphi'' f, j'_F\varphi''\overline{f}) =$$

$$= j'_F\varphi'' P(\widehat{f}, \overline{\widehat{f}}) = j'_F\varphi''(g \circ \widehat{f}) = g \circ \widehat{f}(x)\,.$$

By the Weierstrass–Stone Theorem, the above equality holds for an arbitrary g.

g) Define

$$w : \mathcal{D} \longrightarrow E\,, \quad h \longmapsto h(f(x))\,.$$

By b) and f),

$$w = j'_F \circ \varphi'' \circ \psi\,,$$

so that

$$w'' = j''_F \circ \varphi'''' \circ \psi''\,.$$

Take $(a, h) \in F \times \mathcal{C}''$. Then

$$\langle h, \varphi''' j''_F j_F a\rangle = \langle j'_F\varphi'' h, j_F a\rangle = \langle j'_F\varphi'' h, a\rangle =$$

$$= \langle \varphi'' h, j_F a\rangle = \langle h, \varphi' j_F a\rangle = \langle h, j_{\mathcal{C}'}\varphi' j_F a\rangle\,.$$

We deduce successively that

$$\varphi''' \circ j''_F \circ j_F = j_{\mathcal{C}'} \circ \varphi' \circ j_F\,,$$

$$j'_F \circ j''_F \circ \varphi'''' = j'_F \circ \varphi'' \circ j'_{\mathcal{C}'}\,,$$

$$j'_F \circ w'' = j'_F \circ j''_F \circ \varphi'''' \circ \psi'' = j'_F \circ \varphi'' \circ j'_{\mathcal{C}'} \circ \psi''\,,$$

$$g(f(x)) = j'_F \circ w''(g) = j'_F \circ \varphi''(j'_{\mathcal{C}'}\psi'' g) = (j'_{\mathcal{C}'}\psi'' g)(x)\,.$$

Now we prove the final assertion of g). Let U be the interior of $\overset{-1}{f}(\alpha)$, and take $x' \in \mathcal{C}'$ with

$$\operatorname{Supp} j_{\mathcal{C}'} x' \subset U\,.$$

Then

$$\langle y, \psi' j_{\mathcal{C}'} x' \rangle = \langle \psi y, j_{\mathcal{C}'} x' \rangle = \langle y \circ \widehat{f}, j_{\mathcal{C}'} x' \rangle =$$

$$= \langle y(\alpha) e_{\sigma(\mathcal{C}'')}, j_{\mathcal{C}'} x' \rangle = \langle y, \delta_\alpha \rangle x'(\sigma(x))$$

for every $y \in \mathcal{D}$, so that

$$\psi' j_{\mathcal{C}'} x' = x'(\sigma(x)) \delta_\alpha .$$

It follows that

$$\langle x', j'_{\mathcal{C}'} \psi'' g \rangle = \langle \psi' j_{\mathcal{C}'} x', g \rangle = x'(\sigma(x)) g(\delta_\alpha) .$$

Since $\sigma(\mathcal{C}'')$ is a hyperstonian space (Theorem 6.3.2.1 b),c)),

$$j'_{\mathcal{C}'} \psi'' g = g(\delta_\alpha)$$

on U.

h) Let $(a_n)_{n \in \mathbb{N}}$ be a dense sequence in the predual of E. Then $\varphi' j_F a_n \in \mathcal{C}'$ for every $n \in \mathbb{N}$. Define

$$\mu := \sum_{n \in \mathbb{N}} \frac{1}{2^n (\|\varphi' j_F a_n)\| + 1)} |\varphi' j_F a_n| \in \mathcal{C}' .$$

The map

$$L^1(\mu) \longrightarrow \mathbb{K}, \quad h \longmapsto \langle h \cdot \mu, f \rangle$$

is continuous and so there is a bounded Borel function g on $\sigma(x)$ such that

$$\langle h \cdot \mu, f \rangle = \int h g d\mu$$

for every $h \in L^1(\mu)$.

Take $n \in \mathbb{N}$. By the Radon–Nikodym Theorem, there is a $g_n \in L^\infty(\mu)$ such that

$$\varphi' j_F a_n = g_n \cdot \mu .$$

Then, by a),b), and c),

$$\langle a_n, f(x) \rangle = \langle a_n, uf \rangle = \langle a_n, j'_F \varphi'' f \rangle = \langle \varphi' j_F a_n, f \rangle =$$

$$= \langle g_n \cdot \mu, f \rangle = \int g_n g d\mu = \int g d(\varphi' j_F a_n) = \langle \varphi' j_F a_n, \widetilde{g} \rangle =$$

$$= \langle a_n, j_F' \varphi'' \widetilde{g} \rangle = \langle a_n, u\widetilde{g} \rangle = \langle a_n, g(x) \rangle .$$

Since n is arbitrary and $(a_n)_{n \in \mathbb{N}}$ is dense, it follows that

$$f(x) = g(x) .$$

The final assertion follows by d).

i) $x' \circ \mu$ is obviously a measure. Put

$$\mathcal{F} := \{ f \in \mathcal{C}'' \mid \langle f(x), x' \rangle = \langle f, x' \circ \mu \rangle \} .$$

By a), Theorem 4.4.1.8 c), and Corollary 4.4.4.8 d), $\mathcal{F}$ is an order faithful vector subspace of $\mathcal{C}''$. Since it contains $\mathcal{C}$ (Corollary 4.4.1.11 e)), $\mathcal{F} = \mathcal{C}''$ follows from Example 6.3.2.6 f).

Take $\nu \in \mathcal{C}'''$. Then, using the notation from g), for every $h \in \mathcal{D}$,

$$\langle \psi' \nu, h \rangle = \langle \nu, \psi h \rangle = \langle \nu, h \circ \widehat{f} \rangle = \langle \widehat{f}(\nu), h \rangle$$

so that

$$\psi' \nu = \widehat{f}(\nu) .$$

Take $B \in \mathfrak{B}$ and define

$$\widetilde{B} : \mathcal{D}' \longrightarrow \mathbb{K} , \quad \rho \longmapsto \rho(B) .$$

By the above considerations, given $\lambda \in \mathcal{C}'$,

$$\langle j_{\mathcal{C}'}' \psi'' \widetilde{B}, \lambda \rangle = \langle \psi'' \widetilde{B}, j_{\mathcal{C}'} \lambda \rangle = \langle \widetilde{B}, \psi' j_{\mathcal{C}'} \lambda \rangle =$$

$$= \langle \widetilde{B}, \widehat{f}(j_{\mathcal{C}'} \lambda) \rangle = (\widehat{f}(j_{\mathcal{C}'} \lambda))(B) =$$

$$= (j_{\mathcal{C}'} \lambda)(\overset{-1}{\widehat{f}}(B)) = \widetilde{\overset{-1}{\widehat{f}}}(B)(\lambda) .$$

Hence

$$j_{\mathcal{C}'}' \psi'' \widetilde{B} = \widetilde{\overset{-1}{\widehat{f}}}(B) .$$

By g),

$$\mathfrak{B} \longrightarrow \Pr E , \quad B \longmapsto \widetilde{\overset{-1}{\widehat{f}}}(B)(x)$$

is the spectral measure of $f(x)$. ■

Corollary 6.3.3.2 *Let F be the predual of E. Take $a \in F$ (resp. $z \in E$) and $x, y \in \operatorname{Sn} E$, such that*

$$xa = ay \quad (\textit{resp. } xz = zy).$$

Put

$$\mathcal{C} := \mathcal{C}(\sigma(x) \cup \sigma(y)).$$

Then

$$f(x)a = af(y) \quad (\textit{resp. } f(x)z = zf(y))$$

for every $f \in \mathcal{C}''$.

Define

$$\mathcal{F} := \{f \in \mathcal{C}'' \mid f(x)a = af(y) \quad (\text{resp. } f(x)z = zf(y))\}.$$

Let $\mathcal{G}$ be a nonempty upward (downward) directed subset of $\mathcal{F}$ with supremum (infimum) f in $\mathcal{C}''$ and let $\mathfrak{F}$ be its upper (lower) section filter. By Theorem 6.3.3.1 a) and Corollary 4.4.4.8 d), $f(x)$ is the supremum (infimum) of $(g(x))_{g\in\mathcal{G}}$ in E and $f(y)$ is the supremum (infimum) of $(g(y))_{y\in\mathcal{G}}$ in E. By Corollary 4.4.3.10,

$$f(x)a = af(y)$$

(by Theorem 4.4.1.8 c),

$$\langle b, f(x)z\rangle = \langle zb, f(x)\rangle = \lim_{g\in\mathfrak{F}}\langle zb, g(x)\rangle =$$

$$= \lim_{g,\mathfrak{F}}\langle b, g(x)z\rangle = \lim_{g,\mathfrak{F}}\langle b, zg(y)\rangle = \lim_{g,\mathfrak{F}}\langle bz, g(y)\rangle =$$

$$= \langle bz, f(y)\rangle = \langle b, zf(y)\rangle$$

for every $b \in F$, so that

$$f(x)z = zf(y)).$$

Hence f $\in \mathcal{F}$. Since $\mathcal{C}(\sigma(x) \cup \sigma(y)) \subset \mathcal{F}$ (Theorem 4.1.4.1), we get $\mathcal{F} = \mathcal{C}''$. ∎

Example 6.3.3.3 *Let K be a closed vector subspace of H such that neither K nor $K^\perp$ is separabel. Then for any self-normal compact operator x on H*

$$\pi_K \notin \{f(x) \mid f \in \mathcal{C}(\sigma(x))''\}\,.$$

By Theorem 5.5.6.1 a $\Rightarrow$ e, there is a separable closed vector subspace L of H such that

$$x = \pi_L x \pi_L\,.$$

Since $\pi_L \mathcal{L}(H)\pi_L$ is a W^*-subalgebra of $\mathcal{L}(H)$ (Corollary 4.4.4.12 c) and Corollary 6.1.7.14 b)), it follows that

$$\{\pi_L y \pi_L + \alpha 1 \mid y \in \mathcal{L}(H)\,,\ \alpha \in \mathbb{K}\}$$

is a unital W^*-subalgebra of $\mathcal{L}(H)$. By Theorem 6.3.3.1 d),

$$\{f(x) \mid f \in \mathcal{C}(\sigma(x))''\} \subset \{\pi_L y \pi_L + \alpha 1 \mid y \in \mathcal{L}(H)\,,\ \alpha \in \mathbb{K}\}\,.$$

Assume that

$$\pi_K \in \{f(x) \mid f \in \mathcal{C}(\sigma(x))''\}\,.$$

Then there are $y \in \mathcal{L}(H)$ and $\alpha \in \mathbb{K}$ such that

$$\pi_K = \pi_L y \pi_L + \alpha 1\,.$$

Since L is separable and K is not separable, $\alpha = 1$. Hence

$$\pi_{K^\perp} = 1 - \pi_K = \pi_L y \pi_L\,,$$

which is a contradiction, since $K^\perp$ is not separable. ■

Theorem 6.3.3.4 (0) *Take $x \in \operatorname{Sn} E$, $\mathcal{C} := \mathcal{C}(\sigma(x))$, and let G be the unital W^*-subalgebra of E generated by x. Identify $\mathcal{C}''$ with $\mathcal{C}(\sigma(\mathcal{C}''))$ using the Gelfand transform (Theorem 6.3.2.1 c)).*

a) *Given $f, g \in \mathcal{C}''$ with $\{g \neq 0\} \subset \overline{\{f \neq 0\}}$, $g(x) = 0$ whenever $f(x) = 0$.*

b) *There is a greatest clopen set U of $\sigma(\mathcal{C}'')$ such that*

$$e_U(x) = 0\,.$$

Its complement in $\sigma(\mathcal{C}'')$ is denoted by $S(x)$ and is called ***the support of*** *x.*

c) $S(x)$ *is a compact hyperstonian space and* $\mathcal{C}(S(x))$ *is a* W^**-algebra.*

d) *If* $f \in \mathcal{C}(S(x))$ *and* $f_1, f_2 \in \mathcal{C}''$ *such that*

$$f_1|S(x) = f_2|S(x) = f$$

then

$$f_1(x) = f_2(x)\,.$$

Define

$$f(x) := f_1(x)\,.$$

e) *The map*

$$\mathcal{C}(S(x)) \longrightarrow G\,, \quad f \longmapsto f(x)$$

is an isometry of W^**-algebras.*

f) $e_{S(x)}(x)$ *is the carrier of* x *in* G.

g) $\sigma(f(x)) = f(S(x))$, *whenever* $f \in \mathcal{C}(S(x))$.

h) $g(f(x)) = g \circ f(x)$ *for all* $f \in \mathcal{C}(S(x))$, $g \in \mathcal{C}(f(S(x)))$.

a) We may assume that $|f| \leq 1$. Given $\alpha > 0$,

$$|f|^\alpha(x) = |f(x)|^\alpha = 0$$

(Theorem 6.3.3.1 f), Proposition 4.2.5.4). Define

$$U := \overline{\{f \neq 0\}}\,.$$

U is a clopen set (Theorem 6.3.2.1 c)). Then

$$\bigvee_{\alpha>0} |f|^\alpha = e_U\,,$$

and so, by Theorem 6.3.3.1 a) and Corollary 4.4.4.8 d),

$$e_U(x) = \bigvee_{\alpha>0} |f|^\alpha(x) = 0\,.$$

Thus

$$g(x) = (ge_U)(x) = g(x)e_U(x) = 0\,.$$

b) Put

$$\mathfrak{U} := \{V \text{ clopen set of } \sigma(\mathcal{C}'') \mid e_V(x) = 0\},$$

$$U := \overline{\bigcup_{V \in \mathfrak{U}} V}.$$

Take $V, W \in \mathfrak{U}$. By a),

$$e_{V \setminus W}(x) = 0,$$

so that

$$e_{V \cup W}(x) = e_{V \setminus W}(x) + e_W(x) = 0,$$

$$V \cup W \in \mathfrak{U}.$$

By Theorem 6.3.3.1 a) and Corollary 4.4.4.8 d),

$$e_U(x) = \bigvee_{V \in \mathfrak{U}} e_V(x) = 0.$$

Hence $U \in \mathfrak{U}$. U is obviously the greatest element of $\mathfrak{U}$.

c) $\mathcal{C}(S(x))$ may be identified with $e_{S(x)}\mathcal{C}''$, which, by Corollary 4.4.4.12 c), is a W^*-subalgebra of $\mathcal{C}''$. By Corollary 4.4.1.10, $S(x)$ is a compact hyperstonian space.

d) We have

$$\{f_1 \neq f_2\} \subset \sigma(\mathcal{C}'') \setminus S(x)$$

and so, by a),

$$f_1(x) - f_2(x) = (f_1 - f_2)(x) = 0.$$

e) By Theorem 6.3.3.1. a), it suffices to prove that the map is injective. Take $f \in \mathcal{C}(S(x))$ with

$$f(x) = 0.$$

Define

$$g : \sigma(\mathcal{C}'') \longrightarrow \mathbb{K}, \quad x'' \longmapsto \begin{cases} f(x'') & \text{if } x'' \in S(x) \\ 0 & \text{if } x'' \in \sigma(\mathcal{C}'') \setminus S(x), \end{cases}$$

$$V := \overline{\{g \neq 0\}}\,.$$

Then

$$g(x) = 0$$

and by a),

$$e_V(x) = 0\,.$$

Hence (by b))

$$V \cap S(x) = \emptyset\,,$$

$$V = \emptyset\,,$$

$$f = 0\,.$$

f) follows immediately from e).

g) By e),

$$\sigma(f(x)) = \sigma(f) = f(S(x))\,.$$

h) First suppose that there is a $P \in \mathbb{K}[s,t]$ with

$$g(\alpha) = P(\alpha, \overline{\alpha})$$

for every $\alpha \in f(S(x))$. Then

$$g(f(x)) = P(f(x), f(x)^*) = g \circ f(x)\,.$$

By the Weierstrass–Stone Theorem, the above equality holds for an arbitrary g. ■

Proposition 6.3.3.5 (0) *Take $x \in \operatorname{Sn} E$ and put $\mathcal{C} := \mathcal{C}(\sigma(x))$. Let U be a clopen set of $\sigma(\mathcal{C}'')$. We identify $\mathcal{C}''$ with $\mathcal{C}(\sigma(\mathcal{C}''))$ via the Gelfand transform. Define*

$$j : \sigma(x) \longrightarrow \mathbb{K}\,, \quad \alpha \longmapsto \alpha\,,$$

$$y := ((j_{\mathcal{C}} j) e_U)(x)\,,$$

and let λ and μ denote the spectral measures of x and y, respectively. Let $\mathfrak{B}$ be the σ-algebra of Borel sets of $\sigma(x)$ and for $\rho \in \mathcal{M}_b(\sigma(y))$, define

$$\widetilde{\rho} : \mathfrak{B} \longrightarrow \mathbb{K}\,, \quad A \longmapsto \rho(A \cap (\sigma(y) \backslash \{0\}))\,.$$

a) $\sigma(y) \subset \sigma(x) \cup \{0\}$.

b) *If* $0 \notin \sigma(x)$, *then for every* $x' \in E^\pi$ *and* $f'' \in \mathcal{C}''$,

$$\langle f'', \widetilde{x' \circ \mu} \rangle = \langle f'' e_U , x' \circ \lambda \rangle .$$

c) *If* $0 \in \sigma(x)$ *and if* δ_0 *denotes the Dirac measure on* $\sigma(x)$ *at* 0, *then for every* $x' \in E^\pi$ *and* $f'' \in \mathcal{C}''$ *with* $\langle f'', \delta_0 \rangle = 0$,

$$\langle f'', \widetilde{x' \circ \mu} \rangle = \langle f'' e_U, x' \circ \lambda \rangle .$$

d) $\operatorname{Supp} j_{\mathcal{C}'}(\widetilde{x' \circ \mu}) \subset U$ *for every* $x' \in E^\pi$.

e) *If* x *is selfadjoint and*

$$\begin{aligned} f &: \sigma(x) \longrightarrow \mathbb{K}, \qquad \alpha \longmapsto \sup\{\alpha, 0\}, \\ g &: \sigma(x) \longrightarrow \mathbb{K}, \qquad \alpha \longmapsto \sup\{-\alpha, 0\}, \\ h &: \sigma(x) \longrightarrow \mathbb{K}, \qquad \alpha \longmapsto |\alpha| \end{aligned}$$

then y *is selfadjoint and*

$$y^+ = ((j_{\mathcal{C}} f) e_U)(x), \qquad y^- = ((j_{\mathcal{C}} g) e_U)(x)$$

$$|y| = ((j_{\mathcal{C}} h) e_U)(x) .$$

a) By Theorem 6.3.3.4 g),

$$\sigma(x) = (j_{\mathcal{C}} j)(S(x)),$$

$$\sigma(y) = ((j_{\mathcal{C}} j) e_U)(S(x)) .$$

Hence

$$\sigma(y) \subset \sigma(x) \cup \{0\} .$$

b & c. Define

$$\mathcal{C}_0 := \{ f \in \mathcal{C} \mid 0 \in \sigma(x) \Longrightarrow f(0) = 0 \}$$

and for $f \in \mathcal{C}_0$, define

$$\widetilde{f} : \sigma(y) \longrightarrow \mathbb{K}, \quad \alpha \longmapsto \begin{cases} f(\alpha) & \text{if } \alpha \in \sigma(x) \\ 0 & \text{if } \alpha \notin \sigma(x) \end{cases} .$$

Step 1 $f \in \mathcal{C}_0$, $\rho \in \mathcal{M}_b(\sigma(y)) \Rightarrow \widetilde{f} \in \mathcal{C}(\sigma(y))$, $\langle \widetilde{f}, \rho \rangle = \langle f, \widetilde{\rho} \rangle$

The assertion follows from a).

Step 2 $f \in \mathcal{C}_0 \Rightarrow \widetilde{f} \circ ((j_{\mathcal{C}} j) e_U) = (j_{\mathcal{C}} f) e_U$

The maps

$$\mathcal{C} \longrightarrow \mathcal{C}'', \quad f \longmapsto j_{\mathcal{C}} f,$$

$$\mathcal{C} \longrightarrow \mathcal{C}'', \quad f \longmapsto f \circ (j_{\mathcal{C}} j)$$

are involutive algebra homomorphisms (6.3.2.1 b)). Since they take the same value at $j \in \mathcal{C}$, they coincide. The assertion follows.

Step 3 $f \in \mathcal{C}_0 \Rightarrow \langle f, \widetilde{x' \circ \mu} \rangle = \langle (j_{\mathcal{C}} f) e_U, x' \circ \lambda \rangle$

By the first two steps and Theorem 6.3.3.1 f),i),

$$\langle f, \widetilde{x' \circ \mu} \rangle = \langle \widetilde{f}, x' \circ \mu \rangle = \langle \widetilde{f}(y), x' \rangle =$$

$$= \langle \widetilde{f}(((j_{\mathcal{C}} j) e_U)(x)), x' \rangle = \langle \widetilde{f} \circ ((j_{\mathcal{C}} j) e_U)(x), x' \rangle =$$

$$= \langle ((j_{\mathcal{C}} f) e_U)(x), x' \rangle = \langle (j_{\mathcal{C}} f) e_U, x' \circ \lambda \rangle .$$

Step 4 b & c

Let $\mathcal{F}$ be the set of $f'' \in \mathcal{C}''$ such that

$$\langle f'', \widetilde{x' \circ \mu} \rangle = \langle f'' e_U, x' \circ \lambda \rangle .$$

Then $\mathcal{F}$ is an order faithful vector subspace of $\mathcal{C}''$. By Step 1, it contains $j_{\mathcal{C}}(\mathcal{C}_0)$. If $0 \notin \sigma(x)$ then $\mathcal{C} = \mathcal{C}_0$, $\mathcal{F}$ contains $j_{\mathcal{C}}(\mathcal{C})$, and so $\mathcal{F} = \mathcal{C}''$. Assume that $0 \in \sigma(x)$. By Example 6.3.2.6 e),

$$\{f'' \in \mathcal{C}'' \mid \langle f'', \delta_0 \rangle = 0\} \subset \mathcal{F} .$$

d) follows from b) and c).

e) By Theorem 6.3.2.1 b),

$$(j_{\mathcal{C}} j) e_U \in \mathrm{Re}\, \mathcal{C}'',$$

$$(j_{\mathcal{C}} f) e_U, (j_{\mathcal{C}} g) e_U \in \mathcal{C}''_+,$$

$$j_{\mathcal{C}} f - j_{\mathcal{C}} g = j_{\mathcal{C}}(f - g) = j_{\mathcal{C}} j,$$

$$(j_{\mathcal{C}} f)(j_{\mathcal{C}} g) = j_{\mathcal{C}}(fg) = 0 .$$

Hence

$$y = ((j_C j)e_U)(x) \in \operatorname{Re} E\,,$$

$$((j_C f)e_U)(x)\,,\ ((j_C g)e_U)(x) \in E_+\,,$$

$$((j_C f)e_U)(x) - ((j_C g)e_U)(x) = ((j_C j)e_U)(x) = y\,,$$

$$(((j_C f)e_U)(x))(((j_C g)e_U)(x)) = 0$$

(Theorem 6.3.3.1 e)). Thus

$$((j_C f)e_U)(x) = y^+\,,\quad ((j_C g)e_U)(x) = y^-\,,$$

$$((j_C h)e_U)(x) = ((j_C f)e_U + (j_C g)(e_U))(x) = y^+ + y^- = |y|\,. \qquad \blacksquare$$

Remark. If x is not selfadjoint then the relation

$$|y| = ((j_C h)e_U)(x)$$

in e) may fail (see Remark of Corollary 6.3.3.7).

Theorem 6.3.3.6 (0) *Take $x \in \operatorname{Sn} E$. Let G be the unital W^*-subalgebra of E generated by x. Put $C := C(\sigma(x))$ and let $(\mathcal{M}_\iota)_{\iota\in I}$ be a finite family of pairwise orthogonal bands of C' (identified with $\mathcal{M}_b(\sigma(x))$) such that C' is the vector subspace of C' generated by $\bigcup_{\iota\in I} \mathcal{M}_\iota$. Let $\mathfrak{B}$ denote the σ-algebra of Borel sets of $\sigma(x)$ and for every $y \in G$, let μ_y be its spectral measure. Define*

$$\widetilde{\mu}_y : \mathfrak{B} \longrightarrow Pr\,E\,,\quad A \longmapsto \mu_y(A \cap (\sigma(y)\backslash\{0\}))\,.$$

a) *There is a unique family $(x_\iota)_{\iota\in I}$ in G such that*

$$x = \sum_{\iota\in I} x_\iota\,,$$

$$\iota, \lambda \in I\,,\ \iota \neq \lambda \Longrightarrow x_\iota x_\lambda = 0\,,$$

$$\iota \in I\,,\ x' \in E^\pi \Longrightarrow x' \circ \widetilde{\mu}_{x_\iota} \in \mathcal{M}_\iota\,.$$

x_ι is called ***the component of*** *x* ***on*** *$\mathcal{M}_\iota$ for every $\iota \in I$.*

b)

$$\sigma(x) \subset \bigcup_{\iota \in I} \sigma(x_\iota) \subset \sigma(x) \cup \{0\}\,,$$

$$\operatorname{Card} I \neq 1 \Longrightarrow \bigcup_{\iota \in I} \sigma(x_\iota) = \sigma(x) \cup \{0\}\,.$$

c) *For every* $\iota \in I$, *let* p_ι *be the projection of* $\mathcal{C}'$ *onto* $\mathcal{M}_\iota$ *and define*

$$f_\iota'' : \mathcal{C}' \longrightarrow \mathbb{K}\,, \quad f' \longmapsto \langle e_{\sigma(x)}, p_\iota f' \rangle\,,$$

$$j : \sigma(x) \longrightarrow \mathbb{K}\,, \quad \alpha \longmapsto \alpha\,.$$

Then $f_\iota'' \in \mathcal{C}''$, $f_\iota''(x) \in Pr\,E$, *and*

$$x_\iota = ((j_{\mathcal{C}} j) f_\iota'')(x)$$

for every $\iota \in I$.

d) *We identify* $\mathcal{C}''$ *with* $\mathcal{C}(\sigma(\mathcal{C}''))$ *using the Gelfand transform. Take* $\iota \in I$ *and let* U_ι *be the clopen set of* $\sigma(\mathcal{C}'')$ *such that*

$$\mathcal{M}_\iota = \{f' \in \mathcal{C}' \mid \operatorname{Supp} j_{\mathcal{C}'} f' \subset U_\iota\}$$

(Example 6.3.2.8). Then

$$f_\iota'' = e_{U_\iota}\,.$$

d) follows from Example 6.3.2.8. Given $\iota \in I$, put

$$x_\iota := ((j_{\mathcal{C}} j) f_\iota'')(x)\,.$$

Step 1 $\quad (\iota \in I \Rightarrow f_\iota''(x) \in \operatorname{Pr} E)$ and $\sum_{\iota \in I} f_\iota''(x) = 1$

The assertion follows from d) and Theorem 6.3.3.1 a).

Step 2 $\quad \begin{cases} \iota, \lambda \in I, \iota \neq \lambda \Rightarrow x_\iota \in G, x_\iota x_\lambda = 0 \\ \sum_{\iota \in I} x_\iota = x \end{cases}$

By d) and Theorem 6.3.3.1 a),d),

$$x_\iota = ((j_{\mathcal{C}} j) e_{U_\iota})(x) \in G\,,$$

$$x_\iota x_\lambda = 0\,,$$

and

$$x = \sum_{\iota \in I} x_\iota\,.$$

Step 3 b)

By Theorem 6.3.3.4 g),

$$\sigma(x) = (j_C j)(S(x))\,,$$

$$\sigma(x_\iota) = ((j_C j) e_{U_\iota})(S(x))\,.$$

It follows that

$$\sigma(x) \subset \bigcup_{\iota \in I} \sigma(x_\iota) \subset \sigma(x) \cup \{0\}$$

and

$$\bigcup_{\iota \in I} \sigma(x_\iota) = \sigma(x) \cup \{0\}\,,$$

whenever $\operatorname{Card} I \neq 1$.

Step 4 $\quad \iota \in I\,,\ x' \in E^\pi \Rightarrow x' \circ \widetilde{\mu}_{x_\iota} \in \mathcal{M}_\iota$

By d) and Proposition 6.3.3.5 d),

$$\operatorname{Supp} j_{C'}(x' \circ \widetilde{\mu}_{x_\iota}) \subset U_\iota\,,$$

so that

$$x' \circ \widetilde{\mu}_{x_\iota} \in \mathcal{M}_\iota\,.$$

Step 5 Uniqueness

Let $(y_\iota)_{\iota \in I}$ be a family in G satisfying the properties described in a) for $(x_\iota)_{\iota \in I}$. By Theorem 6.3.3.1 d), for every $\iota \in I$ there is a $g''_\iota \in C''$ with

$$y_\iota = g''_\iota(x)\,.$$

We have

$$\left(\sum_{\iota\in I} g_\iota''\right)(x) = \sum_{\iota\in I} y_\iota = x = (jcj)(x)$$

and

$$(g_\iota'' g_\lambda'')(x) = y_\iota y_\lambda = 0$$

for all distinct $\iota, \lambda \in I$. Hence

$$\sum_{\iota\in I} g_\iota'' = jcj\,, \quad g_\iota'' g_\lambda'' = 0$$

on $S(x)$ for all dinstict $\iota, \lambda \in I$ (Theorem 6.3.3.4 e)). Therefore there is a partition $(V_\iota)_{\iota\in I}$ of $S(x)$ into clopen sets such that

$$g_\iota'' = (jcj)e_{V_\iota}$$

on $S(x)$ for every $\iota \in I$. It follows that

$$\sigma(x) \subset \bigcup_{\iota\in I} \sigma(x_\iota) \subset \sigma(x) \cup \{0\}$$

by Proposition 6.3.3.4 g).

If $0 \in \sigma(x)$, then write δ_0 for the Dirac measure on $\sigma(x)$ at 0. Take $x' \in E^\pi$. By Proposition 6.3.3.5 b),c),

$$\langle h'', x' \circ \widetilde{\mu}_{x_\iota}\rangle = \langle h'' e_{U_\iota}\,,\, x' \circ \mu_x\rangle\,,$$

$$\langle h'', x' \circ \widetilde{\mu}_{y_\iota}\rangle = \langle h'' e_{V_\iota}\,,\, x' \circ \mu_x\rangle$$

for every $\iota \in I$ and $h'' \in \mathcal{C}''$ satisfying the condition

$$\langle h'', \delta_0\rangle = 0$$

if $0 \in \sigma(x)$. It follows that

$$\langle h'', \sum_{\iota\in I} x' \circ \widetilde{\mu}_{x_\iota}\rangle = \langle h'', \sum_{\iota\in I} x' \circ \widetilde{\mu}_{y_\iota}\rangle$$

for every $h'' \in \mathcal{C}''$ satisfying the condition

$$\langle h'', \delta_0\rangle = 0$$

whenever $0 \in \sigma(x)$. Since

$$\widetilde{\mu}_{x_\iota}(\{0\}) = \widetilde{\mu}_{y_\iota}(\{0\}) = 0\,,$$

for every $\iota \in I$ whenever $0 \in \sigma(x)$, it follows that

$$\sum_{\iota \in I} x' \circ \widetilde{\mu}_{x_\iota} = \sum_{\iota \in I} x' \circ \widetilde{\mu}_{y_\iota} .$$

From

$$x' \circ \widetilde{\mu}_{x_\iota} , \; x' \circ \widetilde{\mu}_{y_\iota} \in \mathcal{M}_\iota$$

for every $\iota \in I$, we see that

$$x' \circ \widetilde{\mu}_{x_\iota} = x' \circ \widetilde{\mu}_{y_\iota}$$

for every $\iota \in I$. Since x' is arbitrary, we deduce that

$$\widetilde{\mu}_{x_\iota} = \widetilde{\mu}_{y_\iota}$$

for every $\iota \in I$.

Take $\iota \in I$. If A is a Borel set of $\sigma(x_\iota)\backslash\{0\}$ (of $\sigma(y_\iota)\backslash\{0\}$), then

$$\mu_{x_\iota}(A) = \widetilde{\mu}_{x_\iota}(A) = \widetilde{\mu}_{y_\iota}(A) ,$$

$$(\mu_{y_\iota}(A) = \widetilde{\mu}_{y_\iota}(A) = \widetilde{\mu}_{x_\iota}(A)) .$$

It follows that

$$\sigma(x_\iota)\backslash\{0\} = \sigma(y_\iota)\backslash\{0\}$$

and

$$\mu_{x_\iota} = \mu_{y_\iota} .$$

Hence

$$x_\iota = y_\iota . \qquad \blacksquare$$

Corollary 6.3.3.7 (0) *Take* $x \in \mathrm{Sn}\, E$.

a) *There are uniquely determined* $y, z \in \mathrm{Sn}\, E$ *such that* y *is atomic,* z *has at most one atom at* 0,

$$x = y + z ,$$

and

$$e_\alpha(x) = e_\alpha(y) ,$$

for every $\alpha \in \mathbb{K}\backslash\{0\}$. y *(resp.* z*) is called* ***the atomic (resp. atomless) part of*** x.

b) *y and z belong to the unital W^*-subalgebra of E generated by x.*

c) $yz = zy = 0$.

d) $\sigma(x) \subset \sigma(y) \cup \sigma(z) = \sigma(x) \cup \{0\}$.

e) *Given $\rho \in \mathcal{M}_b(\sigma(x))$, we write ρ_a and ρ_b for the atomic and atomless part of ρ, respectively. Define: $\mathcal{C} := \mathcal{C}(\sigma(x))$,*

$$f'' : \mathcal{C}' \longrightarrow \mathbb{K}, \quad \rho \longmapsto \rho_a(\sigma(x)),$$

$$g'' : \mathcal{C}' \longrightarrow \mathbb{K}, \quad \rho \longmapsto \rho_b(\sigma(x)),$$

$$j : \sigma(x) \longrightarrow \mathbb{K}, \quad \alpha \longmapsto \alpha.$$

Then $f'', g'' \in \mathcal{C}''$, $f''(x), g''(x) \in Pr\, E$,

$$f''(x) + g''(x) = 1, \quad y = ((j\mathcal{C}j)f'')(x), \quad z = ((j\mathcal{C}j)g'')(x).$$

f) *If x is selfadjoint then $|y|$ (resp. y^+, resp. y^-) is the atomic part of $|x|$ (resp. x^+, resp. x^-) and $|z|$ (resp. z^+, resp. z^-) is the atomless part of $|x|$ (resp. x^+, resp. x^-).*

Let $\mathcal{M}_1, \mathcal{M}_2$ be the bands of $\mathcal{C}'$ of atomic and atomless measures on $\sigma(x)$, respectively, and let y and z be the components of x on the bands $\mathcal{M}_1$ and M_2, respectively. By Theorem 6.3.3.6 (and Corollary 4.4.4.3 c $\Leftrightarrow$ d), y and z have the properties described in a),b),c),d), and e). Uniqueness follows from the fact that y is completely determined by the properties in a).

f) follows from a),e), and Proposition 6.3.3.5 e). ■

Remark. Take $E := \mathcal{C}(\mathbf{T})''$,

$$j : \mathbf{T} \longrightarrow \mathbb{C}, \quad \alpha \longmapsto \alpha,$$

$$x := j_{\mathcal{C}(\mathbf{T})} j.$$

Then $|x| = 1$, so that the atomic part of $|x|$ is 1, while the atomic part of x is 0.

Corollary 6.3.3.8 (0) *Assume that $H \neq \{0\}$ and let u be a self-normal operator on H. Then there are uniquely determined self-normal operators v, w on H such that:*

a) $u = v + w$.

b) $\sigma_p(u) \subset \sigma_p(v) \subset \sigma_p(u) \cup \{0\}$, $\sigma_p(w) \subset \{0\}$.

c) *If* $\alpha \in \sigma_p(u)\backslash\{0\}$, *then the eigenspaces of* u *and* v *corresponding to the eigenvalue* α *coincide.*

d) *There is an orthonormal basis* A *of* H *and an* $f \in \ell^\infty(A)$ *such that*

$$v\xi = f(\xi)\xi$$

for every $\xi \in A$.

In addition:

e) v *is atomic.*

f) v *and* w *belong to the* W^**-subalgebra of* $\mathcal{L}(H)$ *generated by* u *(Corollary 6.1.7.14 b)).*

g) $vw = wv = 0$.

h) $\sigma(u) \subset \sigma(v) \cup \sigma(w) = \sigma(u) \cup \{0\}$.

i) v *(resp.* w*) is the atomic (resp. atomless) part of* u.

By Corollary 6.3.3.7, there exist uniquely $v, w \in \mathcal{L}(H)$ such that

$$\alpha \in \mathbb{K}\backslash\{0\} \Longrightarrow e_\alpha(u) = e_\alpha(v), \quad \alpha \in \mathbb{K}\backslash\{0\} \Longrightarrow e_\alpha(w) = 0$$

and a),e),f),g),h), and i) are fulfilled. b) and c) now follow from Theorem 6.1.7.15 j). d) follows from Corollary 6.1.7.18. ■

Corollary 6.3.3.9 (0) *Let* $x \in \operatorname{Re} E$ *and* G *the unital* W^**-subalgebra of* E *generated by* x. *Let* $\mathfrak{B}$ *be the* σ*-algebra of Borel sets of* $\sigma(x)$, λ *the restriction of the Lebesgue measure to* $\mathfrak{B}$, *and for every* $y \in G$, *let* μ_y *be its spectral measure and define*

$$\widetilde{\mu}_y : \mathfrak{B} \longrightarrow \operatorname{Pr} E, \quad A \longmapsto \mu_y(A \cap (\sigma(x)\backslash\{0\})).$$

a) *There are uniquely determined* $x_1, x_2, x_3 \in \operatorname{Re} G$ *such that*

$$x = x_1 + x_2 + x_3,$$

$$i, j \in \mathbb{N}_3, \, i \neq j \Longrightarrow x_i x_j = 0,$$

and for every $x' \in E^\pi$, $x' \circ x_1$ *is atomic,* $x' \circ x_2$ *is absolutely continuous with respect to* λ, *and* $x' \circ x_3$ *is atomless and singular with respect to* λ.

b) $\sigma(x) \subset \bigcup_{i=1}^{3} \sigma(x_i) = \sigma(x) \cup \{0\}$. ■

6.3.4 Von Neumann-Algebras

Definition 6.3.4.1 (0) *The (unital) W^*-subalgebras of $\mathcal{L}(H)$ are called* ***(unital) von Neumann algebras on*** *H (Corollary 6.1.7.14 b)).*

Every von Neumann algebra on H has a unit, which is an orthogonal projection in H (Corollary 4.4.4.9).

Proposition 6.3.4.2 (0) *Let E be a C^*-subalgebra of $\mathcal{L}(H)$. Then the following are equivalent:*

a) *E is a von Neumann algebra on H.*

b) *E is closed in $\mathcal{L}(H)_{\mathcal{L}_f(H)}$.*

c) *E is closed in $\mathcal{L}(H)_{\mathcal{L}^1(H)}$.*

d) *E is closed in $\mathcal{L}(H)_H$.*

If these conditions are fulfilled then:

e) *If $p \in \Pr \overline{E}$, then $p \in \Pr E$, where $\overline{E}$ denotes the closure of E in $\mathcal{L}(H)_H$.*

f) *A linear form u' on E belongs to the predual of E (Corollary 4.4.4.9 a $\Rightarrow$ b), iff there is a $u \in \mathcal{L}^1(H)$ such that $u' = \overset{\infty}{u}|E$.*

g) *If 1_H, $v \in E$, $K := \overline{\operatorname{Im} v}$, $L := (\operatorname{Ker} v)^\perp$ then π_K and π_L belong to E and are the left and the right carrier of v in E, respectively.*

In the complex case $e \Rightarrow d$.

a $\Leftrightarrow$ b $\Leftrightarrow$ c $\Leftrightarrow$ d follows from Corollary 6.3.1.12.

d $\Rightarrow$e is trivial.

e $\Rightarrow$ d. Take $u \in \operatorname{Re} \overline{E}$ and $\varepsilon > 0$. By Theorem 4.4.1.8 f) (and Corollary 6.3.1.12), there is a finite family $((\alpha_\iota, p_\iota))_{\iota \in I}$ in $\sigma(u) \times \Pr \overline{E}$ such that

$$\left\| u - \sum_{\iota \in I} \alpha_\iota p_\iota \right\| < \varepsilon .$$

By e),

$$\sum_{\iota \in I} \alpha_\iota p_\iota \in E .$$

Since ε is arbitrary, $u \in E$. Hence E is closed in $\mathcal{L}(H)_H$.

c $\Rightarrow$ f follows from Corollary 6.1.6.11.

a $\Rightarrow$ g. By Proposition 5.3.2.4, π_k and π_L are the left and the right carrier of v in $\mathcal{L}(H)$. By Proposition 4.4.4.11 a), π_K and π_L belong to E and are the left and the right carrier of v in E. ■

Remark. If $\mathbb{K} = \mathbb{R}$ and H is infinite–dimensional then the implication e $\Rightarrow$ d does hot hold. Let us consider an orthonormal basis of H consisting of two families $(e_\iota)_{\iota\in I}$ and $(f_\iota)_{\iota\in I}$ and denote by $\mathcal{U}, \mathcal{V}$, and $\mathcal{W}$ the set of $u, v, w \in \mathcal{L}(H)$, repsectively, for which there is an $a \in \ell^\infty$, a $b \in c_0$, and a $c \in \ell^\infty$ such that

$$ue_\iota = a_\iota e_\iota\,, \quad ve_\iota = b_\iota f_\iota\,, \quad we_\iota = c_\iota f_\iota$$

$$uf_\iota = a_\iota f_\iota\,, \quad vf_\iota = -b_\iota e_\iota\,, \quad wf_\iota = -c_\iota f_\iota$$

for every $\iota \in I$. Put

$$E := \{u + v \mid (u, v) \in \mathcal{U} \times \mathcal{V}\}\,.$$

Then E is a C^*–subalgebra of $\mathcal{L}(H)$ and

$$\overline{E} = \{u + w \mid (u, w) \in \mathcal{U} \times \mathcal{W}\} \neq E\,,$$

so that d) is not fulfilled. On the other side

$$\Pr \overline{E} = \{u \in \mathcal{U} \mid u^2 = u\} \subset E\,,$$

i.e. e) is satisfied.

Proposition 6.3.4.3 (0) (G.K. Pedersen) *Let E be a unital C^*-subalgebra of $\mathcal{L}(H)$ such that the supremum in $\mathcal{L}(H)^\#_+$ of any commutative increasing sequence in $E^\#_+$ belongs to E (Corollary 6.1.7.14 b), Theorem 4.4.1.8 b)) and let $\overline{E}$ be the closure of E in $\mathcal{L}(H)_H$. Put $p_0 := 1$, $u_0 := 0$. Then, for every $p \in \Pr \overline{E}$ (Corollary 6.3.1.12), any $x \in H$, and any $\varepsilon > 0$, there is a sequence $(u_n)_{n\in\mathbb{N}}$ in $E^\#_+$ and a sequence $(p_n)_{n\in\mathbb{N}}$ in $\Pr E$ such that the following hold for every $n \in \mathbb{N}$:*

1) $\|(u_n - p)p_n px\|^2 < \frac{\varepsilon}{2\cdot 4^n}$, $\|u_n x - px\| < \frac{1}{n}$, $\|u_n px - px\| < \frac{1}{n}$.

2) $p_n(u_n - u_{n-1})^2 p_n \leq \frac{1}{2^{n-1}} 1$.

3) $p_n \leq p_{n-1}$, $\|(p_{n-1} - p_n)px\|^2 < \frac{\varepsilon}{2^{n-1}}$.

Let F be a maximal Gelfand C^*–subalgebra of E, $(v_n)_{n\in\mathbb{N}}$ an increasing sequence in $F_+^{\#}$, and v its supremum in $\mathcal{L}(H)_+^{\#}$. By hypothesis, $v \in E$. By Corollary 4.2.2.20 a), $v \in F$. In particular, F is order σ–complete. By Corollary 4.3.2.5, any $u \in \operatorname{Sn} E$ admits a functional calculus with Borel functions.

We construct the sequences $(u_n)_{n\in\mathbb{N}}$ and $(p_n)_{n\in\mathbb{N}}$ inductively. Put $p_1 := 1$. There is a $u_1 \in E_1^{\#}$, such that 1) is fulfilled (Theorem 6.3.1.10). 2) is then automactically fulfilled.

Take $n \in \mathbb{N}$, and assume that the sequences have been constructed up to n. We have

$$0 \leq p_n(u_n - p)^2 p_n \leq 1$$

(Theorem 4.2.2.1 b $\Rightarrow$ a, Corollary 4.2.2.3, Corollary 4.2.1.16) and the map

$$\mathcal{C}(\sigma(p_n(u_n - p)^2 p_n)) \longrightarrow \mathbb{K}, \quad f \longmapsto \langle f(p_n(u_n - p)^2 p_n)px|px\rangle$$

is a positive linear form (Corollary 4.2.1.12, Corollary 5.3.3.7). Hence, there is an $f \in \mathcal{C}([0,1])_+$ and an open nonempty interval $I \subset \left[\frac{1}{2^{n+1}}, \frac{1}{2^n}\right]$, such that $f = 1$ on I and

$$\langle f(p_n(u_n - p)^2 p_n)px|px\rangle < \frac{\varepsilon}{4^{n+2}}.$$

Take $\alpha_0 \in I$, put $g := e_{]-\infty,\alpha_0[}$, and let h be a real function on $\mathbb{R}$, such that $\operatorname{Supp} h \subset I$, $0 \leq h \leq 1$, and $k := g + h$ is continuous. Define

$$v_u := p_n(u - u_n)^2 p_n, \quad w_u := p_n(u - p)^2 p_n,$$

for every $u \in E_+^{\#}$. Then

$$0 \leq v_u \leq 1, \quad 0 \leq w_u \leq 1$$

(Theorem 4.2.2.1 b $\Rightarrow$ a, Corollary 4.2.2.3, Corollary 4.2.1.16). Hence (using the notation of Corollary 4.3.2.5 b)),

$$g(v_u)w_u g(v_u) = (k(v_u) - h(v_u))w_u(k(v_u) - h(v_u)) =$$

$$= k(v_u)w_u k(v_u) + h(v_u)w_u h(v_u) - k(v_u)w_u h(v_u) - h(v_u)w_u k(v_u) \leq$$

$$\leq 2k(v_u)w_u k(v_u) + 2h(v_u)w_u h(v_u) \leq 2k(v_u)w_u(v_u) + 2h(v_u)^2 \leq$$

$$\leq 2k(v_u)w_uk(v_u) + 2f(v_u)$$

for every $u \in E_+^{\#}$ (Corollary 4.2.2.4, Corollary 4.2.2.3, Corollary 4.3.2.5 a)). Let $\mathfrak{F}$ be the trace on $E_+^{\#}$ of the neighbourhood filter of p in $\mathcal{L}(H)_H$ (Theorem 6.3.1.10). We have

$$\lim_{u,\mathfrak{F}} w_u = 0\,, \quad \lim_{u,\mathfrak{F}} k(v_u)w_uk(v_u) = 0\,,$$

$$\lim_{u,\mathfrak{F}} f(v_u) = f(p_n(p-u_n)^2p_n)$$

in $\mathcal{L}(H)_H$ (Proposition 6.3.1.3 a), Proposition 6.3.1.6). By 1),

$$\lim_{u,\mathfrak{F}}\langle p_n(u_n-u)^2p_npx|px\rangle = \lim_{u,\mathfrak{F}}\langle (u_n-u)p_npx|(u_n-u)p_npx\rangle =$$

$$= \lim_{u,\mathfrak{F}} \|(u_n-u)p_npx\|^2 = \|(u_n-p)p_npx\|^2 < \frac{\varepsilon}{2\cdot 4^n}\,,$$

$$\limsup_{u,\mathfrak{F}}\langle g(v_u)w_ug(v_u)px|px\rangle \leq$$

$$\leq 2\limsup_{u,\mathfrak{F}}\langle k(v_u)w_uk(v_u)px|px\rangle + 2\limsup_{u,\mathfrak{F}}\langle f(v_u)px|px\rangle =$$

$$= 2\langle f(p_n(p-u_n)^2p_n)px|px\rangle < \frac{\varepsilon}{2\cdot 4^{n+1}}$$

(Corollary 5.3.3.7). Hence there is a $u_{n+1} \in E_+^{\#}$ such that

$$\|u_{n+1}x - px\| < \frac{1}{n+1}\,, \quad \|u_{n+1}px - px\| < \frac{1}{n+1}\,,$$

$$\langle p_n(u_n-u_{n+1})^2p_npx|px\rangle < \frac{\varepsilon}{2\cdot 4^n}\,,$$

$$\langle g(v_{u_{n+1}})w_{u_{n+1}}g(v_{u_{n+1}})px|px\rangle < \frac{\varepsilon}{2\cdot 4^{n+1}}\,.$$

Since

$$\sigma(g(v_{u_{n+1}})) \subset \overline{g(\sigma(v_{u_{n+1}}))} \subset \{0,1\}$$

(Corollary 4.3.2.5 c)), $g(v_{u_{n+1}}) \in \Pr E$ (Proposition 4.1.2.21 c $\Rightarrow$ a). Put

$$p_{n+1} := g(v_{u_{n+1}})p_n\,.$$

Since

$$p_n \in \{v_{u_{n+1}}\}^c \subset \{g(v_{u_{n+1}})\}^c$$

(Corollary 4.3.2.5 c)), we have

$$p_{n+1}^2 = p_{n+1}\,,$$

i.e., $p_{n+1} \in \Pr E$. Since

$$p_{n+1}p_n = p_{n+1}\,,$$

we deduce that

$$p_{n+1} \leq p_n$$

(Corollary 4.2.7.6 c $\Rightarrow$ a). We have

$$\|(u_{n+1}-p)p_{n+1}px\|^2 = \langle (u_{n+1}-p)p_{n+1}px|(u_{n+1}-p)p_{n+1}px\rangle =$$

$$= \langle p_{n+1}(u_{n+1}-p)^2 p_{n+1}px|px\rangle =$$

$$= \langle g(v_{u_{n+1}})p_n(u_{n+1}-p)^2 p_n g(v_{u_{n+1}})px|px\rangle =$$

$$= \langle g(v_{u_{n+1}})w_{u_{n+1}}g(v_{u_{n+1}})px|px\rangle < \frac{\varepsilon}{2\cdot 4^{n+1}}\,.$$

Since

$$g(\alpha)\alpha < \frac{1}{2^n}\,, \quad 1-g(\alpha) \leq 2^{n+1}\alpha$$

for every $\alpha \in [0,1]$, we deduce successively that

$$g(v_{u_{n+1}})v_{u_{n+1}} < \frac{1}{2^n}1\,, \quad 1-g(v_{u_{n+1}}) \leq 2^{n+1}v_{u_{n+1}}$$

(Corollary 4.3.2.5 a)),

$$\left\|p_{n+1}(u_{n+1}-u_n)^2 p_{n+1}\right\| = \left\|g(v_{u_{n+1}})p_n(u_{n+1}-u_n)^2 p_n g(v_{u_{n+1}})\right\| =$$

$$= \left\|g(v_{u_{n+1}})v_{u_{n+1}}g(v_{u_{n+1}})\right\| \leq \left\|g(v_{u_{n+1}})v_{u_{n+1}}\right\| < \frac{1}{2^n}$$

(Corollary 4.2.1.16 b $\Rightarrow$ a),

$$p_{n+1}(u_{n+1}-u_n)^2p_{n+1} < \frac{1}{2^n}1$$

(Corollary 4.2.1.16 a $\Rightarrow$ b),

$$p_n - p_{n+1} = p_n\Big(1 - g\big(v_{u_{n+1}}\big)\Big) \leq 2^{n+1}v_{u_{n+1}},$$

$$\big\|(p_n - p_{n+1})px\big\|^2 = \langle (p_n - p_{n+1})px | (p_n - p_{n+1})px\rangle =$$

$$= \langle (p_n - p_{n+1})px | px\rangle \leq 2^{n+1}\big\langle v_{u_{n+1}}px | px\big\rangle =$$

$$= 2^{n+1}\big\langle p_n(u_n - u_{n+1})^2p_npx | px\big\rangle < \frac{\varepsilon}{2^n}$$

(Corollary 5.3.3.7). ■

Proposition 6.3.4.4 (0) *Let E be a unital C^*-subalgebra of $\mathcal{L}(H)$ such that the supremum in $\mathcal{L}(H)^{\#}_{+}$ of any commutative well-ordered set of $E^{\#}_{+}$ belongs to E (Corollary 6.1.7.14 b), Theorem 4.4.1.8 b)). Further take $x \in H$, and put*

$$A := \{p \in \Pr E | px = 0\}.$$

Then A possesses a greatest element.

Take $p, q \in A$ and let K be the closed vector subspace of H generated by $\operatorname{Im} p \cup \operatorname{Im} q$. Then π_K is the supremum of $\{p, q\}$ in $\Pr\mathcal{L}(H)$ (Corollary 5.3.3.8). Since

$$x \in K^{\perp},$$

$$\pi_K x = 0.$$

By Corollary 4.2.7.15 (and Proposition 4.2.7.14), $\pi_K \in E$, so that $\pi_K \in A$, and A is upward directed. By Proposition 4.2.7.17, the supremum p of A in $\mathcal{L}(H)$ belongs to E. Since $\pi_{\{x\}^{\perp}}$ is an upper bound of A in $\mathcal{L}(H)$, we have successively that

$$p \leq \pi_{\{x\}^{\perp}},$$

$$px = 0,$$

$$p \in A,$$

p is the greatest element of A. ■

Proposition 6.3.4.5 (0) *Let E be a unital C^*-subalgebra of $\mathcal{L}(H)$ such that the supremum in $\mathcal{L}(H)^{\#}_{+}$ of any commutative well-ordered set of $E^{\#}_{+}$ belongs to E (Corollary 6.1.7.14 b), Theorem 4.4.1.8 b)) and let $\overline{E}$ be the closure of E in $\mathcal{L}(H)_H$. Then, for every $p \in \Pr \overline{E}$ (Corollary 6.3.1.12) and every $x \in H$, there is a $q \in \Pr E$ such that*

$$px = qx = qpx\,.$$

Take $\varepsilon > 0$. Put $p_0 := 1$, $u_0 := 0$. By Proposition 6.3.4.3, there is a sequence $(u_n)_{n\in\mathbb{N}}$ in $E^{\#}_{+}$ and a sequence $(p_n)_{n\in\mathbb{N}}$ in $\Pr E$, such that the following hold for every $n \in \mathbb{N}$:

1) $\|(u_n - p)p_n px\|^2 < \frac{\varepsilon}{2\cdot 4^n}$, $\|u_n x - px\| < \frac{1}{n}$, $\|u_n px - px\| < \frac{1}{n}$.

2) $p_n(u_n - u_{n-1})^2 p_n \leq \frac{1}{2^{n-1}}1$.

3) $p_n \leq p_{n-1}$, $\|(p_{n-1} - p_n)px\|^2 < \frac{\varepsilon}{2^{n-1}}$.

By 3) and Corollary 4.2.7.6 a $\Rightarrow$ b, $(p_n)_{n\in\mathbb{N}}$ is a commutative decreasing sequence in E_+. Put

$$p_\infty := \bigwedge_{n\in\mathbb{N}} p_n \in \Pr E$$

(Corollary 4.2.7.10). $(p_n)_{n\in\mathbb{N}}$ converges to p_∞ in $\mathcal{L}(H)_H$. (Proposition 6.3.1.4 a $\Rightarrow$ c), and so by 3),

$$\langle p_\infty px|px\rangle = \lim_{n\to\infty}\langle p_n px|px\rangle = \sum_{n=1}^{\infty}\langle (p_n - p_{n-1})px|px\rangle + \langle px|px\rangle =$$

$$= \|px\|^2 - \sum_{n=1}^{\infty}\langle (p_{n-1} - p_n)px|(p_{n-1} - p_n)px\rangle \geq$$

$$\geq \|px\|^2 - \sum_{n=1}^{\infty}\frac{\varepsilon}{2^{n-1}} = \|px\|^2 - 2\varepsilon\,.$$

We have

$$(u_{n+1} - u_n)p_\infty(u_{n+1} - u_n) \leq (u_{n+1} - u_n)p_{n+1}(u_{n+1} - u_n)$$

(Corollary 4.2.2.3), so by 2) and Corollary 4.2.1.18,

$$\|(u_{n+1} - u_n)p_\infty\|^2 = \|(u_{n+1} - u_n)p_\infty(u_{n+1} - u_n)\| \leq$$

$$\leq \|(u_{n+1} - u_n)p_{n+1}(u_{n+1} - u_n)\| = \|(u_{n+1} - u_n)p_{n+1}\|^2 =$$

$$= \|p_{n+1}(u_{n+1} - u_n)^2 p_{n+1}\| < \frac{1}{2^n}$$

for every $n \in \mathbb{N}$. Hence, $(u_n p_\infty)_{n\in\mathbb{N}}$ converges, and so $(u_n p_\infty u_n)_{n\in\mathbb{N}}$ also converges. Put

$$u := \lim_{n\to\infty} u_n p_\infty u_n \in E_+^{\#}.$$

By 1),

$$\|u_n px - px\| < \frac{1}{n},$$

$$\|u_n(x - px)\| \leq \|u_n x - px\| + \|px - u_n px\| < \frac{2}{n}$$

for every $n \in \mathbb{N}$. Thus

$$\lim_{n\to\infty} u_n(x - px) = 0, \quad \lim_{n\to\infty} u_n px = px,$$

$$u(x - px) = 0,$$

$$\langle upx|px\rangle = \lim_{n\to\infty} \langle u_n p_\infty u_n px|px\rangle = \lim_{n\to\infty} \langle p_\infty u_n px|u_n px\rangle =$$

$$= \langle p_\infty px|px\rangle > \|px\|^2 - 2\varepsilon.$$

By Proposition 5.3.3.10 a),

$$u^\alpha(x - px) = 0,$$

and by Proposition 6.3.1.5 a), Corollary 5.3.3.7,

$$u^\alpha \geq u, \quad \langle u^\alpha px|px\rangle \geq \langle upx|px\rangle \geq \|px\|^2 - 2\varepsilon$$

for every $\alpha \in]0, 1[$. Put

$$v := \bigvee_{\alpha>0} u^\alpha \in \Pr E$$

(Proposition 6.3.1.5 b)). By Proposition 6.3.1.5 b).

$$v(x - px) = \lim_{\alpha\to 0} u^\alpha(x - px) = 0.$$

$$\langle vpx|px\rangle = \lim_{\alpha\to 0}\langle u^\alpha px|px\rangle \geq \|px\|^2 - 2\varepsilon\,.$$

Define

$$A := \{r \in \Pr E | r(x-px) = 0\}\,.$$

By Proposition 6.3.4.4, A has a greatest element q. We have

$$qx = qpx\,.$$

Since $v \in A$,

$$q \geq v\,,$$

$$\langle qpx|px\rangle \geq \langle vpx|px\rangle \geq \|px\|^2 - 2\varepsilon$$

(Corollary 5.3.3.7). Since ε is arbitrary, it follows that

$$\|px\|^2 \leq \langle qpx|px\rangle \leq \|px\|^2\,,$$

$$qpx = px$$

(Proposition 5.2.3.1). ■

Theorem 6.3.4.6 (0) *Let* $\mathbb{K} = \mathbb{C}$ *and let* E *be a* C^**-subalgebra of* $\mathcal{L}(H)$ *such that the supremum in* $\mathcal{L}(H)^\#_+$ *of any commutative well-ordered set of* $E^\#_+$ *belongs to* E. *Then* E *is a von Neumann algebra on* H.

Let $\overline{E}$ be the closure of E in $\mathcal{L}(H)_H$ and take $p \in \Pr\overline{E}$ (Corollary 6.3.1.12). For $x \in H$, take $p_x \in \Pr E$ such that

$$px = p_x x = p_x px$$

(Proposition 6.3.4.5), and let q_x be the infimum in $\Pr\mathcal{L}(H)$ of

$$\{p_y | y \in \overset{-1}{p}(px)\}\,.$$

By Proposition 4.2.7.17 (and Proposition 4.2.7.14), $q_x \in \Pr E$.

Take $x \in \operatorname{Im} p$. Then

$$p_y x = p_y px = py = x$$

for every $y \in \overset{-1}{p}(x)$. Take $y, z \in \overset{-1}{p}(x)$ and let $p_y \wedge p_z$ be the infimum of p_y and p_z in $\Pr E$. By Corollary 4.2.7.13,

$$(p_y \wedge p_z)x = 0\,.$$

By Proposition 6.3.1.4,

$$q_x x = x\,.$$

Take $z \in \operatorname{Ker} p$ and put

$$y := x + z\,.$$

Then $y \in \overset{-1}{p}(x)$ and

$$p_y z = p_y y - p_y x = py - x = 0\,.$$

We deduce that

$$\|q_x z\| \leq \|p_y z\| = 0$$

(Proposition 5.2.3.13 a $\Rightarrow$ d). Since z is arbitrary, it follows that

$$\operatorname{Ker} p \subset \operatorname{Ker} q_x\,,$$

and so

$$\operatorname{Im} q_x = (\operatorname{Ker} q_x)^{\perp} \subset (\operatorname{Ker} p)^{\perp} = \operatorname{Im} p\,,$$

$$q_x \leq p$$

(Corollary 5.3.3.8).

By the above considerations, p is the supremum of $\{q_x | x \in H\}$ in $\operatorname{Pr}\mathcal{L}(H)$. By Proposition 4.2.7.17, $p \in E$. By Corollary 6.3.4.2 e $\Rightarrow$ a, E is a W^*-subalgebra of $\mathcal{L}(H)$. ■

Remark. The example from the remark of Proposition 6.3.4.2 shows that the above theorem does not hold for $\mathbb{K} = \mathbb{R}$.

6.3.5 The Commutants

Proposition 6.3.5.1 (0) *Let F be an involutive subalgebra of $\mathcal{L}(H)$ such that*

$$x \in \overline{\{ux | u \in F\}}$$

for every $x \in H$. Then F is dense in F''_H.

Take $u \in F^{cc}$.

Step 1 $\left\{ \begin{array}{l} \text{For every } x \in H, \text{ there is a sequence} \\ (u_n)_{n\in\mathbb{N}} \text{ in } F \text{ with } \lim\limits_{n\to\infty} u_n x = ux. \end{array} \right.$

Define

$$K := \overline{\{vx | v \in F\}}.$$

K is a closed vector subspace of H containing x. K is invariant for every $v \in F$. Hence, by Proposition 5.3.2.9, $\pi_K \in F^c$. Thus

$$\pi_K u = u\pi_K .$$

We get that

$$ux = u\pi_K x = \pi_K ux \in K .$$

Hence, there is a sequence $(u_n)_{n\in\mathbb{N}}$ in F such that

$$\lim_{n\to\infty} u_n x = ux .$$

Step 2 $\displaystyle\bigoplus_{i\in\mathbb{N}_k} u \in \left\{ \bigoplus_{i\in\mathbb{N}_k} v \;\middle|\; v \in F \right\}^{cc}$ for every $k \in \mathbb{N}$

Take

$$w \in \left\{ \bigoplus_{i\in\mathbb{N}_k} v \;\middle|\; v \in F \right\}^{c} .$$

Identifying $\mathcal{L}(\bigoplus\limits_{i\in\mathbb{N}_k} H)$ with the involutive unital algebra of $k \times k$–matrices over $\mathcal{L}(H)$ (Theorem 5.6.6.1 f)), we obtain

$$w_{ij} v = v w_{ij}$$

for all $i, j \in \mathbb{N}_k$ and $v \in F$. Hence, $w_{ij} \in F^c$ and

$$w_{ij}u = uw_{ij}$$

for all $i, j \in \mathbb{N}_k$. Thus

$$w\left(\bigoplus_{i\in\mathbb{N}_k} u\right) = \left(\bigoplus_{i\in\mathbb{N}_k} u\right) w\,.$$

Since w is arbitrary, it follows that

$$\bigoplus_{i\in\mathbb{N}_k} u \in \left\{ \bigoplus_{i\in\mathbb{N}_k} v \;\middle|\; v \in F \right\}^{cc}.$$

Step 3 u belongs to the closure of F in F_H^{cc}

Take $x_1, x_2, \ldots, x_k \in H$. Define

$$x := (x_i)_{i\in\mathbb{N}_k} \in \bigoplus_{i\in\mathbb{N}_k} H\,.$$

By Step 2,

$$\bigoplus_{i\in\mathbb{N}_k} u \in \left\{ \bigoplus_{i\in\mathbb{N}_k} v \;\middle|\; v \in F \right\}^{cc}.$$

By Step 1, there is a sequence $(u_n)_{n\in\mathbb{N}}$ in F such that

$$\lim_{n\to\infty} \left(\bigoplus_{i\in\mathbb{N}_k} u_n\right) x = \left(\bigoplus_{i\in\mathbb{N}_k} u\right) x\,.$$

Thus

$$\lim_{n\to\infty} u_n x_i = u x_i$$

for every $i \in \mathbb{N}_k$. Since $x_1, x_2, \ldots, x_k$ are arbitrary, u belongs to the closure of F in F_H^{cc}. ■

Corollary 6.3.5.2 (0) *$\mathcal{L}_f(H)$ is dense in $\mathcal{L}(H)_H$.*

We have

$$\frac{1}{\|x\|^2}\langle\cdot|x\rangle x \in \mathcal{L}_f(H)\,,$$

$$\left(\frac{1}{\|x\|^2}\langle\cdot|x\rangle x\right) x = x$$

for every $x \in H\backslash\{0\}$. By Corollary 5.5.1.11 and Proposition 5.3.2.13 b),e), $\mathcal{L}_f(H)$ is an involutive subalgebra of $\mathcal{L}(H)$ and by Corollary 5.3.2.14,

$$\mathcal{L}_f(H)^{cc} = \mathcal{L}(H)\,.$$

By Proposition 6.3.5.1, $\mathcal{L}_f(H)$ is dense in $\mathcal{L}(H)_H$. ■

Corollary 6.3.5.3 (0) *If H is separable then $\mathcal{L}(H)_H$ is separable and $\mathcal{L}(H)_H^{\#}$ is metrizable.*

By Proposition 5.5.2.12 c), $\mathcal{K}(H)$ is separable. Hence $\mathcal{K}(H)_H$ is separable. By Corollary 6.3.5.2, $\mathcal{K}(H)$ is dense in $\mathcal{L}(H)_H$, so that $\mathcal{L}(H)_H$ is separable.

Let A be a countable dense set of H. Since $\mathcal{L}(H)^{\#}$ is equicontinuous,

$$\mathcal{L}(H)_H^{\#} = \mathcal{L}(H)_A^{\#}\,.$$

Hence, $\mathcal{L}(H)_H^{\#}$ is metrizable. ■

Proposition 6.3.5.4 (0) *If $\mathcal{F}$ is a subset of $\mathcal{L}(H)$ such that*

$$\mathcal{F} \subset \mathcal{F}^* \cup \mathrm{Sn}\,\mathcal{L}(H)\,,$$

then $\mathcal{F}^c$ is a unital von Neumann algebra on H.

The assertion follows immediately from Corollary 4.4.4.12 e). ■

Corollary 6.3.5.5 (0) (von Neumann, 1929) *A unital C^*-subalgebra E of $\mathcal{L}(H)$ is a von Neumann algebra on H iff*

$$E = E^{cc}\,.$$

If

$$E = E^{cc}\,,$$

then, by Proposition 6.3.5.4, E is a von Neumann algebra on H.

Now suppose that E is a von Neumann algebra on H. By Corollary 6.3.4.2 a $\Rightarrow$ d, E is closed in $\mathcal{L}(H)_H$. By Proposition 6.3.5.1,

$$E = E^{cc}\,.$$ ■

Corollary 6.3.5.6 (0) *Let $\mathcal{F}$ be a subset of $\mathcal{L}(H)$ such that*

$$\mathcal{F} \subset \mathcal{F}^* \cup \mathrm{Sn}\,\mathcal{L}(H)\,.$$

Then $\mathcal{F}^{cc}$ is the unital von Neumann algebra on H generated by $\mathcal{F}$.

By Proposition 6.3.5.4, $\mathcal{F}^{cc}$ is a unital von Neumann algebra on H. Let $\mathcal{G}$ be a unital von Neumann algebra on H containing $\mathcal{F}$. Then

$$\mathcal{F}^{cc} \subset \mathcal{G}^{cc} = \mathcal{G}\,,$$

by Corollary 6.3.5.5. Hence $\mathcal{F}^{cc}$ is the unital von Neumann algebra on H generated by $\mathcal{F}$. ■

Corollary 6.3.5.7 *If $\mathbb{K} = \mathbb{C}$, E is a unital von Neumann algebra on H, and $x \in \mathcal{L}(H)$ then the following are equivalent:*

a) $x \in E$.

b) $y^*xy = x$ *for every unitary element y of E^c.*

c) $px = xp$ *for every $p \in \mathrm{Pr}\,E^c$.*

a $\Rightarrow$ b & c. Since $x \in E^{cc}$,

$$xy = yx\,, \quad px = xp\,,$$

$$y^*xy = y^*yx = x\,.$$

b $\Rightarrow$ a. By Corollary 4.1.3.7 (and Proposition 6.3.5.4), every element of E^c is a linear combination of four unitary elements of E^c. Hence $x \in E^{cc}$ follows from b). By Corollary 6.3.5.5, $x \in E$.

c $\Rightarrow$ a. By Proposition 6.3.5.4, E^c is a unital von Neumann algebra on H. Hence, by Theorem 4.4.1.8 g), the vector subspace of E^c generated by $\mathrm{Pr}\,E^c$ is dense in E^c. Thus $x \in E^{cc}$ follows from c), so that $x \in E$. (Corollary 6.3.5.5). ■

Proposition 6.3.5.8 (0) *Let E be a C^*-subalgebra of $\mathcal{L}(H)$ and consider the following assertions:*

a) *E acts irreducibly on H.*

b) $\mathrm{Re}\,(E^c) = \mathbb{R}1$.

c) *E is dense in $\mathcal{L}(H)_H$.*

Then

$$c \Longrightarrow a \Longleftrightarrow b\,.$$

If $\mathbb{K} = \mathbb{C}$, then all these assertions are equivalent.

a $\Leftrightarrow$ b. An orthogonal projection p in H belongs to E^c iff $\mathrm{Im}\,p$ is E–invariant (Corollary 5.3.2.9).

If a) holds, then $\mathrm{Pr}\,(E^c) \subset \{0,1\}$. Since E^c is a von Neumann algebra on H (Proposition 6.3.5.4), we get

$$\mathrm{Re}\,(E^c) = \mathbb{R}1$$

by Theorem 4.4.1.8 g).

Now assume that b) holds and let K be a closed E–invariant vector subspace of H. Then $\pi_K \in \mathrm{Re}\,(E^c)$ so that $K \in \{\{0\}, H\}$.

c $\Rightarrow$ b. By Corollary 5.3.2.14,

$$\mathbb{R}1 \subset \mathrm{Re}\,(E^c) = \mathrm{Re}\,\mathcal{L}(H)^c \subset \mathrm{Re}\,\{\langle\cdot|x\rangle x | x \in H\}^c = \mathbb{R}1\,,$$

so that

$$\mathrm{Re}\,(E^c) = \mathbb{R}1\,.$$

a & b $\Rightarrow$ c. Take $x \in H\backslash\{0\}$. By a) and Proposition 5.3.2.20 b), x is cyclic for F so that

$$x \in \{ux | u \in F\}^{\perp\perp} = \overline{\{ux | u \in F\}}$$

(Corollary 5.2.3.9). By b),

$$E^c = \mathbb{C}1$$

so that

$$E^{cc} = \mathcal{L}(H)\,.$$

By Proposition 6.3.5.1, E is dense in $\mathcal{L}(H)_H$. ■

Remark. The implication b $\Rightarrow$ c does not hold for $\mathbb{K} = \mathbb{R}$. Take

$$H := \mathbb{R}^2\,,$$

$$E := \left\{ \begin{bmatrix} \alpha & \beta \\ -\beta & \alpha \end{bmatrix} \;\middle|\; \alpha, \beta \in \mathbb{R} \right\}.$$

Then

$$E^c = E, \quad \mathrm{Re}\,(E^c) = \mathrm{Re}\,E = \mathbb{R}1,$$

but E is not dense in $\mathcal{L}(\mathbb{R}^2)$.

6.3.6 Irreducible Representations

Theorem 6.3.6.1 (0) (Kadison, 1957) *Assume* $\mathbb{K} = \mathbb{C}$. *Let* E *be a* C^**-subalgebra of* $\mathcal{L}(H)$ *acting irreducibly on* H *and let* $(\xi_\iota)_{\iota \in I}$ *be a linearly independent finite family in* H.

a) *For every* $\eta \in H^I$ *there is an* $x \in E$ *with*

$$(x\xi_\iota)_{\iota \in I} = \eta .$$

b) $\{0\}$ *and* H *are the only* E*-invariant vector subspaces of* H.

a) Step 1 $\exists y \in \mathcal{L}(H)$, $(y\xi_\iota)_{\iota \in I} = \eta$.

Let K be the (finite-dimensional) vector subspace of H generated by $(\xi_\iota)_{\iota \in I}$. There is a $z \in \mathcal{L}(K, H)$ with

$$(z\xi_\iota)_{\iota \in I} = \eta .$$

If we set

$$y : H \longrightarrow H , \quad \xi \longmapsto z\pi_K \xi$$

then $y \in \mathcal{L}(H)$ and

$$(y\xi_\iota)_{\iota \in I} = \eta .$$

Step 2 $\exists x \in E$, $(x\xi_\iota)_{\iota \in I} = \eta$

Put

$$K := H^I$$

and define

$$u : \mathcal{L}(H) \longrightarrow K , \quad y \longmapsto (y\xi_\iota)_{\iota \in I} .$$

By Step 1, u is surjective. By the Open Mapping Principle, the null point of K is an interior point of $u(\mathcal{L}(H)^\#)$. By Proposition 6.3.5.8 a $\Rightarrow$ c, E is dense in $\mathcal{L}(H)_H$ so that by Kaplanski's Density Theorem (Theorem 6.3.1.10), $E^\#$ is dense in $\mathcal{L}(H)_H^\#$. Hence the null point of K is an interior point of $\overline{u(E^\#)}$. By Proposition 1.4.2.2, the null point of K is an interior point of $u(E^\#)$. Hence

$$u(E) = K$$

and there is an $x \in E$ with

$$(x\xi_\iota)_{\iota \in I} = ux = \eta .$$

b) follows from a). ■

Theorem 6.3.6.2 (0) (Segal, 1947) *The following are equivalent for* $x' \in \tau(E)$ *:*

a) $x' \in \tau_0(E)$.

b) *The representation of* E *associated to* x' *is irreducible.*

If $\mathbb{K} = \mathbb{C}$, *the above conditions are fulfilled, and* (H, φ) *denotes the representation of* E *associated to* x' , *then* $E/N_{x'} = H$.

We use the notation from Theorem 5.4.1.2.
a $\Rightarrow$ b. Take

$$v \in \varphi(E)^c \cap \mathcal{L}(\widetilde{E/F})_+^{\#} .$$

By Proposition 5.4.1.6 a) (and Theorem 4.2.8.2), the map

$$y' : E \longrightarrow \mathbb{K} , \quad x \longmapsto \langle (\varphi x) v\xi \mid \xi \rangle$$

is a positive linear form less than x'. By Proposition 2.3.5.4 a $\Rightarrow$ b,

$$y' = \|y'\| x' .$$

Thus, given $x, y \in E$,

$$\langle vux | uy \rangle = \langle v\widetilde{x}\xi \mid \widetilde{y}\xi \rangle = \langle \widetilde{y}^* \widetilde{x} v\xi \mid \xi \rangle =$$

$$= \langle \varphi(y^* x) v\xi \mid \xi \rangle = y'(y^* x) = \|y'\| x'(y^* x) =$$

$$= \langle \|y'\| ux | uy \rangle$$

(Theorem 5.4.1.2 k)), so that

$$v = \|y'\| 1 .$$

Hence

$$\varphi(E)^c \cap \mathcal{L}(\widetilde{E/F})_+^{\#} \subset \mathbb{K}1 .$$

By Corollary 4.1.4.2 b),

$$\mathrm{Re}\,(\varphi(E)^c) \subset \mathbb{K}1 .$$

By Proposition 6.3.5.8 b $\Rightarrow$ a, $\varphi(E)$ acts irreducibly on $\widetilde{E/F}$, i.e. the representation of E associated to x' is irreducible.

b $\Rightarrow$ a. Take $y' \in E'_+$ with $y' \leq x'$. By Proposition 5.4.1.6 b), there is a $v \in \mathcal{L}(\widetilde{E/F})^{\#}_+ \cap \varphi(E)^c$, such that

$$y'(x) = \langle (\varphi x) v \xi | \xi \rangle$$

for every $x \in E$. By b) and Proposition 6.3.5.8 a $\Rightarrow$ b,

$$\operatorname{Re} \varphi(E)^c = \mathbb{R}1 .$$

Hence there is an $\alpha \in \mathbb{R}_+$ such that

$$v = \alpha 1 .$$

We get

$$y'(x) = \alpha \langle (\varphi x)\xi | \xi \rangle = \alpha \langle ux \mid \xi \rangle = \alpha x'(x)$$

for every $x \in E$ (Theorem 5.4.1.2 k)), so that

$$y' = \alpha x' .$$

By Proposition 2.3.5.4 b $\Rightarrow$ a, $x' \in \tau_0(E)$.

Assume now that the above conditions are fulfilled and $\mathbb{K} = \mathbb{C}$. Since $E/N_{x'}$ is a $\varphi(E)$–invariant vector subspace of H (Theorem 5.4.1.2), we get $E/N_{x'} = H$ by Theorem 6.3.6.1 b). ■

Corollary 6.3.6.3 *For every $x \in E$, there is an irreducible representation (H, φ) such that*

$$\|\varphi x\| = \|x\| .$$

We may assume that $x \neq 0$. By Corollary 4.2.8.5 c), there is an $x' \in \tau_0(E)$ such that

$$x'(x^* x) = \|x^* x\| .$$

Let (H, φ, ξ) be the GNS–triple associated to x'. By Theorem 6.3.6.2 a $\Rightarrow$ b, (H, φ) is irreducible. We have

$$\|x\|^2 = \|x^* x\| = x'(x^* x) = \langle \varphi(x^* x)\xi \mid \xi \rangle = \langle (\varphi x)\xi \mid (\varphi x)\xi \rangle =$$

$$= \|(\varphi x)\xi\|^2 \leq \|\varphi x\|^2 \leq \|x\|^2$$

(Theorem 5.4.1.2 k)), so that

$$\|x\| = \|\varphi x\| . \qquad ■$$

Corollary 6.3.6.4 (0) *There is a faithful representation (H, φ) of E which is the Hilbert sum of irreducible representations. If E is separable, then we may take H separable.*

It is sufficient to take a dense set A of $\tau_0(E)$ and to apply Corollary 5.4.2.5 and Theorem 6.3.6.2 a $\Rightarrow$ b. ■

Corollary 6.3.6.5 (0) (J.M. Wedderburn, 1908) *Assume E is finite-dimensional. If $\mathbb{K} = \mathbb{C}$ then there is a finite family $(n_\iota)_{\iota \in I}$ in $\mathbb{N}$ such that E is isomorphic to $\prod_{\iota \in I} \mathbb{C}_{n_\iota, n_\iota}$ and*

$$\operatorname{Dim} E^c = \operatorname{Dim} \operatorname{Re} E^c = \operatorname{Card} I .$$

If $\mathbb{K} = \mathbb{R}$ then there are finite families $(n_\lambda)_{\lambda \in L}$, $(p_\mu)_{\mu \in M}$, $(q_\nu)_{\nu \in N}$ in $\mathbb{N}$ such that E is isomorphic to

$$\left(\prod_{\lambda \in L} \mathbb{R}_{n_\lambda, n_\lambda} \right) \times \left(\prod_{\mu \in M} \mathbb{C}_{p_\mu, p_\mu} \right) \times \left(\prod_{\nu \in N} \mathbb{H}_{q_\nu, q_\nu} \right)$$

and

$$\operatorname{Dim} E^c = \operatorname{Card} L + 2\operatorname{Card} M + \operatorname{Card} N ,$$

$$\operatorname{Dim} \operatorname{Re} E^c = \operatorname{Card} L + \operatorname{Card} M + \operatorname{Card} N ,$$

These families are unique up to permutations. In particular, if E is simple and $\mathbb{K} = \mathbb{C}$ ($\mathbb{K} = \mathbb{R}$), then E is isomorphic to $\mathbb{C}_{n,n}$ (to $\mathbb{R}_{n,n}$, $\mathbb{C}_{n,n}$, or $\mathbb{H}_{n,n}$) for some $n \in \mathbb{N}$; moreover, in the complex case, E is the complexification of a real C^*-algebra and there are at most two non-isomorphic real C^*-algebras, the complexification of which are isomorphic to E.

First suppose that $\mathbb{K} = \mathbb{C}$. By Corollary 6.3.6.4, there is a faithful representation (H, φ) of E which is the Hilbert sum of a family $((H_\iota, \varphi_\iota))_{\iota \in I}$ of irreducible representations. We may assume that for distinct $\iota, \lambda \in I$, the representation (H_ι, φ_ι) and $(H_\lambda, \varphi_\lambda)$ are not equivalent. Since E is finite-dimensional, it follows that I is finite and H_ι is finite-dimensional for every $\iota \in I$. By Proposition 5.5.6.9,

$$\varphi_\iota(E) = \mathcal{L}(H_\iota)$$

for every $\iota \in I$. By Corollary 5.5.6.13,

$$\varphi(E) = \prod_{\iota \in I} \mathcal{L}(H_\iota) .$$

Hence, putting

$$n_\iota := \dim H_\iota$$

for $\iota \in I$, E is isomorphic to $\prod_{\iota \in I} \mathbb{C}_{n_\iota, n_\iota}$. Let us identify E and $\prod_{\iota \in I} \mathbb{C}_{n_\iota, n_\iota}$ using this isomorphism and for every $\iota \in I$ let 1_ι be the unit of $\mathbb{C}_{n_\iota, n_\iota}$ and φ_ι the canonical inclusion $\mathbb{C}_{n_\iota, n_\iota} \to E$. Then $(\varphi_\iota 1_\iota)_{\iota \in I}$ is a linearly independent family in E^c, so that

$$\operatorname{Dim} E^c \geq \operatorname{Card} I .$$

Take $x \in \operatorname{Re} E^c$. Then for every $\iota \in I$,

$$x_\iota \in (\mathbb{C}_{n_\iota, n_\iota})^c$$

so that

$$x_\iota \in \mathbb{C} 1_\iota$$

(Corollary 5.6.6.9). It follows

$$\operatorname{Dim} E^c \leq \operatorname{Card} I ,$$

$$\operatorname{Dim} \operatorname{Re} E^c = \operatorname{Dim} E^c = \operatorname{Card} I .$$

For $\mathbb{K} = \mathbb{R}$, the assertion follows from the above result and Corollary 5.6.6.10. The equalities

$$\operatorname{Dim} E^c = \operatorname{Card} L + 2\operatorname{Card} M + \operatorname{Card} N ,$$

$$\operatorname{Dim} \operatorname{Re} E^c = \operatorname{Card} L + \operatorname{Card} M + \operatorname{Card} N ,$$

follow as in the complex case using Corollary 5.6.6.9.

By Corollary 5.6.6.9, $\mathbb{R}_{n,n}, \mathbb{C}_{n,n}$, and $\mathbb{H}_{n,n}$ are simple for every $n \in \mathbb{N}$.

The uniqueness of the families follows from Proposition 4.3.5.5.

We prove now the final assertion. By the above results, if E is simple then it is isomorphic to $\mathbb{C}_{n,n}$ (to $\mathbb{R}_{n,n}$, $\mathbb{C}_{n,n}$, or $\mathbb{H}_{n,n}$) for some $n \in \mathbb{N}$. The supplementary assertion in the complex case follows from Corollary 5.5.7.13 a),b). ■

Remark. If E is finite–dimensional then by the above corollary and by Example 4.3.3.23, $\operatorname{Un} E$ is the set of extreme points of $E^\#$.

Corollary 6.3.6.6 *Let E be a finite-dimensional C^*-algebra, A' the set of non-degenerate algebra homomorphisms of E into $\mathbb{K}$, and*

$$\mathcal{P} := \{p \in E^c \cap \operatorname{Pr} E | Ep \quad \textit{is one-dimensional}\}.$$

a) *For every $p \in \mathcal{P}$, there is a unique $x'_p \in A'$ such that*

$$xp = x'_p(x)p$$

for every $x \in E$.

b) *The map*

$$\mathcal{P} \longrightarrow A', \quad p \longmapsto x'_p$$

is bijective.

c) *Every $x' \in A'$ is positive.*

By Corollary 6.3.6.5 (and Corollary 5.6.6.9), E is isomorphic to a C^*-direct product of a family of unital simple C^*-algebras, so that the assertions follow from Proposition 4.2.8.29. ■

Corollary 6.3.6.7 *If E is a Gelfand C^*-algebra and (H, φ) is an irreducible representation of E, then H is one-dimensional and*

$$(\varphi x)\xi = \langle (\varphi x)\xi | \xi \rangle \xi$$

for every $x \in E$ and $\xi \in H$ with $\|\xi\| = 1$.

Take $\xi \in H$ with $\|\xi\| = 1$. Then ξ is cyclic for (H, φ) (Proposition 5.3.2.20 b)). Define

$$x' : E \longrightarrow \mathbb{K}, \quad x \longmapsto \langle (\varphi x)\xi | \xi \rangle .$$

By Proposition 5.4.1.3 a), $x' \in \tau(E)$ and (H, φ, ξ) is equivalent to the GNS-triple of E associated to x'. By Theorem 6.3.6.2 b $\Rightarrow$ a, $x' \in \tau_0(E)$ and by Corollary 4.2.8.11, $x' \in \sigma(E)$. By Theorem 5.4.1.2 j),k), it follows that H is one-dimensional and that

$$(\varphi x)\xi = x'(x)\xi = \langle (\varphi x)\xi | \xi \rangle \xi$$

for every $x \in E$. ■

Corollary 6.3.6.8 *If F is an involutive commutative complex Banach algebra and (H, φ) is an irreducible representation of F, then*

$$\mathrm{Dim}\, H \leq 1 .$$

Let E be the C^*-hull of F and ψ the factorization of φ through E (Proposition 4.1.1.22 f)). Then (H, ψ) is a representation of E. Since E is commutative, the assertion follows from Corollary 6.3.6.7. ■

Proposition 6.3.6.9 *Assume $\mathbb{K} = \mathbb{C}$. Given $x', y' \in \tau_0(E)$, the following are equivalent:*

a) *The representations of E associated to x' and y' are equivalent.*

b) *There is an $x \in E$ such that*

$$x' = xy'x^* .$$

Let (H, φ, ξ) and (K, ψ, η) be the GNS-triples of E associated to x' and y', respectively.

a $\Rightarrow$ b. Let $u : H \to K$ be an isometry such that

$$\varphi y = u^* \circ (\psi y) \circ u$$

for every $y \in E$. By Theorem 6.3.6.2 a $\Rightarrow$ b and Theorem 6.3.6.1 a), there is an $x \in E$ such that

$$(\psi x)\eta = u\xi .$$

By Corollary 5.4.1.4 a),

$$x' = xy'x^* .$$

b $\Rightarrow$ a. By Theorem 5.4.1.2 k), if $y \in E$, then

$$\langle (\varphi y)\xi \mid \xi \rangle = \langle y, x' \rangle = \langle y, xy'x^* \rangle = \langle x^*yx, y' \rangle =$$

$$= \langle \psi(x^*yx)\eta \mid \eta \rangle = \langle (\psi x)^*(\psi y)(\psi x)\eta | \eta \rangle = \langle (\psi y)(\psi x)\eta \mid (\psi x)\eta \rangle .$$

Hence

$$(\psi x)\eta \neq 0 .$$

Since (K, ψ) is irreducible (Theorem 6.3.6.2 a $\Rightarrow$ b), $(\psi x)\eta$ is cyclic for (K, ψ) (Proposition 5.3.2.20 b)). By Corollary 5.4.1.4 $b_2 \Rightarrow b_1$, (H, φ) and (K, ψ) are equivalent. ■

Proposition 6.3.6.10 *Let F be a C^*-subalgebra of E, (H,φ) an (irreducible) representation of F, and ξ a cyclic vector for (H,φ). Then there is an (irreducible) representation (K,ψ) of E, such that:*

1) *H is a subspace of K.*

2) *H is $\psi(F)$-invariant.*

3) *φ is the compression of $\psi|F$ to H.*

4) *ξ is a cyclic vector for (K,ψ).*

If F is hereditary and E unital, then (K,ψ) with the above properties is unique.

We may assume that

$$\|\xi\| = 1\,.$$

We define

$$y' : F \longrightarrow \mathbb{K}\,, \quad x \longmapsto \langle(\varphi x)\xi|\xi\rangle\,.$$

By Proposition 5.4.1.3 a) (and Theorem 6.3.6.2 b $\Rightarrow$ a), $y' \in \tau(F)$, $(y' \in \tau_0(F))$ and (H,φ,ξ) is the GNS-triple of F associated to y'. By Corollary 4.2.8.9 (by Corollary 4.2.8.10), there is an $x' \in \tau(E)$ $(x' \in \tau_0(E))$ such that

$$x'|F = y'\,.$$

Let (K,ψ,η) be the GNS-triple of E associated to x'. By Theorem 6.3.6.2 a $\Rightarrow$ b, (K,ψ) is irreducible if $x' \in \tau_0(E)$. Put

$$L := \overline{\{(\psi x)\eta | x \in F\}}\,.$$

L is $\psi(F)$-invariant and so $L^\perp$ is also $\psi(F)$-invariant (Proposition 5.3.2.9). We have

$$\pi_L\eta \in L\,, \quad \eta - \pi_L\eta \in L^\perp\,,$$

and so, given $x \in F$,

$$(\psi x)\pi_L\eta \in L\,,$$

$$(\psi x)(\eta - \pi_L\eta) = (\psi x)\eta - (\psi x)\pi_L\eta \in L \cap L^\perp\,,$$

$$(\psi x)\eta = (\psi x)\pi_L\eta\,.$$

Hence $\pi_L\eta$ is a cyclic vector for the representation $(L, \psi|F)$. Moreover,

$$\langle(\psi x)\pi_L\eta \mid \pi_L\eta\rangle = \langle(\psi x)\eta \mid \eta\rangle = x'(x) = y'(x) = \langle(\varphi x)\xi \mid \xi\rangle$$

for every $x \in F$ (Theorem 5.4.1.2 k)). By Proposition 5.4.1.3 b), we may assume that

$$L = H\,, \quad \pi_H\eta = \xi\,,$$

and φ is the compression of $\psi|F$ to H. From

$$\|\pi_H\eta\| = \|\xi\| = 1 = \|\eta\|$$

we get that

$$\eta = \pi_H\eta = \xi\,.$$

Now suppose that F is hereditary and E unital. Define

$$z' : E \longrightarrow \mathbb{K}\,, \quad x \longmapsto \langle(\psi x)\xi|\xi\rangle\,.$$

Then $z' \in \tau(E)$ (Proposition 5.4.1.3 a)) and

$$z'(x) = \langle(\psi x)\xi|\xi\rangle = y'(x)$$

for every $x \in F$. By Proposition 4.3.4.12,

$$z' = x'\,,$$

and this proves the uniqueness of (K, ψ) (Corollary 5.4.1.3). ■

Proposition 6.3.6.11 *Let F be an involutive Banach algebra and (H, φ) a non-degenerate representation of F such that*

$$\varphi(F) \subset \mathcal{K}(H)\,.$$

Then (H, φ) is the Hilbert sum of irreducible representations.

$\overline{\varphi(F)}$ is a C^*-subalgebra of $\mathcal{K}(H)$. Let (H_0, ψ) be an irreducible representation of $\overline{\varphi(F)}$ (Theorem 6.3.6.2 a $\Rightarrow$ b). By Proposition 6.3.6.10, there is an irreducible representation of $\mathcal{K}(H)$ extending (H_0, ψ). By Theorem 5.5.1.24, this representation is the identical representation. Hence, there is a

$\varphi(F)$-invariant closed vector subspace K of H such that the compression of φ to K is irreducible.

Let $\mathfrak{K}$ be a maximal set of $\varphi(F)$-invariant closed vector subspaces K of H such that the compression of φ to K is irreducible. Assume that

$$\left(\bigcup_{K\in\mathfrak{K}} K\right)^{\perp} \neq \{0\}.$$

Then, by the above considerations (and Proposition 5.3.2.9), there is a $\varphi(F)$-invariant closed vector subspace L of $\left(\bigcup_{K\in\mathfrak{K}} K\right)^{\perp}$ such that the compression of φ to L is irreducible. This contradicts the maximality of $\mathfrak{K}$. Hence

$$\left(\bigcup_{K\in\mathfrak{K}} K\right)^{\perp\perp} = H,$$

and (H,φ) is the Hilbert sum of irreducible representations. ■

Theorem 6.3.6.12 (A. Rosenberg, 1953)

a) *If all irreducible representations of E are faithful (e.g. they are equivalent), then E is simple.*

b) *If $\mathbb{K} = \mathbb{C}$, all irreducible representations of E are equivalent, and E is separable, then there is a separable Hilbert space H such that E is isometric to $\mathcal{K}(H)$.*

a) If all irreducible representations are equivalent, then by Corollary 6.3.6.4, they are faithful. Let F be a closed proper ideal of E and let $q : E \to E/F$ be the quotient map. Then E/F is a C^*-algebra (Theorem 4.2.6.5). Let (H,φ) be an irreducible representation of E/F (Corollary 6.3.6.3). Then $(H,\varphi\circ q)$ is an irreducible representation of E. Since it is faithful, we see that

$$\|x\| = \|\varphi q x\| = 0$$

for every $x \in F$. Hence $F = \{0\}$ and E is simple.

b) Let (H,φ) be an irreducible representation of E. Let F be a maximal Gelfand C^*-subalgebra of E. Take $x' \in \sigma(F)$ $(= \tau_0(F)$ Corollary 4.2.8.11) and let $(H_{x'},\varphi_{x'})$ be the representation of F associated to x'. By Theorem 6.3.6.2 a $\Rightarrow$ b, $(H_{x'},\varphi_{x'})$ is irreducible. Take $\xi_{x'} \in H_{x'}$ with $\|\xi_{x'}\| = 1$. By Corollary 6.3.6.7, $H_{x'}$ is one-dimensional and

$$(\varphi_{x'}x)\xi_{x'} = \langle(\varphi_{x'}x)\xi_{x'} \mid \xi_{x'}\rangle\xi_{x'} = x'(x)\xi_{x'}$$

for every $x \in F$ (Theorem 5.4.1.2 k)). By Proposition 6.3.6.10, there is an irreducible representation (K, ψ) of E such that $H_{x'}$ is a $\psi(F)$-invariant subspace of K and $\varphi_{x'}$ is the compression of $\psi|F$ to $H_{x'}$. By hypothesis, we may assume that $(K, \psi) = (H, \varphi)$. Thus

$$(\varphi x)\xi_{x'} = x'(x)\xi_{x'}$$

for every $x \in F$.

Take $x', y' \in \sigma(F)$, $x' \neq y'$. Then there is an $x \in F$ such that

$$x'(x) = 0 \neq y'(x)$$

(Theorem 2.4.1.13). We get

$$0 = \langle x'(x)\xi_{x'} \mid \xi_{y'} \rangle = \langle (\varphi x)\xi_{x'} | \xi_{y'} \rangle = \langle \xi_{x'} \mid (\varphi x^*)\xi_{y'} \rangle =$$

$$= \langle \xi_{x'} | y'(x^*)\xi_{y'} \rangle = y'(x)\langle \xi_{x'} | \xi_{y'} \rangle ,$$

$$\xi_{x'} \perp \xi_{y'} .$$

Hence $(\xi_{x'})_{x' \in \sigma(F)}$ is an orthonormal family in H. Since E is separable, H is also separable. Thus $\sigma(F)$ is countable. Hence there is an $x' \in \sigma(F)$ such that $\{x'\}$ is open.

Choose $p \in \operatorname{Pr} F$ such that

$$\overset{F}{\widehat{p}} = e_{x'}^{\sigma(F)} .$$

By Proposition 4.2.7.18, $pEp \subset F$ and so pEp is a Gelfand C^*-algebra. Put

$$K := p(H) ,$$

define for each $u \in \mathcal{L}(H)$,

$$\widetilde{u} : K \longrightarrow K , \quad \xi \longmapsto pu\xi$$

and

$$\widetilde{\varphi} : E \longrightarrow \mathcal{L}(K) , \quad x \longmapsto \widetilde{\varphi x} .$$

Then by Proposition 5.3.2.24, $(K, \widetilde{\varphi})$ is an irreducible representation of pEp. Since pEp is a Gelfand C^*-algebra, K is one-dimensional (Corollary 6.3.6.7). Hence $p \in \mathcal{K}(H)$. By Proposition 5.5.6.9,

$$\mathcal{K}(H) \subset \varphi(E)\,.$$

Since φ is faithful, $\overset{-1}{\varphi}(\mathcal{K}(H))$ is a closed ideal of E. E being simple,

$$\mathcal{K}(H) = \varphi(E)\,,$$

i.e. E is isometric to $\mathcal{K}(H)$. ∎

Remark. b) does not hold for the real C^*-algebras $\mathbb{C}$ and $\mathbb{H}$.

Proposition 6.3.6.13 *Assume* $\mathbb{K} = \mathbb{C}$. *Let* (H, φ) *be a non-degenerate representation of* E *such that* $\varphi(E) \subset \mathcal{K}(H)$. *Then* (H, φ) *is equivalent to the Hilbert sum of a family* $((H_\iota, \varphi_\iota))_{\iota \in I}$ *of irreducible representations such that*

$$\varphi_\iota(E) = \mathcal{K}(H_\iota)$$

for every $\iota \in I$ *and* $(\|\varphi_\iota x\|)_{\iota \in I} \in c_0(I)$ *for every* $x \in E$.

Let $(H_\iota)_{\iota \in I}$ be a maximal family of pairwise orthogonal invariant closed subspaces of H such that the compression (H_ι, φ_ι) of (H, φ) to H_ι is irreducible for every $\iota \in I$. By Proposition 5.5.6.9,

$$\varphi_\iota(E) = \mathcal{K}(H_\iota)$$

for every $\iota \in I$. Put

$$K := \Big(\bigcup_{\iota \in I} H_\iota\Big)^\perp .$$

We must show that $K = \{0\}$. Assume the contrary. Then K is $\varphi(E)$-invariant (Proposition 5.3.2.9) and there is an irreducible representation of the compression of $\varphi(E)$ to K (Corollary 6.3.6.4). By Proposition 6.3.6.10 and Theorem 5.5.1.24, there is a closed $\varphi(E)$-invariant subspace K_0 of K such that the compression of (H, φ) to K_0 is irreducible. This contradicts the maximality of $((H_\iota, \varphi_\iota))_{\iota \in I}$.

Take $x \in E$ and assume that $(\|\varphi_\iota x\|)_{\iota \in I} \notin c_0(I)$. Then there is an orthonormal family $(\xi_\iota)_{\iota \in I}$ in H such that $\xi_\iota \in H_\iota$ for every $\iota \in I$ and $(\|(\varphi_\iota x)\xi_\iota\|)_{\iota \in I} \notin c_0(I)$ and this contradicts Theorem 5.5.1.1 a ⇒ d. ∎

Theorem 6.3.6.14 *The following are equivalent:*

a) *There is a faithful representation* (H, φ) *of* E *such that* $\varphi(E) \subset \mathcal{K}(H)$.

b) *If* $\mathbb{K} = \mathbb{C}$ *then there is a family* $(H_\iota)_{\iota\in I}$ *of Hilbert spaces such that* E *is isometric to the* C^**-direct sum of* $(\mathcal{K}(H_\iota))_{\iota\in I}$. *If* $\mathbb{K} = \mathbb{R}$ *then there are families of sets* $(R_\lambda)_{\lambda\in L}$, $(S_\mu)_{\mu\in M}$, *and* $(T_\nu)_{\nu\in N}$ *such that* E *is isomorphic to*

$$F_{\mathbb{R}} \times F_{\mathbb{C}} \times F_{\mathbb{H}} ,$$

where $F_{\mathbb{R}}$, $F_{\mathbb{C}}$, *and* $F_{\mathbb{H}}$ *are the* C^**-direct sums of the families*

$$\big(\mathcal{K}_{\mathbb{R}}(\ell^2(R_\lambda,\mathbb{R}))\big)_{\lambda\in L} , \ \big(\mathcal{K}_{\mathbb{C}}(\ell^2(S_\mu,\mathbb{C}))\big)_{\mu\in M} , \ \big(\mathcal{K}_{\mathbb{H}}(\ell^2(T_\nu,\mathbb{H}))\big)_{\nu\in N} ,$$

respectively.

If these conditions are fulfilled then E *is* $\mathbb{C}$*-order complete.*

Case 1 $\mathbb{K} = \mathbb{C}$

a $\Rightarrow$ b. By Proposition 6.3.6.13, (H, φ) is equivalent to the Hilbert sum of a family $((H_\iota, \varphi_\iota))_{\iota\in J}$ of irreducible representations such that

$$\varphi_\iota(E) = \mathcal{K}(H_\iota)$$

for every $\iota \in J$ and $(\|\varphi_\iota x\|)_{\iota\in J} \in c_0(J)$ for every $x \in E$. Let I be a maximal subset of I such that if $\iota, \kappa \in I$ are distinct, then the representations (H_ι, φ_ι), $(H_\kappa, \varphi_\kappa)$ are not equivalent. Let F denote the C^*-direct sum of the family $(\mathcal{K}(H_\iota))_{\iota\in I}$ and put

$$\psi : E \longrightarrow F , \quad x \longmapsto (\varphi_\iota x)_{\iota\in I} .$$

Since φ is faithful, ψ is injective. By Corollary 5.5.6.13, ψ is surjective.

b $\Rightarrow$ a follows from Proposition 5.3.2.27.

Case 2 $\mathbb{K} = \mathbb{R}$

a $\Rightarrow$ b. By Case 1 a $\Rightarrow$ b, Proposition 4.3.5.6, and Corollary 5.5.1.13 b), E is the C^*-direct sum of a family $(E_\iota)_{\iota\in I}$ of C^*-algebras such that for every $\iota \in I$ there is a complex Hilbert space K_ι such that either $\overset{\circ}{E}_\iota$ is isomorphic to $\mathcal{K}(K_\iota)$ or E_ι is isomorphic to the real C^*-algebra underlying $\mathcal{K}(K_\iota)$. By Corollary 5.5.7.13, in the first case E_ι is isomorphic either to $F(\mathcal{K}(K_\iota))$ or to $\mathcal{K}(H)$ for some real Hilbert space H. By Proposition 5.6.4.18 b), $F(\mathcal{K}(K_\iota))$ is isomorphic to

$$\mathcal{L}_{\mathbb{H}}(\ell^2(T,\mathbb{H})) \cap \mathcal{K}(\ell^2(T,\mathbb{H}))$$

for some set T. It is easy to see (and follows from Corollary 6.3.9.8) that

$$\mathcal{L}_{\mathbb{H}}(\ell^2(T,\mathbb{H})) \cap \mathcal{K}(\ell^2(T,\mathbb{H})) = \mathcal{K}_{\mathbb{H}}(\ell^2(T,\mathbb{H}))\,.$$

The assertion follows.

b $\Rightarrow$ a. By Corollary 5.5.7.13, there is a family $(K_\iota)_{\iota\in I}$ of complex Hilbert spaces such that $\overset{\circ}{E}$ is isomorphic to the C^*-direct sum of the family $(\mathcal{K}(K_\iota))_{\iota\in I}$. By Case 1 b $\Rightarrow$ a, $\overset{\circ}{E}$ has a faithful representation (K,φ) with $\varphi(\overset{\circ}{E}) \subset \mathcal{K}(H)$. If H denotes the underlying real Hilbert space of K (Proposition 5.6.2.5 a),e)), then $(H,\varphi|E)$ is a faithful representation of E with $\varphi(E) \subset \mathcal{K}(H)$.

The final assertion follows from b) and Corollary 5.6.5.4 b). ■

Remark. This theorem generalizes Corollary 6.3.6.5.

Theorem 6.3.6.15 *Every reflexive C^*-algebra is finite-dimensional.*

We may assume $\mathbb{K} = \mathbb{C}$. Let E be reflexive and M the set of minimal elements of $\Pr E\backslash\{0\}$.

Step 1 { Every commutative, linearly independent subset of $\operatorname{Re} E$ is finite.

Let A be a commutative, linearly independent subset of $\operatorname{Re} E$ and let F be the C^*-subalgebra of E generated by A. Then F is a Gelfand C^*-algebra (Corollary 4.1.2.3) and so it is isometric to $\mathcal{C}_0(\sigma(F))$. Hence $\mathcal{C}_0(\sigma(F))$ is reflexive (Proposition 1.3.8.5 b)), $\sigma(F)$ is finite, F is finite-dimensional, and A is finite.

Step 2 $\quad p \in \Pr E\backslash\{0\} \Rightarrow \exists\, q \in M\,,\ q \leq p$

Let A be a totally ordered set of $\Pr E\backslash\{0\}$. Since A is commutative and linearly independent (Corollary 4.2.7.6 a $\Rightarrow$ b), it is finite by Step 1. The assertion now follows by Zorn's Lemma.

Step 3 There is an $A \in \mathfrak{P}_f(M)$ such that { $\sum_{p\in A} p = 1$; $p,q \in A \Rightarrow pq = \delta_{pq}p$.

E is a W^*-algebra, so it is unital (Theorem 4.4.1.8 h)). Put

$$\mathfrak{A} := \{A \in \mathfrak{P}(M) \mid p,q \in A \Longrightarrow pq = \delta_{pq}p\}\,.$$

By Zorn's Lemma, $\mathfrak{A}$ has a maximal element A. Since A is commutative and linearly independent, it is finite by Step 1. Assume that

$$\sum_{p\in A} p \neq 1 .$$

Then

$$1 - \sum_{p\in A} p \in \Pr E\backslash\{0\}$$

and by Step 2, there is a $q \in M$,

$$q \leq 1 - \sum_{p\in A} p .$$

We have $A \cup \{q\} \in \mathfrak{A}$, which contradicts the maximality of A. Hence

$$\sum_{p\in A} p = 1 .$$

Step 4 (H, φ) irreducible representation of $E \Rightarrow \operatorname{Dim} H < \infty$

$\varphi(E)$ is isometric to $E/\operatorname{Ker}\varphi$ (Theorem 4.2.6.6), so it is reflexive (Proposition 1.3.8.6 b)). By Step 2 (and Theorem 4.4.1.8 h)), M is nonempty. By Corollary 5.5.6.10 (and Theorem 4.4.1.8 c)),

$$\mathcal{K}(H) \subset \varphi(E) .$$

Hence $\mathcal{K}(H)$ is reflexive (Proposition 1.3.8.5 b)) and therefore H is finite-dimensional (Corollary 6.1.6.9 c $\Rightarrow$ a).

Step 5 E is finite-dimensional

We may assume $\mathbb{K} = \mathbb{C}$. By Step 3, there is a finite subset A of M such that

$$\sum_{p\in A} p = 1 ,$$

$$p, q \in A \Longrightarrow pq = \delta_{pq} p .$$

For $p \in A$, let x'_p be the unique element of $\tau_0(E)$ with

$$x'_p(p) = 1$$

(Corollary 4.3.4.13 a), Theorem 4.4.1.8 c)) and (H_p, φ_p) the representation of E associated to x'_p. By Theorem 6.3.6.2 a $\Rightarrow$ b, for every $p \in A$ the representation (H_p, φ_p) is irreducible. Thus by Step 4, H_p is finite-dimensional.

Let (H, φ) be the representation of E associated to A, i.e. the Hilbert sum of the representations $((H_p, \varphi_p))_{p \in A}$. Take $x \in E$, and $p \in A$. Then

$$px^*xp = x'_p(px^*xp)p = x'_p(x^*x)p$$

(Corollary 4.3.4.13), so

$$\|\varphi_p x\|^2 \geq x'_p(x^*x) = \|x'_p(x^*x)p\| = \|px^*xp\| = \|xp\|^2$$

(Theorem 5.4.1.2 g)). Hence if $\varphi x = 0$, then $\varphi_p x = 0$ and $xp = 0$ for every $p \in A$. Since

$$x = \sum_{p \in A} xp\,,$$

$x = 0$ and thus φ is injective. Since H is finite-dimensional, E is finite-dimensional as well. ■

Corollary 6.3.6.16 *Let E be an infinite-dimensional W^*-algebra.*

a) *There is a strictly increasing sequence in* $\Pr E$.

b) *E is not separable.*

a) By Theorem 6.3.6.15, E is not reflexive. Hence the predual of E and the dual of E are distinct (Proposition 1.3.8.4). By Theorem 4.4.4.2 c $\Rightarrow$ a, there is a strictly increasing sequence in $\Pr E$.

b) By a), there is a strictly increasing sequence $(p_n)_{n \in \mathbb{N}}$ in $\Pr E$. For every $A \subset \mathbb{N}$, denote by p_A the supremum in $\Pr E$ of

$$\{\sum_{n \in B}(p_{n+1} - p_n) \mid B \in \mathfrak{P}_f(A)\}$$

(Theorem 4.4.1.8 i)). If A and B are distinct subsets of $\mathbb{N}$, then

$$(p_A - p_B)^+ = p_A - p_{A \cap B} \in \Pr E\,, \quad (p_A - p_B)^- = p_B - p_{A \cap B} \in \Pr E$$

(Corollary 4.2.7.6 a $\Rightarrow$ g), so that

$$\|p_A - p_B\| = \sup\{\|p_A - p_{A \cap B}\|\,,\ \|p_B - p_{A \cap B}\|\} = 1$$

(Theorem 4.2.2.9 b), Proposition 4.1.2.21). Since $\{p_A \mid A \subset \mathbb{N}\}$ is uncountable, E is not separable. ■

Proposition 6.3.6.17 *If $\mathbb{K} = \mathbb{C}$ then the following are equivalent:*

a) *E is finite-dimensional.*

b) *There is an $\alpha \in \mathbb{R}_+$ such that*

$$\sum_{\iota \in I} \|x_\iota\| \leq \alpha \| \sum_{\iota \in I} x_\iota \|$$

for every finite family $(x_\iota)_{\iota \in I}$ in E_+.

c) *$\sigma(x)$ is finite for every $x \in E$.*

d) *$\sigma(x)$ is finite for every $x \in E_+$.*

e) *Every Gelfand C^*-subalgebra of E is finite-dimensional.*

a $\Rightarrow$ b follows from the Corollaries 4.2.1.8 and 1.7.1.7.

a $\Rightarrow$ c follows from Proposition 2.2.1.17.

c $\Rightarrow$ d is trivial.

d $\Rightarrow$ e follows from the isomorphism of the Gelfand transform.

b $\Rightarrow$ e. Let F be an infinite-dimensional Gelfand C^*-subalgebra of E. Then $\sigma(F)$ is infinite. Hence there is a sequence $(x_n)_{n \in \mathbb{N}}$ in F_+ such that

$$\|x_n\| = 1\,, \quad x_m x_n = 0$$

for any dinstict $m, n \in \mathbb{N}$. Then

$$\sum_{n \in M} \|x_n\| = \left\| \sum_{n \in M} x_n \right\| \operatorname{Card} M$$

for every $M \in \mathfrak{P}_f(\mathbb{N})$ which contradicts b).

e $\Rightarrow$ a. Let A be an upward directed commutative subset of $E_+^\#$. By Corollary 4.1.4.2 c), the C^*-subalgebra F of E generated by A is Gelfand. By e), F is finite-dimensional. It follows that the upper section filter of A converges to an $x \in F$. Let (H, φ) be a faithful representation of E. By Proposition 1.7.2.2, φx is the supremum of $\varphi(A)$ in $\mathcal{L}(H)$. By Theorem 6.3.4.6, E is a W^*-algebra. If E is infinite-dimensional then by Corollary 6.3.6.16 a), there is a strictly increasing sequence $(p_n)_{n \in \mathbb{N}}$ in $\operatorname{Pr} E$. By Corollary 4.2.7.6 a $\Rightarrow$ b, $(p_n)_{n \in \mathbb{N}}$ is commutative. Hence the C^*-subalgebra G of E generated by $(p_n)_{n \in \mathbb{N}}$ is Gelfand (Corollary 4.1.4.2 c)). G has to be infinite-dimensional and this contradicts e). ■

Proposition 6.3.6.18 *The following are equivalent for every nonempty set T:*

a) *E is reflexive.*

b) *$\ell^2(T,E)$ is reflexive (Example 5.6.4.2 a)).*

a $\Rightarrow$ b. By Theorem 6.3.6.15, E is finite-dimensional. Let $(x_\iota)_{\iota\in I}$ be an algebraic basis of E with $\|x_\iota\| = 1$ for every $\iota \in I$ and $(x'_\iota)_{\iota\in I}$ an algebraic basis of E' such that

$$\langle x_\iota, x'_\lambda\rangle = \delta_{\iota\lambda}$$

for all $\iota, \lambda \in I$ (Corollary 1.3.3.2 a)). For every $\iota \in I$ put

$$H_\iota := \{\xi \in \ell^2(T,E) | t \in T \Longrightarrow \xi_t \in \mathbb{K}x_\iota\}\,,$$

$$\varphi_\iota : \ell^2(T) \longrightarrow H_\iota\,, \quad \alpha \longmapsto \alpha x_\iota$$

and denote by π_ι the natural projection

$$\ell^2(T,E) \longrightarrow H_\iota\,.$$

For every $(x',\xi) \in E' \times \ell^2(T,E)$ put

$$\widetilde{(x',\xi)} : \ell^2(T,E) \longrightarrow \mathbb{K}\,, \quad \eta \longmapsto \langle\langle\eta|\xi\rangle, x'\rangle\,;$$

$\widetilde{(x',\xi)}$ is linear and continuous.

Let $\xi' \in (\ell^2(T,E))'$. For $\iota \in I$, the map

$$\ell^2(T) \longrightarrow \mathbb{K}\,, \quad \alpha \longmapsto \xi'(\alpha x_\iota)$$

is linear and continuous. By the Theorem of Fréchet–Riesz (Theorem 5.2.5.2), there is an $\alpha_\iota \in \ell^2(T)$ for every $\iota \in I$ such that

$$\xi'(\alpha x_\iota) = \langle\alpha|\alpha_\iota\rangle$$

for every $\alpha \in \ell^2(T)$. For every $\eta \in \ell^2(T,E)$ and $\iota \in I$,

$$\widetilde{(x'_\iota, \alpha_\iota 1)}(\eta) = \sum_{\lambda\in I} \widetilde{(x'_\iota, \alpha_\iota 1)}(\pi_\lambda\eta) = \sum_{\lambda\in I}\langle\langle\pi_\lambda\eta|\alpha_\iota 1\rangle, x'_\iota\rangle =$$

$$= \langle\langle\pi_\iota\eta|\alpha_\iota 1\rangle, x'_\iota\rangle = \xi'(\pi_\iota\eta)$$

so that

$$\xi'(\eta) = \sum_{\iota\in I}\xi'(\pi_\iota\eta) = \sum_{\iota\in I}\widetilde{(x'_\iota, \alpha_\iota 1)}(\eta)\,,$$

$$\xi' = \sum_{\iota \in I} \widetilde{(x'_\iota, \alpha_\iota 1)} .$$

By Proposition 5.6.4.6 d) (and Proposition 1.3.6.27), the predual of $\ell^2(T,E)$ is reflexive so that $\ell^2(T,E)$ is also reflexive (Proposition 1.3.8.4).

b $\Rightarrow$ a. Take $t \in T$ and put

$$\varphi : \ell^2(T,E) \longrightarrow E , \quad \xi \longmapsto \xi_t ,$$

$$\psi : E \longrightarrow \ell^2(T,E) , \quad x \longmapsto x e_t .$$

Then $\varphi \circ \psi$ is the identity map $E \to E$ and so $\varphi'' \circ \psi''$ is the identity map $E'' \to E''$ (Corollaries 1.3.4.4, 1.3.4.5). Thus φ'' is surjective. Since $\ell^2(T,E)$ is reflexive, $j_{\ell^2(T,E)}$ and

$$j_E \circ \varphi = \varphi'' \circ j_{\ell^2(T,E)}$$

are surjective (Proposition 1.3.6.16). Hence j_E is surjective, i.e. E is reflexive. ■

Proposition 6.3.6.19 *Let E be a finite-dimennsional C^*-algebra and $n,p \in \mathbb{N}$. The following are equivalent:*

a) $\frac{p}{n} \in \mathbb{N}$.

b) *$E_{n,n}$ is isomorphic to a unital C^*- subalgebra of $E_{p,p}$.*

c) *There is an involutive algebra homomorphism $E_{n,n} \to E_{p,p}$.*

a $\Rightarrow$ b $\Rightarrow$ c is trivial.

c $\Rightarrow$ a. We may assume $\mathbb{K} = \mathbb{C}$ (Corollary 5.6.6.8 c)). By Corollary 6.3.6.5, there is a finite family $(m_k)_{k \in I}$ in $\mathbb{N}$ such that E is isomorphic to $\prod_{k \in I} \mathbb{C}_{m_k,m_k}$. By Corollary 5.6.6.8 b), $E_{n,n}$ $E_{p,p}$ are isomorphic to

$$\prod_{k \in I} \mathbb{C}_{nm_k,nm_k} \text{ and } \prod_{k \in I} \mathbb{C}_{pm_k,pm_k} ,$$

respectively. Take $k \in I$. Since $\mathbb{C}_{nm_k,nm_k}$ is simple (Theorem 5.6.6.7 b)), there is exactly one $k' \in I$ and a unital algebra homomorphism $\mathbb{C}_{nm_k,nm_k} \to \mathbb{C}_{pm_{k'},pm_{k'}}$. Moreover

$$I = \{k' | k \in I\} .$$

By Corollary 2.1.4.13 c),

$$\frac{pm_k}{nm_{k'}} \in \mathbb{N}$$

for every $k \in \mathbb{N}$. Hence

$$(\frac{p}{n})^{\operatorname{Card} I} = \prod_{k \in I} \left(\frac{pm_k}{nm_{k'}} \right) \in \mathbb{N}.$$

It follows

$$\frac{p}{n} \in \mathbb{N}. \qquad \blacksquare$$

Proposition 6.3.6.20 *Let E be a C^*-algebra and $p \in \operatorname{Pr} E$. The following are equivalent:*

a) $p \in E^c$.

b) pE *is an ideal of* E.

c) Ep *is an ideal of* E.

d) pEp *is an ideal of* E.

e) *For every irreducible representation* (H, φ) *of* E, $\varphi p \in \{0, 1\}$.

a $\Rightarrow$ b & c is trivial.

b $\Rightarrow$ c. Take $x, y \in E$. By b), there is a $z \in E$ such that

$$y^* p x^* = pz.$$

It follows

$$xpy = z^* p \in Ep$$

so that Ep is an ideal of E.

c $\Rightarrow$ b. Take $x, y \in E$. By c), there is a $z \in E$ such that

$$x^* p y^* = zp.$$

Then

$$ypx = pz^* \in pE,$$

i.e. pE is an ideal of E.

b & c $\Rightarrow$ d. Take $x, y \in E$. By b) (by c)), there is a $z \in E$ such that

$$xpy = pz \qquad (xpy = zp).$$

Then

$$xpyp = pzp \in pEp \qquad (pxpy = pzp \in pEp)$$

and pEp is an ideal of E.

d $\Rightarrow$ e. Let $x \in E$. By d), there is a $y \in E$ such that

$$xp = xp^3 = pyp\,.$$

It follows

$$(\varphi x)(\varphi p) = (\varphi p)(\varphi(yp))\,.$$

Hence $\mathrm{Im}\,(\varphi p)$ is $\varphi(E)$–invariant. By Proposition 5.3.2.9, $\mathrm{Im}\,(\varphi p)$ reduces $\varphi(E)$. Since (H, φ) is irreducible, $\varphi p \in \{0, 1\}$.

e $\Rightarrow$ a. Take $x \in E$. By e), for every irreducible representation (H, φ) of E,

$$\varphi(xp) = (\varphi x)(\varphi p) = (\varphi p)(\varphi x) = \varphi(px)\,,$$

$$\varphi(xp - px) = 0\,.$$

By Corollary 6.3.6.3,

$$xp - px = 0\,.$$

Hence

$$xp = px$$

and $p \in E^c$. ■

6.3.7 Commutative von Neumann Algebras

Proposition 6.3.7.1 (0) *Let* $\mathbb{K} = \mathbb{C}$ $(\mathbb{K} = \mathbb{R})$ *and* E *a Gelfand* C^**-subalgebra of* $\mathcal{L}(H)$. *Then the following are equivalent:*

a) $E = E^c$ $(E = \mathrm{Re}\,(E^c))$.

b) E *is a maximal Gelfand* C^**-subalgebra of* $\mathcal{L}(H)$.

c) E *is a maximal Gelfand von Neumann algebra on* H.

If these conditions are fulfilled, then:

d) E *acts non-degenerately on* H.

e) *If* H *is separable then* E *has a cyclic vector.*

a $\Leftrightarrow$ b follows from Proposition 4.2.2.14.
b $\Leftrightarrow$ c follows from Corollary 4.4.4.12 b).
a $\Rightarrow$ d. Put

$$K := \{ux \mid (u,x) \in E \times H\}^{\perp\perp}.$$

Then K is the closed vector subspace of H generated by $\{ux \mid (u,x) \in E \times H\}$ (Corollary 5.2.3.9) and so K is E-invariant. By Proposition 5.3.2.9, $\pi_K \in E^c$, so that $1 - \pi_K \in E^c$. By a), $1 - \pi_K \in E$. Take $x \in K^\perp$. Then

$$x = (1 - \pi_K)x \in K,$$

$$x = 0,$$

$$K^\perp = \{0\},$$

$$K = H.$$

Thus E acts non-degenerately on H.

a & d $\Rightarrow$ e. By Proposition 5.3.2.23, E has a separating vector. By a) and d), this vector is cyclic for E (Proposition 5.3.2.22 b)). ■

Example 6.3.7.2 (0) *Let* T *be a locally compact space and* μ *a positive Radon measure on* T. *Given* $x \in L^\infty(\mu)$ *and* $\mathcal{F} \subset L^\infty(\mu)$, *define*

$$\widetilde{x} : L^2(\mu) \longrightarrow L^2(\mu), \quad y \longmapsto xy$$

and

$$\widetilde{\mathcal{F}} := \{\widetilde{x} \mid x \in \mathcal{F}\}.$$

a) $\widetilde{L^\infty(\mu)}$ *is a maximal Gelfand von Neumann algebra on* $L^2(\mu)$ *(Example 2.3.2.32). It is called* ***the von Neumann algebra on*** $L^2(\mu)$ ***of multiplication operators.***

b) $\widetilde{\mathcal{K}(T)}$ *is strongly dense in* $\widetilde{L^\infty(\mu)}$ *and* $\widetilde{L^\infty(\mu)}$ *is the von Neumann algebra on* $L^2(\mu)$ *generated by* $\widetilde{\mathcal{K}(T)}$.

c) *If* T *is compact then* e_T *is cyclic for* $\widetilde{\mathcal{C}(T)}$ *and separating for* $\widetilde{L^\infty(\mu)}$.

a) By Example 2.2.2.22 d),

$$\widetilde{L^\infty(\mu)} = \left(\widetilde{L^\infty(\mu)}\right)^c$$

and the assertion now follows from Proposition 6.3.7.1 a $\Rightarrow$ c (in the real case $\widetilde{L^\infty(\mu)} \subset \mathrm{Re}\,\mathcal{L}(L^2(\mu))$).

b) By Example 2.2.2.22 e),

$$\left(\widetilde{\mathcal{K}(T)}\right)^c = \left(\widetilde{L^\infty(\mu)}\right)^c = \widetilde{L^\infty(\mu)},$$

so that

$$(\widetilde{\mathcal{K}(T)})^{cc} = \widetilde{L^\infty(\mu)},$$

and the assertion now follows from Proposition 6.3.5.1 and Corollary 6.3.1.12.

c) is easy to see. ■

Example 6.3.7.3 *Let* A *be an orthonormal basis of* H. *Given* $x \in \ell^\infty(A)$, *define*

$$\widetilde{x} : H \longrightarrow H, \quad \xi \longmapsto \sum_{\eta \in A} x(\eta)\langle \xi | \eta \rangle \eta .$$

Then

$$\{\widetilde{x} \mid x \in \ell^\infty(A)\}$$

is a maximal Gelfand von Neumann algebra on H *and the von Neumannn algebra on* H *generated by* $\{\widetilde{x} \mid x \in c_0(A)\}$.

The assertion follows immediately from Example 6.3.7.2 a),b) , by identifying H with $\ell^2(A)$. ■

Theorem 6.3.7.4 (0) *Let E be a Gelfand unital von Neumann algebra on a Hilbert space H. Let E admit a cyclic vector. Then there are a compact space T, a positive Radon measure μ on T, and an isometry of Hilbert spaces $u : H \to L^2(\mu)$ such that uEu^{-1} is the von Neumann algebra of multiplication operators on $L^2(\mu)$ (Example 6.3.7.2). If H is separable then we may take T metrizable.*

Let F be a unital C^*-subalgebra of E dense in E_H. By Theorem 2.4.1.3 c), $T := \sigma(F)$ is compact. Let φ be the Gelfand transform of F and ξ a cyclic vector for E. The map

$$\mathcal{C}(T) \longrightarrow \mathbb{K}, \quad x \longmapsto \langle (\varphi^{-1}x)\xi|\xi\rangle$$

is a positive linear form (Corollary 5.3.3.7) and so there is a positive Radon measure μ on T such that

$$\int x d\mu = \langle (\varphi^{-1}x)\xi|\xi\rangle$$

for every $x \in \mathcal{C}(T)$. Given $x \in L^\infty(\mu)$ and $\mathcal{F} \subset L^\infty(\mu)$, define

$$\widetilde{x} : L^2(\mu) \longrightarrow L^2(\mu), \quad y \longmapsto xy,$$

and

$$\widetilde{\mathcal{F}} := \{\widetilde{x} \mid x \in \mathcal{F}\}.$$

We have

$$\|\varphi(v)\|_{2,\mu}^2 = \int |\varphi(v)|^2 d\mu = \int \varphi(v^*v) d\mu =$$

$$= \langle (\varphi^{-1}\varphi(v^*v))\xi|\xi\rangle = \langle v^*v\xi|\xi\rangle = \langle v\xi|v\xi\rangle = \|v\xi\|^2$$

for every $v \in F$. Put

$$K := \{v\xi \mid v \in F\},$$

and

$$u : K \longrightarrow \mathcal{C}(T), \quad v\xi \longmapsto \varphi(v).$$

Then u is linear and preserves norms. Since ξ is cyclic for E and F is dense in E_H, K is dense in H. $\mathcal{C}(T)$ being dense in $L^2(\mu)$, u may be extended to an isometry $H \longrightarrow L^2(\mu)$. Define

$$\psi : E \longrightarrow \mathcal{L}(L^2(\mu)), \quad v \longmapsto u \circ v \circ u^{-1}.$$

Take $v \in F$ and $x \in C(T)$. There is a $w \in F$ such that

$$x = \varphi(w).$$

We get

$$(\psi v)x = uvu^{-1}\varphi(w) = uvw\xi = \varphi(vw) =$$

$$= \varphi(v)\varphi(w) = \widetilde{\varphi(v)}x,$$

so that

$$\psi v = \widetilde{\varphi(v)} \in \widetilde{C(T)},$$

$$\psi(F) = \widetilde{C(T)}.$$

Since F is strongly dense in E and $\widetilde{C(T)}$ is strongly dense in $\widetilde{L^\infty(\mu)}$ (Example 6.3.7.2 b)),

$$\psi(E) = \widetilde{L^\infty(\mu)}.$$

Now assume that H is separable. By Corollary 6.3.5.3, $\mathcal{L}(H)_H^\#$ is separable and metrizable. Hence $E_H^\#$ is separable and metrizable. Let A be a countable dense set of $E_H^\#$ and F the unital C^*–subalgebra of E generated by A. Then F is separable and dense in E_H. By Theorem 2.4.1.3 d), $T = \sigma(F)$ is metrizable. ■

Corollary 6.3.7.5 (0) *Let E be a Gelfand von Neumann algebra on a separable Hilbert space H. Then there is a compact space T, a positive Radon measure μ on T, and a bijective involutive algebra homomorphism of E onto the von Neumann algebra on $L^2(\mu)$ of multiplication operators.*

The unit of E being an orthogonal projection in H (Corollary 4.4.4.9), we may assume that $1_H \in E$. Then E acts non–degenerately on H and by Proposition 5.3.2.23, E has a separating vector x. Put

$$K := \overline{\{ux | u \in E\}}$$

and for each $u \in E$, define

$$\widetilde{u} : K \longrightarrow K, \quad y \longmapsto uy$$

(K is E-invariant). The map

$$E \longrightarrow \mathcal{L}(K), \quad u \longmapsto \widetilde{u}$$

is an involutive unital algebra homomorphism and it is injective, since x is separating for E. The map

$$E_{\mathcal{L}^1(H)} \longrightarrow \mathcal{L}(K)_{\mathcal{L}^1(K)}, \quad u \longmapsto \widetilde{u}$$

is obviously continuous. Since

$$E_{\mathcal{L}^1(H)} = E_{\ddot{E}}$$

(Proposition 6.3.4.2), the map

$$E \longrightarrow \mathcal{L}(K), \quad u \longmapsto \widetilde{u}$$

is a W^*-homomorphism (Proposition 4.4.4.6), so $\{\widetilde{u} | u \in E\}$ is a von Neumann algebra on K (Corollary 4.4.4.8 b)). x is obviously a cyclic vector for $\{\widetilde{u} \mid u \in E\}$. Replacing E by $\{\widetilde{u} \mid u \in E\}$ if necessary, we may assume that E has a cyclic vector. The assertion now follows from Theorem 6.3.7.4. ■

6.3.8 Representations of W^*–Algebras

Proposition 6.3.8.1 (0) *Assume E unital and $\mathbb{K} = \mathbb{C}$ ($\mathbb{K} = \mathbb{R}$). Take $x' \in \tau(E)$, let (H, φ) be the representation of E associated to x' and A a downward directed set of E_+ such that*

$$\inf_{y \in A} x'(x^* y x) = 0$$

for every unitary (invertible) element x of E. Then 0 is the infimum of $\varphi(A)$ in $\mathcal{L}(H)$.

$\varphi(A)$ is downward directed (Corollary 4.2.1.4). Let v be the infimum of $\varphi(A)$ in $\mathcal{L}(H)$ (Corollary 6.1.7.14 b), Theorem 4.4.1.8 c)) and x a unitary (invertible) element of E. Put

$$F := \{y \in E \mid x'(y^* y) = 0\},$$

and let w denote the quotient map $E \to E/F$. Then

$$\left\| v^{\frac{1}{2}} wx \right\|^2 = \left\langle v^{\frac{1}{2}} wx \mid v^{\frac{1}{2}} wx \right\rangle = \langle vwx \mid wx \rangle =$$

$$= \inf_{y \in A} \langle (\varphi y) wx | wx \rangle = \inf_{y \in A} \langle w(yx) \mid wx \rangle = \inf_{y \in A} x'(x^* y x) = 0$$

(Proposition 6.3.1.4 a $\Rightarrow$ b), so that

$$wx \in \operatorname{Ker} v^{\frac{1}{2}} = \operatorname{Ker} v$$

(Proposition 5.3.3.10 a)). Since every element of E is a linear combination of four unitary (of two invertible) elements of E (Corollary 4.1.3.7), v vanishes on $\operatorname{Im} w$. Since $\operatorname{Im} w$ is dense in H, v is identically 0. ■

Corollary 6.3.8.2 (0) *Assume E unital and $\mathbb{K} = \mathbb{C}$ ($\mathbb{K} = \mathbb{R}$). Let $\emptyset \neq A \subset \tau(E)$, let (H, φ) be the representation of E associated to A, and let B be a downward directed set of E_+ such that*

$$\inf_{y \in B} x'(x^* y x) = 0$$

for every $x' \in A$ and for every unitary (invertible) element x of E. Then 0 is the infimum of $\varphi(B)$ in $\mathcal{L}(H)$.

For each $x' \in A$, let $\varphi_{x'}$ denote the algebra homomorphism associated to x'.

Take $\xi \in H$. Then

$$\langle (\varphi y)\xi | \xi \rangle = \sum_{x' \in A} \langle (\varphi_{x'} y)\xi_{x'} | \xi_{x'} \rangle$$

for every $y \in B$. By Proposition 6.3.8.1,

$$\inf_{y \in B} \langle (\varphi y)\xi | \xi \rangle = 0 .$$

Since ξ is arbitrary, 0 is the infimum of $\varphi(B)$ in $\mathcal{L}(H)$ (Proposition 6.3.1.4 b $\Rightarrow$ a). ■

Corollary 6.3.8.3 (0) *Assume E unital and $\mathbb{K} = \mathbb{C}$ ($\mathbb{K} = \mathbb{R}$). Let $\emptyset \neq A \subset \tau(E)$, (H, φ) the representation of E associated to A, and $\mathfrak{B}$ a set of upward directed subsets of E_+ such that:*

1) *every $B \in \mathfrak{B}$ possesses a supremum $\bigvee_{x \in B} x$ in E_+.*

2) $B \in \mathfrak{B}$, $x' \in A \Rightarrow x'\Big(\bigvee_{x \in B} x \Big) = \sup_{x \in B} x'(x)$.

3) *$x^* B x \in \mathfrak{B}$ for every $B \in \mathfrak{B}$ and for every unitary (invertible) element x of E.*

Then for every $B \in \mathfrak{B}$, $\varphi\Big(\bigvee_{x \in B} x \Big)$ is the supremum of $\varphi(B)$ in $\mathcal{L}(H)$.

Given $B \in \mathfrak{B}$, define

$$\widetilde{B} := \{ \bigvee_{y \in B} y - x \mid x \in B \}$$

and put

$$\widetilde{\mathfrak{B}} := \{ \widetilde{B} \mid B \in \mathfrak{B} \} .$$

Take $B \in \mathfrak{B}$. Then $\widetilde{B}$ is a downward directed subset of E_+,

$$\inf_{x \in \widetilde{B}} x'(x) = 0$$

for every $x' \in A$, and $x^* \widetilde{B} x = \widetilde{x^* B x} \in \widetilde{\mathfrak{B}}$ for every unitary (invertible) element x of E (Proposition 4.2.2.21). By Corollary 6.3.8.2, 0 is the infimum of $\varphi(\widetilde{B})$ in $\mathcal{L}(H)$. Hence $\varphi\Big(\bigvee_{x \in B} x \Big)$ is the supremum of $\varphi(B)$ in $\mathcal{L}(H)$. ■

Theorem 6.3.8.4 (0) *Assume E unital, $\mathbb{K} = \mathbb{C}$, and $\mathfrak{A}$ a set of upward directed subsets of E_+ such that:*

1) *Every $A \in \mathfrak{A}$ has a supremum in E_+.*

2) *$x^*Ax \in \mathfrak{A}$ for every $A \in \mathfrak{A}$ and every unitary element x of E.*

3) *Every commutative well-ordered set of E_+, which has a supremum in E_+, belongs to $\mathfrak{A}$.*

Let B be a nonempty dense set of

$$\{x' \in \tau(E) \mid A \in \mathfrak{A} \Longrightarrow x'\Big(\bigvee_{x \in A}\Big) = \sup_{x \in A} x'(x)\}\,.$$

Let (H, φ) be the representation of E associated to B and F the closed vector subspace of E' generated by

$$\{yx'z \mid y, z \in E\,,\ x' \in B\}\,.$$

Then the following are equivalent:

a) *E is a W^*-algebra.*

b) *E is order-complete and E_F is Hausdorff.*

c) *E_F is Hausdorff and every commutative well-ordered set of $E_+^{\#}$ has a supremum in E.*

d) *φ is injective and every commutative well-ordered set of $E_+^{\#}$ has a supremum in E.*

e) *φ is injective and $\varphi(E)$ is a von Neumann algebra on H.*

f) *E is a W^*-algebra and φ is an injective W^*-homomorphism.*

a $\Rightarrow$ b follows from Theorem 4.4.1.8 c).

b $\Rightarrow$ c is trivial.

c $\Rightarrow$ d follows from Proposition 5.4.2.7.

d $\Rightarrow$ e. By Theorem 4.2.6.6, $\varphi(E)$ is a C^*-subalgebra of $\mathcal{L}(H)$ and the map

$$E \longrightarrow \varphi(E)\,, \quad x \longmapsto \varphi x$$

is an isometry of C^*-algebras. Let A be a commutative well-ordered set of $\varphi(E)_+^{\#}$ and let v be its supremum in $\mathcal{L}(H)$ (Corollary 6.1.7.14 b), Theorem

4.4.1.8 c)). Then $\overset{-1}{\varphi}(A)$ is a commutative well-ordered subset of $E_+^{\#}$. Let x be the supremum of $\overset{-1}{\varphi}(A)$ in E. By Corollary 6.3.8.3,

$$v = \varphi x \in \varphi(E)\,.$$

By Theorem 6.3.4.6, $\varphi(E)$ is a von Neumann algebra on H.

e $\Rightarrow$ f. By Corollary 4.4.4.9 a $\Rightarrow$ b, $\varphi(E)$ is a W^*-algebra and the inclusion map $\varphi(E) \to \mathcal{L}(H)$ is a W^*-homomorphism. By Theorem 4.2.6.6, the map

$$E \longrightarrow \varphi(E)\,, \quad x \longmapsto \varphi x$$

is an isometry of C^*-algebras. Thus E is a W^*-algebra and the above map is a W^*-homomorphism. Hence φ is a W^*-homomorphism.

f $\Rightarrow$ a is trivial. ■

Corollary 6.3.8.5 (0) *If E is a W^*-algebra, then there is a Hilbert space H and an injective W^*-homomorphism $\varphi : E \to \mathcal{L}(H)$ such that $\varphi(E)$ is a von Neumann algebra on H.*

Put

$$A := \tau(E) \cap \ddot{E}$$

and denote by (H, φ) the representation of E associated to A.

Step 1 φ is injective

Let $x \in E$. By Corollary 4.4.1.5, there is an $x' \in A$ such that

$$\langle xx^*xx^*, x' \rangle \neq 0\,.$$

Denote by $q : E \to E/N_{x'}$ the quotient map and by $(H_{x'}, \varphi_{x'})$ the representation of E associated to x'. Then $xx^* \notin N_{x'}$, so $q(xx^*) \neq 0$. It follows

$$(\varphi_{x'}x)qx^* = q(xx^*) \neq 0\,, \quad \varphi_{x'}x \neq 0\,.$$

Thus φ is injective.

Step 2 φ is a W^*-homomorphism

By Corollary 4.4.1.9, A is dense in $\tau(E)$ and by Theorem 4.4.4.2 a $\Rightarrow$ b, x' is order continuous for every $x' \in A$. By Theorem 6.3.8.4 a $\Rightarrow$ f, φ is a W^*-homomorphism in the complex case.

Assume now $\mathbb{K} = \mathbb{R}$ and put

$$B := \{(x', 0) \mid x' \in A\}.$$

By Proposition 5.4.2.14,

$$\overset{\circ}{E} \longrightarrow \mathcal{L}(\overset{\circ}{H}), \quad (x, y) \longmapsto (\varphi x, \varphi y)$$

is equivalent to the representation of E associated to B. By Theorem 4.3.6.4 (and Corollary 4.4.4.4), $B \subset \overset{\circ}{\overset{\circ}{E}}$. By Corollary 6.3.8.3 (and Step 1 and Theorem 4.4.4.2 a $\Rightarrow$ b), the supremum of every upward directed subset of $\overset{\circ}{\varphi}(\overset{\circ}{E})_+$ belongs to $\overset{\circ}{\varphi}(\overset{\circ}{E})$ so that by Theorem 6.3.4.6, $\overset{\circ}{\varphi}(\overset{\circ}{E})$ is a W^*–subalgebra of $\mathcal{L}(\overset{\circ}{H})$. By Step 1 and Theorem 4.2.6.6, the map

$$\overset{\circ}{\varphi} : \overset{\circ}{E} \longrightarrow \overset{\circ}{\varphi}(\overset{\circ}{E})$$

is an isomorphism of C^*–algebras, so it is a W^*–homomorphism. It follows that $\overset{\circ}{\varphi}$ is a W^*–homomorphism. Thus φ is a W^*–homomorphism (Proposition 4.4.4.7).

Step 3 $\varphi(E)$ is a von Neumann algebra on H

The assertion follows from Step 2 and Corollary 4.4.4.8 b). ■

Corollary 6.3.8.6 (0) *Let E be a complex W^*–algebra and F a C^*–subalgebra of E such that the supremum in E of any commutative well–ordered set of $F_+^{\#}$ belongs to F. Then F is a W^*–subalgebra of E.*

By Corollary 6.3.8.5, there is a Hilbert space H and an injective W^*–homomorphism $\varphi : E \to \mathcal{L}(H)$ such that $\varphi(E)$ is a von Neumann algebra on H. Then the map

$$E \longrightarrow \varphi(E), \quad x \longmapsto \varphi x$$

is an isometry of C^*–algebras (Theorem 4.2.6.6) and $\varphi(E)$ is an order faithful C^*–subalgebra of $\mathcal{L}(H)$ (Theorem 4.4.1.8 d)). Hence $\varphi(F)$ is a C^*–subalgebra of $\mathcal{L}(H)$. Let A be a commutative well–ordered set of $\varphi(F)_+^{\#}$ and v its supremum in $\mathcal{L}(H)$. $\overset{-1}{\varphi}(A)$ is a commutative well–ordered set of $F_+^{\#}$ and x, its supremum in E, belongs to F. Since v belongs to $\varphi(E)$, we get

$$v = \varphi x \in \varphi(F).$$

Hence $\varphi(F)$ is a W^*–subalgebra of $\mathcal{L}(H)$ (Theorem 6.3.4.6) and so a W^*–subalgebra of $\varphi(E)$ (Corollary 4.4.4.10). Thus F is a W^*–subalgebra of E. ■

Corollary 6.3.8.7 (0) *Let E be a W^*-algebra, F a C^*-subalgebra of E, and G the W^*-subalgebra of E generated by F. Then $F^{\#}$ is dense in $G^{\#}_{\ddot{E}}$.*

By Corollary 6.3.8.5, we may assume E is a von Neumann algebra on some Hilbert space H. By Corollary 6.3.1.12 (and Corollary 4.4.4.10), G is the closure of F in $\mathcal{L}(H)_H$. Let $x \in G^{\#}$. By Kaplanski's Theorem (Theorem 6.3.1.10), there is a filter $\mathfrak{F}$ on $\mathcal{L}(H)$ converging strongly to x such that $F^{\#} \in \mathfrak{F}$. Then $\mathfrak{F}$ converges to x in $\mathcal{L}(H)_{\mathcal{L}^1(H)}$. Since $\ddot{E}$ may be identified with a quotient space of $\mathcal{L}^1(H)$ (Corollary 4.4.4.9), the restriction of $\mathfrak{F}$ to $G^{\#}$ converges to x in $G^{\#}_{\ddot{E}}$. Hence $F^{\#}$ is dense in $G^{\#}_{\ddot{E}}$. ■

Proposition 6.3.8.8 *If $\aleph$ is the Hilbert dimension of H, then there is a faithful unital representation (K, φ) of $\mathcal{L}(H)''$ (Theorem 6.3.2.1 b)) such that the Hilbert dimension of K is at most $2^{\aleph}$ and φ is a W^*-homomorphism.*

Let (L, ψ) be the complex universal representation of $\mathcal{L}(H)$. By Corollary 5.5.2.2 and Proposition 5.5.2.18 b), the Hilbert dimension of L is at most $2^{\aleph}$. By Theorem 6.3.2.1 b), the strong closure $\overline{\psi(\mathcal{L}(H))}$ of $\mathcal{L}(H)$ in $\mathcal{L}(K)$ is isometric to $\mathcal{L}(H)''$. Thus by Proposition 6.3.4.2 c $\Rightarrow$ a, $\mathcal{L}(H)''$ may be identified with a unital W^*-subalgebra of $\mathcal{L}(K)$. The assertion now follows from Corollary 4.4.4.9 a $\Rightarrow$ b. ■

Proposition 6.3.8.9 *Let E be a W^*-subalgebra of a W^*-algebra F and $a \in \operatorname{Re} \ddot{E}$. TRhen there is a $b \in \ddot{F}_+$ such that*

$$a \leq b|E.$$

By Corollary 4.4.4.9, there is a $c \in \operatorname{Re} \ddot{F}$ such that

$$a = c|E.$$

Then

$$b := c^+$$

possesses the desired property. ■

Proposition 6.3.8.10 *Let E be a von Neumann algebra on a Hilbert space H. For every $a \in \ddot{E}_+$, there is a $u \in \mathcal{L}^1(H)_+$ such that*

$$\overset{\infty}{\ddot{u}}|E = a.$$

By Proposition 6.3.8.9, there is a $v \in \mathcal{L}^1(H)_+$ such that

$$a \leq \overset{\infty}{\widetilde{v}}|E\,.$$

Let A be a set of pairwise orthogonal elements of $H\backslash\{0\}$ such that

$$\sum_{\xi\in A} \|\xi\|^2 < \infty$$

and

$$v = \sum_{\xi\in A} \langle\cdot|\xi\rangle\xi\,.$$

We denote by (K, φ, X), the GNS–triple associated to $\overset{\infty}{\widetilde{v}}|E$. By Proposition 5.4.1.6 b), there is a $w \in \mathcal{L}(K)_+^{\#} \cap \varphi(E)^c$ such that

$$\langle x, a\rangle = \langle(\varphi x)wX|X\rangle$$

for every $x \in E$. Put

$$\zeta := w^{\frac{1}{2}}X\,.$$

Then

$$\langle x, a\rangle = \langle(\varphi x)\zeta|\zeta\rangle$$

for every $x \in E$.

In the sequel we use the notation of Example 5.4.3.3. Then

$$\zeta_\xi \in H_\xi \subset H$$

for every $\xi \in A$ and

$$\sum_{\xi\in A} \|\zeta_\xi\|^2 < \infty\,.$$

Define

$$u := \sum_{\xi\in A} \langle\cdot|\zeta_\xi\rangle\zeta_\xi \in \mathcal{L}^1(H)_+\,.$$

For $x \in E$,

$$\langle x, a\rangle = \langle(\varphi x)\zeta|\zeta\rangle = \sum_{\xi\in A} \langle x\zeta_\xi|\zeta_\xi\rangle = \langle x, \overset{\infty}{\widetilde{u}}\rangle\,,$$

so that

$$a = \overset{\infty}{\widetilde{u}}|E\,.$$

■

Corollary 6.3.8.11 *Let E be a W^*-subalgebra of a W^*-algebra F. Then for every $a \in \ddot{E}_+$, there is a $b \in \ddot{F}_+$ such that $b|E = a$.*

By Corollary 6.3.8.5, we may assume that F is a von Neumann algebra on a Hilbert space H. Then E is also a von Neumann algebra on H (Corollary 4.4.4.10). By Proposition 6.3.8.10, there is a $u \in \mathcal{L}^1(H)_+$ such that

$$a = \overset{\infty}{\widetilde{u}}|E.$$

If we put

$$b := \overset{\infty}{\widetilde{u}}|F$$

then $b \in \ddot{F}_+$ and $b|E = a$. ■

6.3.9 Finite–dimensional C^*–algebras

Throughout this subsection we use the following notation:

a) E is a finite–dimensional C^*–algebra.

b) $(F)_{F\in\mathfrak{F}}$ is a finite family in $\{\mathbb{R},\mathbb{C},\mathbb{H}\}$ and $n\in\mathbb{N}^{\mathfrak{F}}$ such that E is the C^*–direct product of the family $(F_{n(F),n(F)})_{F\in\mathfrak{F}}$ of C^*–algebras (Corollary 6.3.6.5).

c) For every $F\in\mathfrak{F}$:

c$_1$) $F_{n(F),n(F)}$ is identified with the corresponding closed ideal of E.

c$_2$) 1_F denotes the unit of $F_{n(F),n(F)}$.

c$_3$) $B(F) := \begin{cases} \{1_F\} & \text{if } F=\mathbb{K} \\ \{1_F, 1_F i\} & \text{if } \mathbb{K}=\mathbb{R} \text{ and } F=\mathbb{C} \\ \{1_F, 1_F i, 1_F j, 1_F k\} & \text{if } \mathbb{K}=\mathbb{R} \text{ and } F=\mathbb{H}. \end{cases}$

c$_4$) $\omega(F) := n(F)\operatorname{Card} B(F)$.

c$_5$) $\psi_F : F\to\mathbb{K}, \quad \alpha\longmapsto \begin{cases} \alpha & \text{if } \mathbb{K}=\mathbb{C} \\ \operatorname{re}\alpha & \text{if } \mathbb{K}=\mathbb{R}. \end{cases}$

c$_6$) For $r,s\in\mathbb{N}_{n(F)}$,

$$e_{rs}^F := [\delta_{ri}\delta_{sj}]_{i,j\in\mathbb{N}_{n(F)}} \in F_{n(F),n(F)}.$$

d) $\operatorname{Prm} E :=$ the set of minimal elements of $\operatorname{Pr} E\backslash\{0\}$.

e) $\varphi : E\to\mathbb{K}, \quad x\mapsto \sum_{F\in\mathfrak{F}}\sum_{r=1}^{n(F)} \psi_F(x_F)_{rr}$.

f) For every Hilbert right E-module H, $\widetilde{H}$ denotes the Hilbert space obtained by endowing the underlying vector space of H with the scalar product

$$H\times H\longrightarrow\mathbb{K}, \quad (\xi,\eta)\longmapsto\varphi\langle\xi|\eta\rangle$$

(Proposition 5.6.2.5 a),e),g),h), Proposition 5.6.6.17 a)).

g) G, H, I are Hilbert right E–modules.

In this subsection we present some aspects of the theory of Hilbert right C^*–modules over finite–dimensional C^*– algebras, which is not very far from the classical theory of Hilbert spaces. In particular, the main results of the theory of $\mathcal{L}^p$–spaces (presented in Section 6.1) still hold in this case.

Proposition 6.3.9.1 (8)

a) $x, y \in E \Rightarrow \varphi(xy) = \varphi(yx)$.

b) *An element* x *of* E *belongs to* $\operatorname{Prm} E$ *iff there are* $F \in \mathfrak{F}$ *and* $\xi \in F^{n(F)}$ *with*

$$x = \xi\langle\cdot|\xi\rangle\,, \quad \langle\xi|\xi\rangle = 1_F\,.$$

Moreover, F *is determined by* x .

c) $x \in \operatorname{Prm} E \Rightarrow \varphi x = 1$.

a) follows from Proposition 6.1.5.19.

b) First assume $x \in \operatorname{Prm} E$. Take $F \in \mathfrak{F}$. Since $1_F \in E^c \cap \operatorname{Pr} E$,

$$x1_F = 1_F x \in \operatorname{Pr} E$$

$$x1_F \leq x\,, \quad x1_F \leq 1_F$$

(Corollary 4.2.7.6 c $\Rightarrow$ a). Since x is a minimal element of $\operatorname{Pr} E\backslash\{0\}$, it follows

$$x1_F \in \{0, x\}\,.$$

Since

$$x = \sum_{F\in\mathfrak{F}} x1_F\,,$$

it follows that there is exactly one $F \in \mathfrak{F}$ with

$$x = x1_F \leq 1_F\,.$$

Hence x is a minimal element of $\operatorname{Pr} F_{n(F),n(F)}\backslash\{0\}$. By Proposition 5.6.5.11 b), there is a $\xi \in F^{n(F)}$ with

$$x = \xi\langle\cdot|\xi\rangle\,, \quad \langle\xi|\xi\rangle = 1_F\,.$$

The reverse implication follows from Proposition 5.6.6.18 a).

c) follows from b) and Theorem 5.6.6.1 k). ■

Proposition 6.3.9.2 *If we put*

$$\widetilde{a} : E \longrightarrow \mathbb{K}, \quad x \longmapsto \varphi(ax)$$

for every $a \in E$ *then* $\widetilde{a} \in \mathcal{L}(E, \mathbb{K})$ *for every* $a \in E$ *and the map*

$$E \longrightarrow \mathcal{L}(E, \mathbb{K}), \quad a \longmapsto \widetilde{a}$$

is linear and bijective.

It is obvious that the map is linear. Take $a \in E$ with $\widetilde{a} = 0$. Then

$$0 = \widetilde{a}(a^*) = \varphi(aa^*)$$

and this implies $a = 0$. Hence the map is injective. Since E and $\mathcal{L}(E, \mathbb{K})$ have the same finite dimension the map is bijective. ∎

Proposition 6.3.9.3 (8)

a) *H is self-dual and reflexive and the norms of H and $\widetilde{H}$ are equivalent.*

b) $\mathcal{L}_E(G, H) = \widehat{\mathcal{L}}_E(G, H)$.

c) *$\mathcal{L}_E(H)$ is a unital von Neumann algebra on $\widetilde{H}$.*

d) *$\mathcal{K}_E(H)$ is a closed ideal and a hereditary C^*-subalgebra of $\mathcal{L}_E(H)$ and so a $\mathbb{C}$-order complete C^*-algebra.*

e) *Let $p \in \Pr \mathcal{K}_E(H)$ and let A be a Fourier basis of* $\operatorname{Im} p$. *Then A is finite and*

$$p = \sum_{\xi \in A} \xi \langle \cdot | \xi \rangle .$$

a) follows from Proposition 5.6.2.5 a),d).

b) follows from a) and Proposition 5.6.2.4.

c) By Proposition 5.6.2.5 c),i), $\mathcal{L}_E(H)$ is a unital C^*-subalgebra of $\mathcal{L}(\widetilde{H})$. By b),

$$\mathcal{L}_E(H) = \widehat{\mathcal{L}}_E(H) = \{u \in \mathcal{L}(\widetilde{H}) \mid (\xi, x) \in H \times E \Longrightarrow u(\xi x) = (u\xi)x\},$$

so that $\mathcal{L}_E(H)$ is a strongly closed set of $\mathcal{L}(\widetilde{H})$. By Proposition 6.3.4.2 d $\Rightarrow$ a, $\mathcal{L}_E(H)$ is a unital von Neumann algebra on $\widetilde{H}$.

d) By Corollary 5.6.5.4 b), $\mathcal{K}_E(H)$ is a closed ideal and a hereditary C^*–subalgebra of $\mathcal{L}_E(H)$. By c) and Theorem 4.4.1.8 c), $\mathcal{L}_E(H)$ is $\mathbb{C}$–order complete so that $\mathcal{K}_E(H)$ is also $\mathbb{C}$–order complete.

e) By Proposition 5.6.5.11 a) (and Corollary 4.2.7.6 a $\Rightarrow$ c & d), the set

$$\{\xi\langle\cdot|\xi\rangle \mid \xi \in A\} \cup \{p\}$$

is commutative. Since it is contained in $\mathrm{Re}\,\mathcal{K}_E(H)$, the C^*–subalgebra J of $\mathcal{K}_E(H)$ generated by it is Gelfand (Corollary 4.1.4.2 c)). By Corollary 5.6.5.4 d), $\mathcal{K}_E(H) \subset \mathcal{K}(\widetilde{H})$ so that by Proposition 5.3.1.17, $\sigma(J)$ is discrete. Since p is the unit of J, $\sigma(J)$ is finite. It follows that A is finite. By Proposition 5.6.5.11 c),

$$p = \sum_{\xi \in A} \xi\langle\cdot|\xi\rangle\,. \qquad \blacksquare$$

Definition 6.3.9.4 (8) *A Fourier set A of H is called* ***extreme*** *if*

$$\{\langle\xi|\xi\rangle \mid \xi \in A\} \subset \mathrm{Prm}\,E\,.$$

Theorem 6.3.9.5 (8) *Let u be a compact self-normal element of $\mathcal{L}_E(H)$.*

a) *$u \in \mathcal{K}_E(H)$ and there is an extreme Fourier basis A of H and an $f \in c_0(A)$ such that*

$$u = \sum_{\xi \in A} f(\xi)\xi\langle\cdot|\xi\rangle \qquad (\textit{in } \mathcal{L}_E(H))\,,$$

$$f(A) \subset \sigma_{\mathcal{L}_E(H)}(u) \subset f(A) \cup \{0\}\,.$$

Moreover, for every $g \in \mathcal{C}\left(\sigma_{\mathcal{L}_E(H)}(u)\right)$ with

$$0 \in \sigma_{\mathcal{L}_E(H)}(u) \Longrightarrow g(0) = 0\,,$$

$$g(u) = \sum_{\xi \in A} g(f(\xi))\xi\langle\cdot|\xi\rangle \qquad (\textit{in } \mathcal{K}_E(H))\,.$$

b) *If we put $B := \{f \neq 0\}$ then*

$$H \longrightarrow H\,, \quad \eta \longmapsto \sum_{\xi \in B} \xi\langle\eta|\xi\rangle$$

(Theorem 5.6.3.13 d)) is the carrier of u in $\mathcal{L}_E(H)$.

a) By Theorem 4.1.2.12 and Proposition 5.6.2.5 c),

$$\sigma_{\mathcal{L}_E(H)}(u) = \sigma_{\mathcal{L}(\tilde{H})}(u)\,.$$

Put

$$C := \begin{cases} \sigma_{\mathcal{L}(\tilde{H})}(u)\backslash\{0\} & \text{if } H \text{ is infinite-dimensional} \\ \sigma_{\mathcal{L}(\tilde{H})}(u) & \text{if } H \text{ is finite-dimensional.} \end{cases}$$

By Theorem 3.1.5.1 a),c), C is discrete and $\mathrm{Ker}(\alpha 1 - u)$ is finite-dimensional for every $\alpha \in \mathbb{K}\backslash\{0\}$. By Theorem 4.1.4.6 a $\Rightarrow$ b, there is a family $(p_\alpha)_{\alpha\in C}$ in $\mathrm{Pr}\,\mathcal{L}_E(H)$ such that:

1) $u = \sum_{\alpha\in C} \alpha p_\alpha$;

2) $\alpha, \beta \in C \Rightarrow p_\alpha p_\beta = \delta_{\alpha\beta} p_\alpha$;

3) H is finite-dimensional $\Rightarrow \sum_{\alpha\in C} p_\alpha = 1$.

Take $\alpha \in C$ and $\xi \in \mathrm{Im}\, p_\alpha$. Then

$$p_\beta \xi = p_\beta p_\alpha \xi = \delta_{\alpha\beta} p_\alpha \xi = \delta_{\alpha\beta} \xi$$

for every $\beta \in C$. It follows successively

$$u\xi = \sum_{\beta\in C} \beta p_\beta \xi = \alpha\xi\,,$$

$$(\alpha 1 - u)\xi = 0\,,$$

$$\xi \in \mathrm{Ker}(\alpha 1 - u)\,,$$

$$\mathrm{Im}\, p_\alpha \subset \mathrm{Ker}(\alpha 1 - u)\,.$$

Hence $\mathrm{Im}\, p_\alpha$ is a finite-dimensional Hilbert right E-module (Corollary 4.1.2.6). By Proposition 5.6.3.16 (and Proposition 5.6.3.13 a)), there is a finite extreme Fourier basis A_α of $\mathrm{Im}\, p_\alpha$. By Proposition 5.6.5.11 c),

$$p_\alpha = \sum_{\xi\in A_\alpha} \xi\langle\cdot|\xi\rangle\,.$$

Take distinct elements α, β of C and $\xi \in A_\alpha, \eta \in A_\beta$. Then

$$\langle \xi | \eta \rangle = \langle p_\alpha \xi | p_\beta \eta \rangle = \langle \xi | p_\alpha p_\beta \eta \rangle = 0$$

so that $\bigcup_{\alpha \in C} A_\alpha$ is an extreme Fourier set of H. By Proposition 5.6.3.16, there is an extreme Fourier basis A of H containing $\bigcup_{\alpha \in C} A_\alpha$.

If we define

$$f : A \longrightarrow \mathbb{R}, \quad \beta \longmapsto \begin{cases} \alpha & \text{if } \beta \in A_\alpha \text{ for some } \alpha \in C \\ 0 & \text{if } \beta \in A \setminus \bigcup_{\alpha \in C} A_\alpha \end{cases}$$

then

$$f(A) \subset \sigma_{\mathcal{L}_E(H)}(u) \subset f(A) \cup \{0\}$$

and by Proposition 5.6.3.14 b) and Theorem 4.1.4.6 a $\Rightarrow$ e (and Proposition 6.3.9.3 a)),

$$u = \sum_{\alpha \in C} \alpha p_\alpha = \sum_{\alpha \in C} \alpha \sum_{\xi \in A_\alpha} \xi \langle \cdot | \xi \rangle = \sum_{\xi \in A} f(\xi) \xi \langle \cdot | \xi \rangle \qquad (\text{in } \mathcal{K}_E(H)),$$

$$g(u) = \sum_{\xi \in A} g(f(\xi)) \xi \langle \cdot | \xi \rangle \qquad (\text{in } \mathcal{L}_E(H)).$$

b) follows from a) and Proposition 5.6.3.15 b). ■

Proposition 6.3.9.6 (8) *Let $u \in \mathcal{L}_E(G,H)$, A an extreme Fourier basis of G, and $f \in c_0(A)_+$ such that*

$$u^* \circ u = \sum_{\xi \in A} f(\xi) \xi \langle \cdot | \xi \rangle \qquad (\text{in } \mathcal{K}_E(G)).$$

Put

$$B := \{ f \neq 0 \}, \qquad g := \sqrt{f|B} \in c_0(B)_+,$$

and

$$\widetilde{\xi} := \frac{1}{g(\xi)} u \xi$$

for every $\xi \in B$.

a) $\xi, \eta \in B \Rightarrow \langle \widetilde{\xi} | \widetilde{\eta} \rangle = \langle \xi | \eta \rangle$.

b) $u = \sum_{\xi \in B} g(\xi) \widetilde{\xi} \langle \cdot | \xi \rangle \qquad (\text{in } \mathcal{K}_E(G,H)),$

$$\|u\| = \|g\|_\infty.$$

a) $\langle\widetilde{\xi}|\widetilde{\eta}\rangle = \frac{1}{g(\xi)g(\eta)}\langle u\xi|u\eta\rangle = \frac{1}{g(\xi)g(\eta)}\langle u^*u\xi|\eta\rangle =$

$$= \frac{1}{g(\xi)g(\eta)}\langle f(\xi)\xi|\eta\rangle = \langle\xi|\eta\rangle\,.$$

b) For $\xi \in A\backslash B$,

$$\langle u\xi|u\xi\rangle = \langle u^*u\xi|\xi\rangle = \langle f(\xi)\xi|\xi\rangle = 0$$

so that

$$u\xi = 0\,.$$

Take $\eta \in G$. By Theorem 5.6.3.13 d), f),

$$\eta = \sum_{\xi\in A}\xi\langle\eta|\xi\rangle$$

so that

$$u\eta = \sum_{\xi\in A}(u\xi)\langle\eta|\xi\rangle = \sum_{\xi\in B} g(\xi)\widetilde{\xi}\langle\eta|\xi\rangle\,.$$

By a), Proposition 5.6.3.14 b), and Proposition 6.3.9.3 a),b),

$$u = \sum_{\xi\in B} g(\xi)\widetilde{\xi}\langle\cdot|\xi\rangle \qquad (\text{in } \mathcal{K}_E(G,H))$$

and

$$\|u\| = \|g\|_\infty\,. \qquad \blacksquare$$

Theorem 6.3.9.7 (8) *Let u be a compact element of $\mathcal{L}_E(G,H)$.*

a) *$u \in \mathcal{K}_E(G,H)$ and there are an extreme Fourier set A of G, a family $(\widetilde{\xi})_{\xi\in A}$ in H such that*

$$\langle\xi|\eta\rangle = \langle\widetilde{\xi}|\widetilde{\eta}\rangle$$

for all $\xi,\eta \in A$, and an $f \in c_0(A)_+$ such that

$$u = \sum_{\xi\in A} f(\xi)\widetilde{\xi}\langle\cdot|\xi\rangle\rangle \qquad (\text{in } \mathcal{K}_E(G,H))\,.$$

Moreover, for every such representation of u,

$$\|u\| = \|f\|_\infty\,.$$

b) *Take* $g \in \ell^\infty(A)$ *and put*

$$v : H \longrightarrow G\,, \quad \eta \longmapsto \sum_{\xi \in A}^{\ddot{H}} g(\xi)\xi\langle\eta|\widetilde{\xi}\rangle$$

(Proposition 5.6.3.8 a), Theorem 5.6.3.12 b), Proposition 6.3.9.3 a)). Then

$$v \circ u = \sum_{\xi \in A} f(\xi)g(\xi)\xi\langle\cdot|\xi\rangle \qquad (in\ \mathcal{K}_E(G))\,.$$

a) By Theorem 6.3.9.5 a), there is an extreme Fourier basis B of G and a $g \in c_0(B)$ such that

$$u^* \circ u = \sum_{\xi \in B} g(\xi)\xi\langle\cdot|\xi\rangle \qquad (\text{in } \mathcal{K}_E(G))\,,$$

$$g(B) \subset \sigma_{\mathcal{L}_B(G)}(u^* \circ u) \cup \{0\} \subset \mathbb{R}_+$$

(Corollary 5.6.1.12). Now the assertion follows from Proposition 6.3.9.6.

b) Take $\zeta \in G$. Then

$$\langle u\zeta|\widetilde{\xi}\rangle = f(\xi)\langle\zeta|\xi\rangle$$

for every $\xi \in A$ so that

$$(v \circ u)\zeta = v(u\zeta) = \sum_{\xi \in A}^{\ddot{H}} g(\xi)\xi\langle u\zeta|\widetilde{\xi}\rangle = \sum^{\ddot{H}} f(\xi)g(\xi)\xi\langle\zeta|\xi\rangle\,.$$

By Proposition 5.6.3.14 (and Proposition 6.3.9.3 a)),

$$v \circ u = \sum_{\xi \in A} f(\xi)g(\xi)\xi\langle\cdot|\xi\rangle \qquad (\text{in } \mathcal{K}_E(G))\,. \qquad \blacksquare$$

Corollary 6.3.9.8 (8) $\mathcal{K}_E(G,H) = \mathcal{L}_E(G,H) \cap \mathcal{K}(G,H)$.

The inclusion from the left to the right follows from Corollary 5.6.5.4 d). The reverse inclusion follows from Theorem 6.3.9.7 a). $\blacksquare$

Definition 6.3.9.9 (8) *Let* $\xi \in H$ *such that* $\langle\xi|\xi\rangle \in \operatorname{Prm} E$. *By Proposition 6.3.9.1 b), there is a unique* $F \in \mathfrak{F}$ *such that*

$$\langle\xi|\xi\rangle \leq 1_F\,.$$

We define

$$\omega(\xi) := \omega(F)\,.$$

Theorem 6.3.9.10 (8) *Let $\xi \in H$ such that $\langle \xi|\xi\rangle \in \operatorname{Prm} E$.*

a) *There is an $F \in \mathfrak{F}$ and a $\zeta \in F^{n(F)}$ such that*

$$\langle \zeta|\zeta\rangle = 1_F\,, \quad \langle \xi|\xi\rangle = \zeta\langle\cdot|\zeta\rangle \in \operatorname{Pr} F_{n(F),n(F)}\,.$$

Put

$$x_r := \zeta\langle\cdot|e_r\rangle \in F_{n(F),n(F)}\,,$$

$$\xi_r := \xi x_r$$

for every $r \in \mathbb{N}_{n(F)}$ and

$$A_\xi := \{\xi_r y | r \in \mathbb{N}_{n(F)}, y \in B(F)\}\,.$$

b) $r, s \in \mathbb{N}_{n(F)} \Rightarrow x_r x_s^* = \delta_{rs}\langle \xi|\xi\rangle$.

c) $r, s \in \mathbb{N}_{n(F)}, y, z \in B(F) \Rightarrow \varphi\langle \xi_r y|\xi_s z\rangle = \delta_{rs}\delta_{yz}$.

d) $\operatorname{Card} A_\xi = \omega(\xi)$.

e) $\eta \in A_\xi^\perp$ *(in $\widetilde{H}$)* $\Rightarrow \langle \xi|\eta\rangle = 0$.

f) *For every $u \in \mathcal{L}_E(H)$,*

$$\varphi\left(\sum_{\eta\in A_\xi}\langle u\eta|\eta\rangle\right) = \omega(\xi)\varphi\langle u\xi|\xi\rangle\,.$$

a) follows from Proposition 6.3.9.1 b).

b) By Proposition 5.6.5.2 a),f),

$$x_r x_s^* = (\zeta\langle\cdot|e_r\rangle)\circ(e_s\langle\cdot|\zeta\rangle) =$$

$$(\zeta\langle e_s|e_r\rangle)\langle\cdot|\zeta\rangle = \delta_{rs}\langle\xi|\xi\rangle\,.$$

c) By b),

$$\varphi\langle\xi_r|\xi_s\rangle = \varphi(x_s^*\langle\xi|\xi\rangle x_r) = \varphi(\langle\xi|\xi\rangle x_r x_s^*) =$$

$$= \delta_{rs}\varphi(\langle\xi|\xi\rangle\langle\xi|\xi\rangle) = \delta_{rs}$$

(Proposition 6.3.9.1 a),c)). It follows

$$\varphi\langle\xi_r y|\xi_s z\rangle = \varphi(z^*\langle\xi_r|\xi_s\rangle y) =$$

$$= \varphi(\langle\xi_r|\xi_s\rangle yz^*) = \delta_{rs}\varphi(yz^*) = \delta_{rs}\delta_{yz}$$

(Proposition 6.3.9.1 a)).

d) follows from c).

e) For $r \in \mathbb{N}_{n(F)}$ and $y \in B(F)$,

$$0 = \varphi\langle\xi x_r y|\eta\rangle = \varphi(\langle\xi|\eta\rangle x_r y) = \varphi(\langle\xi|\eta\rangle\circ(\zeta\langle\cdot|e_r\rangle)y) =$$

$$= \varphi((\langle\xi|\eta\rangle\zeta)\langle\cdot|e_r\rangle)y) = \psi((\langle\xi|\eta\rangle\zeta)_r y)$$

(Proposition 5.6.5.2 d), Theorem 5.6.6.1 k)). It follows successively

$$(\langle\xi|\eta\rangle\zeta)_r = 0\,,$$

$$\langle\xi|\eta\rangle\zeta = 0\,,$$

$$\langle\xi|\eta\rangle = \langle\xi\langle\xi|\xi\rangle|\eta\rangle = \langle\xi|\eta\rangle\langle\xi|\xi\rangle = \langle\xi|\eta\rangle\circ(\zeta\langle\cdot|\zeta\rangle) =$$

$$= (\langle\xi|\eta\rangle\zeta)\langle\cdot|\zeta\rangle = 0$$

(Proposition 5.6.3.9 a $\Rightarrow$ b, Proposition 5.6.5.2 d)).

f) For $r \in \mathbb{N}_{n(F)}$ and $y \in B(F)$, by b),

$$\varphi\langle u(\xi_r y)|\xi_r y\rangle = \varphi(y^* x_r^*\langle u\xi|\xi\rangle x_r y) = \varphi(\langle u\xi|\xi\rangle x_r x_r^*) =$$

$$= \varphi(\langle u\xi|\xi\rangle\langle\xi|\xi\rangle) = \varphi(\langle u(\xi\langle\xi|\xi\rangle)|\xi\rangle = \varphi\langle u\xi|\xi\rangle$$

(Proposition 6.3.9.1 a), Proposition 6.3.9.3 b), Proposition 5.6.3.9 a $\Rightarrow$ b). Hence by d),

$$\varphi\left(\sum_{\eta\in A_\xi}\langle u\eta|\eta\rangle\right) = \omega(\xi)\varphi\langle u\xi|\xi\rangle\,. \qquad \blacksquare$$

Definition 6.3.9.11 (8) *For* $p \in \{0\}\cup[1,\infty]$ *define*

$$\mathcal{L}_E^p(G,H) = \mathcal{L}_E(G,H)\cap\mathcal{L}^p(\widetilde{G},\widetilde{H})\,,$$

$$\mathcal{L}_E^p(H) = \mathcal{L}_E^p(H,H)\,.$$

$\mathcal{L}_E^p(H)$ is an involutive unital $\mathcal{L}_E(H)$–module (Corollary 6.1.3.13 b), Proposition 5.6.2.5 b)).

Corollary 6.3.9.12 (8) *Let A be an extreme Fourier basis of H.*

a) $\bigcup_{\xi\in A} A_\xi$ *is an orthonormal basis of $\widetilde{H}$, where A_ξ was defined in Theorem 6.3.9.10 a).*

b) $u \in \mathcal{L}_E^1(H) \Rightarrow tr\, u = \varphi\left(\sum_{\xi\in A} \omega(\xi)\langle u\xi|\xi\rangle\right)$.

c) $\xi \in A$, $\eta \in H \Rightarrow tr(\eta\langle\cdot|\xi\rangle) = \omega(\xi)\varphi\langle\eta|\xi\rangle$.

a) By Theorem 6.3.9.10 c), $\bigcup_{\xi\in A} A_\xi$ is an orthonormal set of $\widetilde{H}$. Take $\eta \in \left(\bigcup_{\xi\in A} A_\xi\right)^\perp$. By Theorem 6.3.9.10 e),

$$\langle\eta|\xi\rangle = 0$$

for every $\xi \in A$. By Theorem 5.6.3.13 b), $\eta = 0$. Hence $\bigcup_{\xi\in A} A_\xi$ is an orthonormal basis of $\widetilde{H}$.

b) By a) and Theorem 6.3.9.10 f),

$$tr\, u = \sum_{\xi\in A}\sum_{\eta\in A_\xi} \varphi\langle u\eta|\eta\rangle = \sum_{\xi\in A} \omega(\xi)\varphi\langle u\xi|\xi\rangle .$$

c) By b),

$$tr\,(\eta\langle\cdot|\xi\rangle) = \varphi\left(\sum_{\zeta\in A} \omega(\zeta)\langle\eta\langle\zeta|\xi\rangle|\xi\rangle\right) = \omega(\xi)\varphi\langle\eta\langle\xi|\xi\rangle|\xi\rangle =$$

$$= \omega(\xi)\varphi(\langle\eta|\xi\rangle\langle\xi|\xi\rangle) = \omega(\xi)\varphi\langle\eta|\xi\rangle$$

(Proposition 6.3.9.1 a), Proposition 5.6.3.9 a $\Rightarrow$ c). ■

Theorem 6.3.9.13 (8) *Let $\mathfrak{A}$ be the set of Fourier sets A of H such that*

$$\{\langle\xi|\xi\rangle \mid \xi \in A\} \subset \{e_{11}^F \mid F \in \mathfrak{F}\} .$$

a) *Every maximal element of $\mathfrak{A}$ $\{\mathfrak{A}$ ordered by inclusion$\}$ is a Fourier basis of H.*

b) *There is a unique family* $(\aleph_F)_{F\in\mathfrak{F}}$ *of cardinal numbers such that* H *is isomorphic to*

$$\bigoplus_{F\in\mathfrak{F}} \ell^2(\aleph_F, e_{11}^F E)\,.$$

c) *If we identify* H *with*

$$\bigoplus_{F\in\mathfrak{F}} \ell^2(\aleph_F, e_{11}^F E)$$

using the isomorphism of b) then

$$\{e_{11}^F e_t^{\aleph_F} y \mid F\in\mathfrak{F}\,,\ t\in\aleph_F\,,\ y\in B(F)\}$$

is an orthonormal basis of $\widetilde{H}$.

Let A be a maximal element of $\mathfrak{A}$. For every $F\in\mathfrak{F}$ put

$$A_F := \{\xi\in A \mid \langle\xi|\xi\rangle = e_{11}^F\}\,.$$

By Proposition 5.6.4.12 b), A_F is a Fourier basis of $H1_F$ for every F. Since H is isomorphic to $\bigoplus_{F\in\mathfrak{F}} H1_F$ (Proposition 5.6.4.13 a),b)) and $A = \bigcup_{F\in\mathfrak{F}} A_F$, it follows that A is a Fourier basis of H.

b) follows from Proposition 5.6.4.12 c) and Proposition 5.6.4.10 c), and c) follows from Corollary 6.3.9.12 a). ■

Remark. The family $(\aleph_F)_{F\in\mathfrak{F}}$ may be considered as a kind of Hilbert dimension of H.

Theorem 6.3.9.14 (8) *Let* A *be an extreme Fourier set of* G, $(\widetilde{\xi})_{\xi\in A}$ *a family in* H *such that*

$$\langle\widetilde{\xi}|\widetilde{\eta}\rangle = \langle\xi|\eta\rangle$$

for all $\xi,\eta\in A$, $f\in c_0(A)$, $p\in[1,\infty[$, *and*

$$u := \sum_{\xi\in A} f(\xi)\widetilde{\xi}\langle\cdot|\xi\rangle \qquad (\text{in } \mathcal{K}_E(G,H))\,.$$

Then $u\in\mathcal{L}_E^p(G,H)$ *iff* $f\in\ell^p(A)$ *and in this case*

$$\|u\|_p^p = \sum_{\xi\in A}\omega(\xi)\,|f(\xi)|^p\,,$$

$$u = \sum_{\xi\in A} f(\xi)\widetilde{\xi}\langle\cdot|\xi\rangle \qquad (\text{in } \mathcal{L}_E^p(G,H))\,.$$

By Proposition 6.3.9.1 b), for every $\xi \in A$, there are $F \in \mathfrak{F}$ and $\eta \in F^{n(F)}$ with

$$\langle \xi | \xi \rangle = \eta \langle \cdot | \eta \rangle , \quad \langle \eta | \eta \rangle = 1_F$$

and the assertion follows from Proposition 5.6.6.19 b),d),e) and Corollary 6.1.2.12. ■

Definition 6.3.9.15 (8) *We denote for every* $u \in \mathcal{L}(G,H)$,

$$\vec{u} : G \longrightarrow H , \quad \xi \longmapsto \sum_{F \in \mathfrak{F}} \frac{1}{\omega(F)} \sum_{y \in B(F)} \sum_{s,t=1}^{n(F)} u(\xi e_{st}^F y) e_{ts}^F y^* .$$

Theorem 6.3.9.16 (8) *Let* $p \in \{0\} \cup [1,\infty]$.

a) $\vec{u} \in \mathcal{L}_E^p(G,H)$ *for every* $u \in \mathcal{L}^p(\widetilde{G},\widetilde{H})$ *and the map*

$$\mathcal{L}^p(\widetilde{G},\widetilde{H}) \longrightarrow \mathcal{L}^p(\widetilde{G},\widetilde{H}) , \quad u \longmapsto \vec{u}$$

is a projection of $\mathcal{L}^p(\widetilde{G},\widetilde{H})$ *onto* $\mathcal{L}_E^p(G,H)$.

b) $u \in \mathcal{L}(G,H)$, $v \in \mathcal{L}_E(I,G) \Rightarrow \overrightarrow{u \circ v} = \vec{u} \circ v$.

c) *If* $u \in \mathcal{L}(H)$, $\xi \in H$, $F \in \mathfrak{F}$, *and* $r \in \mathbb{N}_{n(F)}$ *such that*

$$\langle \xi | \xi \rangle = e_{rr}^F$$

then

$$\varphi(\langle \vec{u}\xi | \xi \rangle) = \frac{1}{\omega(\xi)} \sum_{\eta \in A_\xi} \varphi \langle u\eta | \eta \rangle ,$$

where A_ξ *was defined in Theorem 6.3.9.10 a).*

d) $u \in \mathcal{L}^1(\widetilde{H}) \Rightarrow tr\, u = tr\, \vec{u}$.

a) Take $\xi \in G$ and $x \in E$. For $F \in \mathfrak{F}$, $y \in B(F)$, and $s', t' \in \mathbb{N}_{n(F)}$,

$$\sum_{s,t=1}^{n(F)} u(\xi e_{s't'}^F e_{st}^F y) e_{ts}^F = \sum_{s,t=1}^{n(F)} \delta_{t's} u(\xi e_{s't}^F y) e_{ts}^F =$$

$$= \sum_{t=1}^{n(F)} u(\xi e_{s't}^F y) e_{tt'}^F = \sum_{s,t=1}^{n(F)} u(\xi e_{st}^F y) \delta_{ss'} e_{tt'}^F =$$

$$= \sum_{s,t=1}^{n(F)} u(\xi e_{st}^F y) e_{ts}^F e_{s't'}^F .$$

Since

$$\sum_{y \in B(F)} \sum_{s,t=1}^{n(F)} u(\xi e_{st}^F iy) e_{ts}^F y^* =$$

$$= \sum_{y \in B(F)} \sum_{s,t=1}^{n(F)} u(\xi e_{st}^F y) e_{ts}^F y^* i$$

for $F \neq \mathbb{K}$ it follows

$$\vec{u}(\xi x) = (\vec{u}\xi)x .$$

By Proposition 6.3.9.3 b),

$$\vec{u} \in \widehat{\mathcal{L}}_E(G,H) = \mathcal{L}_E(G,H) .$$

Since the construction of $\vec{u}$ from u involves a finite number of steps and every step is linear and continuous, it follows

$$\vec{u} \in \mathcal{L}_E^p(G,H)$$

and the map

$$\mathcal{L}^p(\widetilde{G},\widetilde{H}) \longrightarrow \mathcal{L}_E^p(G,H) , \quad u \longmapsto \vec{u}$$

is linear and continuous.

Now assume $u \in \mathcal{L}_E^p(G,H)$ and take $\xi \in G$. Then for every $F \in \mathfrak{F}$ and $y \in B(F)$,

$$\sum_{s,t=1}^{n(F)} u(\xi e_{st}^F y) e_{ts}^F y^* = \sum_{s,t=1}^{n(F)} (u\xi) e_{st}^F e_{ts}^F =$$

$$= n(F) \sum_{s=1}^{n(F)} (u\xi) e_{ss}^F = n(F)(u\xi) 1_F$$

so that

$$\sum_{y \in B(F)} \sum_{s,t=1}^{n(F)} u(\xi e_{st}^F y) e_{ts}^F y^* = \omega(F)(u\xi) 1_F .$$

It follows

$$\vec{u}\xi = u\xi\,,$$

$$\vec{u} = u\,.$$

Hence the map

$$\mathcal{L}^p(\widetilde{G},\widetilde{H}) \longrightarrow \mathcal{L}^p(\widetilde{G},\widetilde{H})\,, \quad u \longmapsto \vec{u}$$

is a projection of $\mathcal{L}^p(\widetilde{G}.\widetilde{H})$ onto $\mathcal{L}^p_E(G,H)$.

b) For $\xi \in I$,

$$\overrightarrow{u\circ v}\xi = \sum_{F\in\mathfrak{F}} \frac{1}{\omega(F)} \sum_{y\in B(F)} \sum_{s,t=1}^{n(F)} uv(\xi e^F_{st}y)e^F_{ts}y^* =$$

$$= \sum_{F\in\mathfrak{F}} \frac{1}{\omega(F)} \sum_{y\in B(F)} \sum_{s,t=1}^{n(F)} u((v\xi)e^F_{st}y)e^F_{ts}y^* = \vec{u}(v\xi)$$

(Proposition 6.3.9.3 b)) so that

$$\overrightarrow{u\circ v} = \vec{u}\circ v\,.$$

c) By Proposition 5.6.3.9 a $\Rightarrow$ b,

$$\xi = \xi\langle\xi|\xi\rangle = \xi e^F_{rr}$$

so that

$$A_\xi = \{\xi e^F_{rt}y \mid t \in \mathbb{N}_{n(F)}, y \in B(F)\}\,.$$

By Proposition 6.3.9.1 a),

$$\sum_{\eta\in A_\xi} \varphi\langle u\eta|\eta\rangle = \sum_{y\in B(F)} \sum_{t=1}^{n(F)} \varphi\langle u(\xi e^F_{rt}y) \mid \xi e^F_{rt}y\rangle =$$

$$= \sum_{y\in B(F)} \sum_{t=1}^{n(F)} \varphi(e^F_{tr}y^*\langle u(\xi e^F_{rt}y) \mid \xi\rangle) =$$

$$= \sum_{y\in B(F)} \sum_{t=1}^{n(F)} \varphi(\langle u(\xi e^F_{rt}y) \mid \xi\rangle e^F_{tr}y^*) =$$

$$= \varphi \left(\sum_{y \in B(F)} \sum_{t=1}^{n(F)} \langle u(\xi e_{rt}^F y) e_{tr}^F y^* \mid \xi \rangle \right) ,$$

$$\varphi \langle \vec{u} \xi | \xi \rangle = \sum_{F' \in \mathfrak{F}} \frac{1}{\omega(F')} \varphi \left(\sum_{y \in B(F')} \sum_{s,t=1}^{n(F')} \langle u(\xi e_{st}^{F'} y) e_{ts}^{F'} y^* \mid \xi \rangle \right) =$$

$$= \sum_{F' \in \mathfrak{F}} \frac{1}{\omega(F')} \varphi \left(\sum_{y \in B(F')} \sum_{s,t=1}^{n(F')} \left\langle u \left(\xi e_{rr}^F e_{st}^{F'} y \right) e_{ts}^{F'} y^* \mid \xi \right\rangle \right) =$$

$$= \sum_{F' \in \mathfrak{F}} \frac{1}{\omega(F')} \varphi \left(\sum_{y \in B(F')} \sum_{s,t=1}^{n(F')} \delta_{FF'} \delta_{rs} \left\langle u \left(\xi e_{rt}^F y \right) e_{ts}^F y^* \mid \xi \right\rangle \right) =$$

$$= \frac{1}{\omega(F)} \varphi \left(\sum_{y \in B(F)} \sum_{t=1}^{n(F)} \left\langle u \left(\xi e_{rt}^F y \right) e_{tr}^F y^* \mid \xi \right\rangle \right) =$$

$$= \frac{1}{\omega(\xi)} \sum_{\eta \in A_\xi} \varphi \langle u\eta | \eta \rangle .$$

d) By Theorem 6.3.9.13 a), there is a Fourier basis A of H such that

$$\{ \langle \xi | \xi \rangle \mid \xi \in A \} \subset \{ e_{11}^F \mid F \in \mathfrak{F} \} .$$

By a),c), and Corollary 6.3.9.12 a),b),

$$tr\, \vec{u} = \varphi \left(\sum_{\xi \in A} \omega(\xi) \langle \vec{u} \xi | \xi \rangle \right) = \sum_{\xi \in A} \sum_{\eta \in A_\xi} \varphi \langle u\eta | \eta \rangle = tr\, u . \qquad \blacksquare$$

Proposition 6.3.9.17 (8) *For every* $u \in \mathcal{L}_E(G,H)$,

$$\|u\|^2 = \sup \{ \varphi \langle u\xi | u\xi \rangle \mid \xi \in G, \exists\, F \in \mathfrak{F},\ \exists\, r \in \mathbb{N}_{n(F)}, \langle \xi | \xi \rangle = e_{rr}^F \} .$$

Put

$$\alpha := \sup \{ \varphi \langle u\xi | u\xi \rangle \mid \xi \in G, \exists\, F \in \mathfrak{F},\ \exists\, r \in \mathbb{N}_{n(F)}, \langle \xi | \xi \rangle = e_{rr}^F \} .$$

Take $\eta \in G$. For all $F \in \mathfrak{F}$ and $r \in \mathbb{N}_{n(F)}$ put

$$\eta_r^F := \begin{cases} (\langle \eta|\eta\rangle_F)_{rr}^{-\frac{1}{2}} \eta e_{rr}^F & \text{if } (\langle \eta|\eta\rangle_F)_{rr} \neq 0 \\ 0 & \text{if } (\langle \eta|\eta\rangle_F)_{rr} = 0 \end{cases}$$

(Theorem 5.6.6.1 g)). If $F \in \mathfrak{F}$ and $r \in \mathbb{N}_{n(F)}$ such that

$$(\langle \eta|\eta\rangle_F)_{rr} \neq 0$$

then

$$\langle \eta_r^F|\eta_r^F\rangle = (\langle \eta|\eta\rangle_F)_{rr}^{-1} e_{rr}^F \langle \eta|\eta\rangle e_{rr}^F =$$

$$= (\langle \eta|\eta\rangle_F)_{rr}^{-1} (\langle \eta|\eta\rangle_F)_{rr} e_{rr}^F = e_{rr}^F$$

so that

$$(\langle u\eta|u\eta\rangle_F)_{rr} = \varphi(e_{rr}^F \langle u\eta|u\eta\rangle e_{rr}^F) =$$

$$= \varphi\langle u(\eta e_{rr}^F) \mid u(\eta e_{rr}^F)\rangle =$$

$$= (\langle \eta|\eta\rangle_F)_{rr} \varphi\langle u\eta_r^F \mid u\eta_r^F\rangle \leq \alpha(\langle \eta|\eta\rangle_F)_{rr} .$$

Hence

$$\varphi\langle u\eta|u\eta\rangle = \sum_{F\in\mathfrak{F}} \sum_{r=1}^{n(F)} (\langle u\eta|u\eta\rangle_F)_{rr} \leq$$

$$\leq \alpha \sum_{F\in\mathfrak{F}} \sum_{r=1}^{n(F)} (\langle \eta|\eta\rangle_F)_{rr} = \alpha\varphi\langle \eta|\eta\rangle ,$$

$$\|u\|^2 \leq \alpha$$

(Proposition 6.3.9.3 c)). The reverse inequality is trivial. ■

Theorem 6.3.9.18 (8) *Let $p \in \{0\} \cup [1, \infty]$, q the conjugate exponent of p, and*

$$\theta : \mathcal{L}_E^q(H, G) \longrightarrow (\mathcal{L}_E^p(G, H))', \quad u \longmapsto \overset{p}{\widetilde{u}} \mid \mathcal{L}_E^p(G, H) .$$

a) *θ preserves the norms.*

b) *If $p \neq \infty$ then θ is an isometry.*

c) *Assume* $p \neq \infty$, *denote by* i *the inclusion map*

$$\mathcal{L}^p_E(G,H) \longrightarrow \mathcal{L}^p(\widetilde{G},\widetilde{H}),$$

and identify $\mathcal{L}^q_E(H,G)$ *and* $\mathcal{L}^q(\widetilde{H},\widetilde{G})$ *with* $(\mathcal{L}^p_E(G,H))'$ *and* $(\mathcal{L}^p(\widetilde{G},\widetilde{H}))'$, *respectively (b) and Theorem 6.1.6.6). Then*

$$i'u = \vec{u}$$

for every $u \in \mathcal{L}^q(\widetilde{H},\widetilde{G})$.

d) *If* $G = H$ *then* θ *is an involutive homomorphism of* $\mathcal{L}_E(H)$*–modules.*

e) *If* $G = H$ *and if we identify* $\mathcal{L}^1_E(H)$ *and* $\mathcal{L}^1(\widetilde{H})$ *with the preduals of* $\mathcal{L}_E(H)$ *and* $\mathcal{L}(\widetilde{H})$, *respectively (b),d), and Corollary 6.1.7.14 b)) then*

$$\mathcal{L}^1(\widetilde{H}) \longrightarrow \mathcal{L}^1_E(H), \quad u \longmapsto \vec{u}$$

is the pretranspose of the inclusion map $\mathcal{L}_E(H) \longrightarrow \mathcal{L}(\widetilde{H})$.

a) Case 1 $q \neq \infty$

Take $u \in \mathcal{L}^q_E(H,G)$. By Theorem 6.3.9.7 a) and Theorem 6.3.9.14, there are an extreme Fourier set A of H, a family $(\widetilde{\xi})_{\xi\in A}$ in G such that

$$\langle \xi|\eta\rangle = \langle \widetilde{\xi}|\widetilde{\eta}\rangle$$

for all $\xi,\eta \in A$, and an $f \in \ell^q(A)$ such that

$$u = \sum_{\xi\in A} f(\xi)\widetilde{\xi}\langle\cdot|\xi\rangle \quad (\text{in } \mathcal{L}^q_E(H,G)),$$

$$\|u\|_q^q = \sum_{\xi\in A} \omega(\xi)|f(\xi)|^q.$$

Take $B \in \mathfrak{P}_f(A)$ and put

$$v := \sum_{\substack{\xi\in B\\ f(\xi)\neq 0}} \frac{|f(\xi)|^q}{f(\xi)}\xi\langle\cdot|\widetilde{\xi}\rangle.$$

If $q \neq 1$ then by Theorem 6.3.9.14,

$$\|v\|_p^p = \sum_{\xi\in B} \omega(\xi)|f(\xi)|^{pq-p} = \sum_{\xi\in B} \omega(\xi)|f(\xi)|^q \leq \|u\|_q^q.$$

If $q = 1$ then by Theorem 6.3.9.7 a), $\|v\|_p \leq 1$. By Theorem 6.3.9.7 b),

$$u \circ v = \sum_{\xi \in B} |f(\xi)|^q \xi \langle \cdot | \xi \rangle$$

so that by Corollary 6.3.9.12 b) (and Proposition 6.3.9.1 c)),

$$\overset{p}{\widetilde{u}}(v) = tr\,(u \circ v) = \sum_{\xi \in B} \omega(\xi) |f(\xi)|^q .$$

Since B is arbitrary,

$$\|\overset{p}{\widetilde{u}} | \mathcal{L}_E^p(G,H)\|\, \|u\|_q^{\frac{q}{p}} \geq |\overset{p}{\widetilde{u}}(v)| \geq \|u\|_q^q ,$$

$$\|u\|_q = \|\overset{p}{\widetilde{u}}\| \geq \|\overset{p}{\widetilde{u}} | \mathcal{L}_E^p(G,H)\| \geq \|u\|_q^{q-\frac{q}{p}} = \|u\|_q$$

if $q \neq 1$ and

$$\|u\|_q = \|\overset{p}{\widetilde{u}}\| \geq \|\overset{p}{\widetilde{u}} | \mathcal{L}_E^p(G,H)\| \geq |\overset{p}{\widetilde{u}}(v)| \geq \|u\|_q$$

if $q = 1$ (Theorem 6.1.6.6). Hence

$$\|\overset{p}{\widetilde{u}} | \mathcal{L}_E^p(G,H)\| = \|u\|_q$$

and θ preserves the norms.

Case 2 $\quad q = \infty$

Let $u \in \mathcal{L}_E(H,G)$. Take $\xi \in H$, $F \in \mathfrak{F}$, and $r \in \mathbb{N}_{n(F)}$ with

$$\langle \xi | \xi \rangle = e_{rr}^F$$

and put

$$v := \xi \langle \cdot | u\xi \rangle \in \mathcal{L}^1(G,H) .$$

For $w \in \mathcal{K}_E(H,G)^{\#}$,

$$|\overset{0}{\widetilde{v}}(w)| = |tr\,(v \circ w)| = |tr\,(\xi \langle \cdot | w^* u \xi \rangle)| = |tr((w^* u \xi) \langle \cdot | \xi \rangle| =$$

$$= |\omega(\xi) \varphi \langle w^* u \xi | \xi \rangle| = \omega(\xi) |\varphi \langle u\xi | w\xi \rangle| =$$

$$= \omega(\xi) |\varphi(\langle \xi | \xi \rangle \langle u\xi | w\xi \rangle \langle \xi | \xi \rangle)| = w(\xi) |\varphi(\langle u\xi | w\xi \rangle_F)_{rr}| \leq$$

$$\leq \omega(\xi)\|\langle u\xi|w\xi\rangle_F\| \leq \omega(\xi)\|\langle u\xi|w\xi\rangle\| \leq \omega(\xi)\|u\xi\|\,\|w\xi\| \leq \omega(\xi)\|u\|$$

(Corollary 6.1.5.3, Proposition 5.6.5.2 a),e), Corollary 6.3.9.12 c), Proposition 5.6.3.9 a $\Rightarrow$ b, Proposition 6.3.9.3 b), Theorem 5.6.6.1 a)). By Case 1,

$$\|v\|_1 = \|\overset{0}{\widetilde{v}}|\mathcal{K}_E(H,G)\| \leq \omega(\xi)\|u\|\,,$$

so that

$$\|\overset{1}{\widetilde{u}}|\mathcal{L}^1_E(G,H)\|\omega(\xi)\|u\| \geq |\overset{1}{\widetilde{u}}(v)| = |tr\,(v\circ u)| = |tr(\xi\langle\cdot|u^*u\xi\rangle)| =$$

$$= |tr\,(u^*u\xi)\langle\cdot|\xi\rangle| = |\omega(\xi)\varphi\langle u^*u\xi|\xi\rangle| = \omega(\xi)\varphi\langle u\xi|u\xi\rangle$$

(Proposition 5.6.5.2 a),e), Corollary 6.1.5.3, Corollary 6.3.9.12 c)). By Proposition 6.3.9.17,

$$\|\overset{1}{\widetilde{u}}|\mathcal{L}^1_E(G,H)\|\,\|u\| \geq$$

$$\geq \sup\{\varphi\langle u\xi|u\xi\rangle|\xi \in H\,,\ \exists\, F \in \mathfrak{F}\,,\ \exists\, r \in \mathbb{N}_{n(F)}\,,\ \langle\xi|\xi\rangle = e^F_{rr}\} =$$

$$= \|u\|^2$$

so that

$$\|u\| = \|\overset{1}{\widetilde{u}}\| \geq \|\overset{1}{\widetilde{u}}|\mathcal{L}^1_E(G,H)\| \geq \|u\|\,,$$

(Theorem 6.1.6.6, Proposition 6.3.9.3 b)),

$$\|u\| = \|\overset{1}{\widetilde{u}}|\mathcal{L}^1_E(G,H)\|\,.$$

Hence θ preserves the norms.

b) Take $x' \in (\mathcal{L}^p_E(G,H))'$. By the Hahn–Banach Theorem, there is a $y' \in (\mathcal{L}^p(\widetilde{G},\widetilde{H}))'$ with

$$y'|\mathcal{L}^p_E(G,H) = x'\,.$$

By Theorem 6.1.6.6, there is a $u \in \mathcal{L}^q(\widetilde{H},\widetilde{G})$ such that

$$\overset{p}{\widetilde{u}} = y'\,.$$

By Theorem 6.3.9.16 a),b),d), $\overrightarrow{u} \in \mathcal{L}^q_E(H,G)$ and

$$x'(v) = y'(v) = \overset{p}{\widetilde{u}}(v) = tr\,(u \circ v) = tr\,\overrightarrow{u \circ v} = tr\,(\vec{u} \circ v) = \overset{p}{\vec{\widetilde{u}}}(v)$$

for any $v \in \mathcal{L}_E^p(G, H)$ so that

$$x' = \overset{p}{\vec{\widetilde{u}}}\,.$$

Hence θ is surjective.

c) For $v \in \mathcal{L}_E^p(G, H)$,

$$\langle iv, u \rangle = tr\,(u \circ v) = tr\,\overrightarrow{u \circ v} = tr(\vec{u} \circ v) = \langle v, \overset{p}{\vec{\widetilde{u}}} \rangle$$

(Theorem 6.3.9.16 a), b), d)), so that

$$iu' = \vec{u}\,.$$

d) follows from Corollary 6.1.7.9.

e) Denote by i the inclusion map $\mathcal{L}_E(H) \to \mathcal{L}(\widetilde{H})$. Then for $u \in \mathcal{L}^1(\widetilde{H})$ and $v \in \mathcal{L}_E(H)$,

$$\langle u, \overset{1}{\widetilde{iv}} \rangle = tr(u \circ v) = tr\,\overrightarrow{u \circ v} = tr(\vec{u} \circ v) = \langle \vec{u}, v \rangle$$

(Theorem 6.3.9.16 a),b),d)). ■

6.3.10 A generalization

Proposition 6.3.10.1 *Let E be a finite-dimensional C^*-algebra, H a Hilbert right E-module, $\mathcal{U}$ an upward directed upper bounded set of $\mathcal{K}_E(H)_+$, and $\mathfrak{F}$ the upper section filter of $\mathcal{U}$. Then*

$$v := \bigvee_{u \in \mathcal{U}} u$$

exists and $\mathfrak{F}$ converges to v in $\mathcal{K}_E(H)$.

Let w be an upper bound of $\mathcal{U}$ and let $\mathfrak{F}_1$, $\mathfrak{F}_2$ be ultrafilters on $\mathcal{U}$ finer than $\mathfrak{F}$. By Proposition 5.6.5.13 b), $\mathfrak{F}_1$ and $\mathfrak{F}_2$ converge in $(\mathcal{K}_E(H))_{\ddot{H}}$ to v_1 and v_2, respectively. By Proposition 5.6.5.13 c),

$$u \leq v_1 \leq w$$

for every $u \in \mathcal{U}$, so that

$$v_2 \leq v_1 .$$

By symmetry,

$$v_1 = v_2 .$$

Hence $\mathfrak{F}$ converges in $(\mathcal{K}_E(H))_{\ddot{H}}$ to the supremum v of $\mathcal{U}$.

By Theorem 6.3.9.5 a), there is an extreme Fourier basis A of H and an $f \in c_0(H)_+$ such that

$$v = \sum_{\xi \in A} f(\xi)\xi\langle\cdot|\xi\rangle \quad (\text{in } \mathcal{K}_E(H)) .$$

Take $\varepsilon > 0$ and put

$$B := \{\xi \in A \mid f(\xi) \geq \varepsilon\} \in \mathfrak{P}_f(A) ,$$

$$p := \sum_{\xi \in B} \xi\langle\cdot|\xi\rangle .$$

$p \in \Pr \mathcal{K}_E(H)$ (Proposition 5.6.5.11 a)) and by Proposition 4.2.2.21 (and Proposition 6.3.9.3 d)),

$$\bigvee_{u \in \mathcal{U}} (pup) = pvp .$$

Since $\operatorname{Im} p$ is finite-dimensional,

$$\lim_{u,\mathfrak{F}} pup = pvp$$

in $\mathcal{K}_E(H)$. Hence there is a $u_0 \in \mathcal{U}$ such that

$$\|pup - pvp\| < \varepsilon$$

for every $u \in \mathcal{U}$, $u \geq u_0$.

Take $u \in \mathcal{U}$, $u \geq u_0$, and $\eta \in H^{\#}$ and put

$$w := v - u$$

$$\eta_1 := p\eta\,, \quad \eta_2 := (1-p)\eta\,.$$

By Theorem 5.6.1.11 c),

$$\|\langle w\eta_1|\eta_1\rangle\| = \|\langle wp\eta|p\eta\rangle\| = \|\langle pwp\eta|\eta\rangle\| \leq \|pwp\| < \varepsilon\,.$$

By Theorem 5.6.1.11 c) and Proposition 5.6.3.12 b),

$$0 \leq \langle w\eta_2|\eta_2\rangle \leq \langle v\eta_2|\eta_2\rangle = \sum_{\xi\in A} f(\xi)\langle\xi\langle\eta_2|\xi\rangle|\eta_2\rangle =$$

$$= \sum_{\xi\in A\setminus B} f(\xi)\langle\xi|\eta_2\rangle\langle\eta_2|\xi\rangle \leq \varepsilon \sum_{\xi\in A\setminus B} \langle\xi|\eta_2\rangle\langle\eta_2|\xi\rangle \leq \varepsilon\langle\eta_2|\eta_2\rangle\,,$$

so that

$$\|\langle w\eta_2|\eta_2\rangle\| < \varepsilon$$

(Corollary 4.2.1.18). By Proposition 5.6.1.2 b),

$$\langle w\eta|\eta\rangle = \langle w(\eta_1+\eta_2)|\eta_1+\eta_2\rangle \leq 2(\langle w\eta_1|\eta_1\rangle + \langle w\eta_2|\eta_2\rangle)\,,$$

so that

$$\|\langle w\eta|\eta\rangle\| \leq 2(\|\langle w\eta_1|\eta_1\rangle\| + \|\langle w\eta_2|\eta_2\rangle\|) < 4\varepsilon\,.$$

Since η is arbitrary it follows

$$\|u - v\| = \|w\| \leq 4\varepsilon$$

(Theorem 5.6.1.11 c)). Hence $\mathfrak{F}$ converges to v in $\mathcal{K}_E(H)$. ■

Proposition 6.3.10.2 *Let $(E_\iota)_{\iota\in I}$ be a family of C^*-algebras and let E be its C^*- direct sum. For every $\iota \in I$, we identify E_ι with the corresponding closed ideal of E and denote by $\mathfrak{F}_\iota$ an approximate unit of E_ι. Let F, G, H be Hilbert right E-modules and for every $\iota \in I$ put*

$$F_\iota := \{\xi \in F \mid \xi = \lim_{x,\mathfrak{F}_\iota} \xi x\}\,, \quad G_\iota := \{\xi \in G \mid \xi = \lim_{x,\mathfrak{F}_\iota} \xi x\}\,,$$

$$H_\iota := \{\xi \in H \mid \xi = \lim_{x,\mathfrak{F}_\iota} \xi x\}\,.$$

a) *For every $\iota \in I$, H_ι is (in a natural way) a Hilbert right E_ι-module. Define*

$$\widetilde{H} := \{\xi \in \prod_{\iota\in I} H_\iota \mid (\|\xi_\iota\|)_{\iota\in I} \in c_0(I)\}\,.$$

b) *$\widetilde{H}$ endowed with the right multiplication*

$$\widetilde{H} \times E \longrightarrow \widetilde{H}\,, \quad (\xi, x) \longmapsto (\xi_\iota x_\iota)_{\iota\in I}$$

and with the inner-product

$$\widetilde{H} \times \widetilde{H} \longrightarrow E\,, \quad (\xi, \eta) \longmapsto (\langle \xi_\iota | \eta_\iota \rangle)_{\iota\in I}$$

is a Hilbert right E-module such that

$$\|\xi\| = \sup_{\iota\in I} \|\xi_\iota\|$$

for every $\xi \in \widetilde{H}$.

c) *$\sum_{\iota\in I} \xi_\iota \in H$ for every $\xi \in \widetilde{H}$ and the map*

$$\widetilde{H} \longrightarrow H\,, \quad \xi \longmapsto \sum_{\iota\in I} \xi_\iota$$

is an isomorphism of Hilbert right E-modules. We identify H and $\widetilde{H}$ using this isomorphism.

d) *Take $u \in \mathcal{L}_E(G, H)$ and $\iota \in I$. Then $u\xi \in H_\iota$ for every $\xi \in G_\iota$. Put*

$$u_\iota : G_\iota \longrightarrow H_\iota\,, \quad \xi \longmapsto u\xi\,.$$

Then $u_\iota \in \mathcal{L}_{E_\iota}(G_\iota, H_\iota)$ and

$$u_\iota^* = (u^*)_\iota\,.$$

Moreover

$$\|u\| = \sup_{\iota\in I} \|u_\iota\|\,.$$

e) *If* $u \in \mathcal{L}_E(F,G)$ *and* $v \in \mathcal{L}_E(G,H)$, *then, with the notation of d),*

$$(v \circ u)_\iota = v_\iota \circ u_\iota .$$

f) *If* $(\xi,\eta) \in G \times H$, *then, with the notation of d),*

$$(\eta\langle\cdot|\xi\rangle)_\iota = \eta_\iota\langle\cdot|\xi_\iota\rangle$$

for every $\iota \in I$ *and*

$$(\|\eta_\iota\langle\cdot|\xi_\iota\rangle\|)_{\iota\in I} \in c_0(I) .$$

g) *Put*

$$\mathcal{K} := \{u \in \prod_{\iota\in I} \mathcal{K}_{E_\iota}(G_\iota,H_\iota) \mid (\|u_\iota\|)_{\iota\in I} \in c_0(I)\}$$

and endow $\mathcal{K}$ *with the norm*

$$\mathcal{K} \longrightarrow \mathbb{R}_+ , \quad u \longmapsto \sup_{\iota\in I} \|u_\iota\| .$$

Then (with notation of d)) $(u_\iota)_{\iota\in I} \in \mathcal{K}$ *for every* $u \in \mathcal{K}_E(G,H)$ *and the map*

$$\mathcal{K}_E(G,H) \longrightarrow \mathcal{K} , \quad u \longmapsto (u_\iota)_{\iota\in I}$$

is an isomorphism of Banach spaces.

h) *If* $\mathcal{K}$ *denotes the* C^**-direct sum of the family* $\mathcal{K}_{E_\iota}(H_\iota))_{\iota\in I}$ *then (with the notation of d))* $(u_\iota)_{\iota\in I} \in \mathcal{K}$ *for every* $u \in \mathcal{K}_E(H)$ *and the map*

$$\mathcal{K}_E(H) \longrightarrow \mathcal{K} , \quad u \longmapsto (u_\iota)_{\iota\in I}$$

is an isomorphism of C^**-algebras.*

a), b), and c) are easy to see.

d) We have

$$u\xi = \lim_{x,\mathfrak{F}_\iota} u(\xi x) = \lim_{x,\mathfrak{F}_\iota} (u\xi)x ,$$

so that $u\xi \in H_\iota$. For $(\xi,\eta) \in G_\iota \times H_\iota$,

$$\langle u_\iota\xi|\eta\rangle = \langle u\xi|\eta\rangle = \langle\xi|u^*\eta\rangle = \langle\xi|(u^*)_\iota\eta\rangle ,$$

so that $u_\iota \in \mathcal{L}_{E_\iota}(G_\iota,H_\iota)$ and

$$u_\iota^* = (u^*)_\iota .$$

The equality

$$\|u\| = \sup_{\iota \in I} \|u_\iota\|$$

is obvious.

e) is easy to see.

f) For $\zeta \in G_\iota$,

$$(\eta\langle\cdot|\xi\rangle)_\iota\zeta = \eta\langle\zeta|\xi\rangle = \eta\langle\zeta|\xi_\iota\rangle =$$

$$= \lim_{x,\mathfrak{F}_\iota} \eta(x\langle\zeta|\xi_\iota\rangle) = \lim_{x,\mathfrak{F}_\iota} (\eta x)\langle\zeta|\xi_\iota\rangle = \eta_\iota\langle\zeta|\xi_\iota\rangle ,$$

so that

$$(\eta\langle\cdot|\xi\rangle)_\iota = \eta_\iota\langle\cdot|\xi_\iota\rangle .$$

By Proposition 5.6.5.2 a),

$$\|\eta_\iota\langle\cdot|\xi_\iota\rangle\| \leq \|\eta_\iota\| \, \|\xi_\iota\| ,$$

so that

$$(\|\eta_\iota\langle\cdot|\xi_\iota\rangle\|)_{\iota\in I} \in c_0(I) .$$

g) Take $\varepsilon > 0$. There is a finite family $((\xi_\lambda, \eta_\lambda))_{\lambda\in L}$ in $G \times H$ such that

$$\|u - \sum_{\lambda\in L} \eta_\lambda\langle\cdot|\xi_\lambda\rangle\| < \varepsilon .$$

By d) and f), for every $\iota \in I$,

$$\|u_\iota - \sum_{\lambda\in L} \eta_{\lambda\iota}\langle\cdot|\xi_{\lambda\iota}\rangle\| < \varepsilon ,$$

so that $u_\iota \in \mathcal{K}_{E_\iota}(G_\iota, H_\iota)$ and

$$\|u_\iota\| \leq \left\| \sum_{\lambda\in L} \eta_{\lambda\iota}\langle\cdot|\xi_{\lambda\iota}\rangle \right\| + \varepsilon \leq$$

$$\leq \sum_{\lambda\in L} \|\eta_{\lambda\iota}\langle\cdot|\xi_{\lambda\iota}\rangle\| + \varepsilon .$$

Since ε is arbitrary it follows from f)

$$(\|u_\iota\|)_{\iota\in I} \in c_0(I)\,,$$

i.e. $(u_\iota)_{\iota\in I} \in \mathcal{K}$.

Take $v \in \mathcal{K}$ and $\xi \in G$. Then

$$(\|v_\iota\|)_{\iota\in I}\,, \quad (\|\langle\xi_\iota|\xi_\iota\rangle\|)_{\iota\in I} \in c_0(I)$$

and from

$$\iota \in I \Longrightarrow \|v_\iota\xi_\iota\| \leq \|v_\iota\|\,\|\xi_\iota\|$$

it follows

$$(v_\iota\xi_\iota)_{\iota\in I} \in \widetilde{H} = H\,.$$

We put

$$u : G \longrightarrow H\,, \quad \xi \longmapsto (v_\iota\xi_\iota)_{\iota\in I}\,.$$

If for every $\iota \in I$,

$$p_\iota : G \longrightarrow G_\iota\,, \quad q_\iota : H_\iota \longrightarrow H$$

denote the canonical projection and the canonical injection, respectively, then

$$q_\iota \circ v_\iota \circ p_\iota \in \mathcal{K}_E(G,H)$$

for every $\iota \in I$ and

$$u = \sum_{\iota\in I}(q_\iota \circ v_\iota \circ p_\iota) \quad (\text{in } \mathcal{K}_E(G,H))\,.$$

Since

$$u_\iota = v_\iota$$

for every $\iota \in I$, the map

$$\mathcal{K}_E(G,H) \longrightarrow \mathcal{K}\,, \quad u \longmapsto (u_\iota)_{\iota\in I}$$

is surjective. By d), it is an isomorphism of Banach spaces.

h) follows from d), e), and g). ■

Proposition 6.3.10.3 *We use the notation of Proposition 6.3.10.2 and assume in addition that E_ι is finite-dimensional for every $\iota \in I$.*

a) *Let $\mathcal{U}$ be an upward directed, upper bounded set of $\mathcal{K}_E(H)_+$ and let $\mathfrak{F}$ be its upper section filter. Then $\mathfrak{F}$ converges in $\mathcal{K}_E(H)$ to the supremum of $\mathcal{U}$ in $\mathcal{K}_E(H)$.*

b) *Let $\mathfrak{A}$ be the set of extreme Fourier bases of H and for every $\iota \in I$ let $\mathfrak{A}_\iota$ be the set of extreme Fourier bases of H_ι. Then for every $A \in \mathfrak{A}$,*

$$(A \cap H_\iota)_{\iota \in I} \in \prod_{\iota \in I} \mathfrak{A}_\iota$$

and the map

$$\mathfrak{A} \longrightarrow \prod_{\iota \in I} \mathfrak{A}_\iota\,, \quad A \longmapsto (A \cap H_\iota)_{\iota \in I}$$

is bijective.

c) *If A is an extreme Fourier basis of H then*

$$\langle \eta | \zeta \rangle = \sum_{\xi \in A} \langle \xi | \zeta \rangle \langle \eta | \xi \rangle \quad (\textit{in } E)\,,$$

$$\eta = \sum_{\xi \in A} \xi \langle \eta | \xi \rangle$$

for every $\eta, \zeta \in H$.

d) *If A is an infinite extreme Fourier basis of H and if $\mathfrak{F}$ denotes the filter of cofinite sets of A then*

$$\lim_{\xi, \mathfrak{F}} \langle \xi | \eta \rangle = 0\,, \quad \lim_{\xi, \mathfrak{F}} u\xi = 0$$

for every $\eta \in H$ and $u \in \mathcal{K}_E(H)$.

e) *If $p \in \Pr \mathcal{K}_E(H)$ and if A is a Fourier basis of $\operatorname{Im} p$ then A is finite and*

$$p = \sum_{\xi \in A} \xi \langle \cdot | \xi \rangle\,.$$

a) By Proposition 6.3.10.2 g), $\mathcal{K}_E(H)$ may be identified with the C^*–direct sum of the family $\mathcal{K}_{E_\iota}(H_\iota))_{\iota \in I}$ and the assertion follows from Proposition 6.3.10.1.

b) Let p be a minimal element of $\Pr \mathcal{K}_E(H) \backslash \{0\}$. Then, identifying $\mathcal{K}_E(H)$ with the C^*–direct sum of the family $(\mathcal{K}_{E_\iota}(H_\iota))_{\iota \in I}$ (Proposition 6.3.10.2 g)) it

follows $p_\iota \leq p$ for every $\iota \in I$. Since p is a minimal element of $\Pr \mathcal{K}_E(H)\backslash\{0\}$ it follows that there is exactly one $\iota \in I$ with $p_\iota = p$ and $p_\lambda = 0$ for every $\lambda \in I\backslash\{\iota\}$. This proves the assertion.

c) By b) and Theorem 5.6.3.13 f),

$$\langle \eta|\zeta\rangle_\iota = \sum_{\xi \in A \cap H_\iota} \langle \xi|\zeta\rangle\langle \eta|\xi\rangle$$

for every $\iota \in I$. It follows

$$\langle \eta|\zeta\rangle = \sum_{\xi \in A} \langle \xi|\zeta\rangle\langle \eta|\xi\rangle .$$

Take $\varepsilon > 0$. By Proposition 6.3.10.2, there is a $J \in \mathfrak{F}_f(I)$ such that

$$\|\eta_\iota\| < \frac{\varepsilon}{2}$$

for every $\iota \in I\backslash J$. By b) and Theorem 5.6.3.13 d),f),

$$\eta_\iota = \sum_{\xi \in A \cap H_\iota} \xi\langle \eta|\xi\rangle$$

for every $\iota \in I$. Hence for every $\iota \in I$, there is an $A_\iota \in \mathfrak{P}_f(A \cap H_\iota)$ such that

$$\|\eta_\iota - \sum_{\xi \in B} \xi\langle \eta|\xi\rangle\| < \varepsilon$$

for every $B \in \mathfrak{P}_f(A \cap H_\iota)$, $B \supset A_\iota$. Put

$$\widetilde{A} := \bigcup_{\iota \in J} A_\iota \in \mathfrak{P}_f(A)$$

and let $B \in \mathfrak{P}_f(A)$, $B \supset \widetilde{A}$. Then

$$B \cap H_\iota \in \mathfrak{P}_f(A \cap H_\iota), \quad B \cap H_\iota \supset A_\iota$$

and so

$$\left\| \eta_\iota - \sum_{\xi \in B \cap H_\iota} \xi\langle \eta|\xi\rangle \right\| < \varepsilon$$

for every $\iota \in J$. Take $\iota \in I\backslash J$. Then by Proposition 5.6.3.12 a), b),

$$\left\langle \sum_{\xi \in B \cap H_\iota} \xi\langle \eta|\xi\rangle \,\middle|\, \sum_{\xi \in B \cap H_\iota} \xi\langle \eta|\xi\rangle \right\rangle =$$

$$= \sum_{\xi \in B \cap H_\iota} \langle \xi | \eta \rangle \langle \eta | \xi \rangle \leq \langle \eta_\iota | \eta_\iota \rangle$$

and so (by Corollary 4.2.1.18)

$$\left\| \sum_{\xi \in B \cap H_\iota} \xi \langle \eta | \xi \rangle \right\| \leq \|\eta_\iota\| ,$$

$$\left\| \eta_\iota - \sum_{\xi \in B \cap H_\iota} \xi \langle \eta | \xi \rangle \right\| \leq \|\eta_\iota\| + \|\eta_\iota\| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon .$$

Hence by Proposition 6.3.10.2 b), c),

$$\left\| \eta - \sum_{\xi \in B} \xi \langle \eta | \xi \rangle \right\| = \sup_{\iota \in I} \left\| \left(\eta - \sum_{\xi \in B} \xi \langle \eta | \xi \rangle \right)_\iota \right\| \leq \varepsilon .$$

Since ε is arbitrary

$$\eta = \sum_{\xi \in A} \xi \langle \eta | \xi \rangle .$$

d) By c),

$$\lim_{\xi, \mathfrak{F}} \langle \xi | \eta \rangle = 0 .$$

Let $\varepsilon > 0$. There are finite families $(\eta_\lambda)_{\lambda \in L}$ and $(\zeta_\lambda)_{\lambda \in L}$ in H such that

$$\left\| u - \sum_{\lambda \in L} \eta_\lambda \langle \cdot | \zeta_\lambda \rangle \right\| < \varepsilon .$$

By the above,

$$\lim_{\xi, \mathfrak{F}} \sum_{\lambda \in L} \eta_\lambda \langle \xi | \zeta_\lambda \rangle = 0 .$$

Since ε is arbitrary,

$$\lim_{\xi, \mathfrak{F}} u\xi = 0 .$$

e) By d), A is finite. Put

$$q := \sum_{\xi \in A} \xi \langle \cdot | \xi \rangle .$$

Then $q \in \Pr\ \mathcal{K}_E(H)$ and

$$pq = q\,,$$

so that

$$\operatorname{Im} q \subset \operatorname{Im} p\,.$$

Since A is a Fourier basis of $\operatorname{Im} p$ it follows

$$\operatorname{Im} q = \operatorname{Im} p$$

and $p = q$. ■

Theorem 6.3.10.4 *We use the notation and assumptions of Proposition 6.3.10.3 and denote by u a self-normal element of $\mathcal{K}_E(H)$.*

a) *There is an extreme Fourier basis A of H and an $f \in c_0(A)$ such that*

$$u = \sum_{\xi \in A} f(\xi)\xi\langle\cdot|\xi\rangle \quad (\text{in } \mathcal{K}_E(H))\,,$$

$$f(A) \subset \sigma_{\mathcal{L}_E(H)}(u) \subset f(A) \cup \{0\}\,.$$

Moreover, for every $g \in \mathcal{C}(\sigma_{\mathcal{L}_E(H)}(u))$ with

$$0 \in \sigma_{\mathcal{L}_E(H)}(u) \Longrightarrow g(0) = 0\,,$$

$$g(u) = \sum_{\xi \in A} g(f(\xi))\xi\langle\cdot|\xi\rangle \quad (\text{in } \mathcal{L}_E(H))$$

b) *If we put $B := \{f \neq 0\}$ then*

$$H \longrightarrow H\,, \quad \eta \longmapsto \sum_{\xi \in B} \xi\langle\eta|\xi\rangle$$

is the carrier of u in $\mathcal{L}_E(H)$.

Let $\iota \in I$. By Proposition 6.3.10.2 g), u_ι is a self-normal element of $\mathcal{K}_{E_\iota}(H_\iota)$. By Theorem 6.3.9.5 a) and Corollary 6.3.9.8, there is an extreme Fourier basis A_ι of H_ι and an $f_\iota \in c_0(A_\iota)$ such that

$$u_\iota = \sum_{\xi \in A_\iota} f_\iota(\xi)\xi\langle\cdot|\xi\rangle \quad (\text{in } \mathcal{K}_{E_\iota}(H_\iota))\,,$$

$$f_\iota(A_\iota) \subset \sigma_{\mathcal{L}_{E_\iota}(H_\iota)}(u_\iota) \subset f_\iota(A_\iota) \cup \{0\}\,,$$

$$g(u_\iota) = \sum_{\xi \in A_\iota} g(f_\iota(\xi))\xi\langle\cdot|\xi\rangle \quad (\text{in } \mathcal{L}_{E_\iota}(H_\iota)) .$$

Put

$$A := \bigcup_{\iota \in I} A_\iota ,$$

$$f : A \longrightarrow \mathbb{K}, \quad \xi \longmapsto f_\iota(\xi) \quad \text{if } \xi \in A_\iota .$$

By Proposition 6.3.10.3 b), A is an extreme Fourier basis of H. Since

$$(\|u_\iota\|)_{\iota \in I} \in c_0(I) ,$$

$f \in c_0(A)$. The assertions follow. ■

Theorem 6.3.10.5 *We use the notation and assumptions of Proposition 6.3.10.3. Let G, H be Hilbert right E-modules and $u \in \mathcal{K}_E(G,H)$.*

There are an extreme Fourier set A of G, a family $(\widetilde{\xi})_{\xi \in A}$ in H such that

$$\langle \xi|\eta\rangle = \langle \widetilde{\xi}|\widetilde{\eta}\rangle$$

for all $\xi, \eta \in A$, and an $f \in c_0(A)_+$ such that

$$u = \sum_{\xi \in A} f(\xi)\widetilde{\xi}\langle\cdot|\xi\rangle \quad (\textit{in } \mathcal{K}_E(G,H)) .$$

Moreover, for every such representation of u,

$$\|u\| = \|f\|_\infty .$$

Let $\iota \in I$ and put

$$u_\iota : G_\iota \longrightarrow H_\iota , \quad \xi \longmapsto u\xi .$$

By Proposition 6.3.10.2 d),f), u_ι is well-defined and belongs to $\mathcal{K}_{E_\iota}(G_\iota, H_\iota)$. By Theorem 6.3.9.7 a) (and Corollary 6.3.9.8), there are an extreme Fourier set A_ι of G_ι, a family $(\widetilde{\xi})_{\xi \in A_\iota}$ in H_ι such that

$$\langle \xi|\eta\rangle = \langle \widetilde{\xi}|\widetilde{\eta}\rangle$$

for all $\xi, \eta \in A_\iota$, and an $f_\iota \in c_0(A_\iota)_+$ such that

$$u_\iota = \sum_{\xi \in A_\iota} f_\iota(\xi)\widetilde{\xi}\langle\cdot|\xi\rangle \quad (\text{in } \mathcal{K}_{E_\iota}(G_\iota, H_\iota)) ,$$

$$\|u_\iota\| = \|f_\iota\|_\infty .$$

Put

$$A := \bigcup_{\iota \in I} A_\iota ,$$

$$f : A \longrightarrow \mathbb{K}, \quad \xi \longmapsto f_\iota(\xi) \quad \text{if } \xi \in A_\iota .$$

Since $(\|u_\iota\|)_{\iota \in I} \in c_0(I)$ and

$$\|u\| = \sup_{\iota \in I} \|u_\iota\|$$

(Proposition 6.3.10.2 g)), it follows $f \in c_0(A)_+$ and

$$\|u\| = \|f\|_\infty .$$

Let $\varepsilon > 0$. There is a $J \in \mathfrak{P}_F(I)$ such that

$$\|u_\iota\| < \frac{\varepsilon}{2}$$

for every $\iota \in I \backslash J$. For every $\iota \in I$ there is a $B_\iota \in \mathfrak{P}_f(A_\iota)$ such that

$$\|u_\iota - \sum_{\xi \in C} f_\iota(\xi) \widetilde{\xi\langle \cdot | \xi \rangle}\| < \varepsilon$$

for every $C \in \mathfrak{P}_f(A_\iota)$, $C \supset B_\iota$. Put

$$B := \bigcup_{\iota \in J} B_\iota \in \mathfrak{P}_f(A)$$

and take $C \in \mathfrak{P}_f(A)$, $C \supset B$. For $\iota \in J$

$$\left\| \left(u - \sum_{\xi \in C} f(\xi) \widetilde{\xi\langle \cdot | \xi \rangle} \right)_\iota \right\| =$$

$$= \left\| u_\iota - \sum_{\xi \in C \cap A_\iota} f_\iota(\xi) \widetilde{\xi\langle \cdot | \xi \rangle} \right\| < \varepsilon .$$

For $\iota \in I \backslash J$,

$$\left\| \left(u - \sum_{\xi \in C} f(\xi) \widetilde{\xi\langle \cdot | \xi \rangle} \right)_\iota \right\| = \left\| u_\iota - \sum_{\xi \in C \cap A_\iota} f_\iota(\xi) \widetilde{\xi\langle \cdot | \xi \rangle} \right\| \leq$$

$$\leq \|u_\iota\| + \left\| \sum_{\xi \in C \cap A_\iota} f_\iota(\xi)(\xi)\widetilde{\xi}\langle \cdot | \xi \rangle \right\| \leq \|u_\iota\| + \|f_\iota\|_\infty < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

(Theorem 6.3.9.7 a)). Hence (Proposition 6.3.10.2 g))

$$\left\| u - \sum_{\xi \in C} f(\xi)\widetilde{\xi}\langle \cdot | \xi \rangle \right\| = \sup_{\iota \in I} \left\| \left(u - \sum_{\xi \in C} f(\xi)\widetilde{\xi}\langle \cdot | \xi \rangle \right)_\iota \right\| \leq \varepsilon .$$

Since ε is arbitrary,

$$u = \sum_{\xi \in A} f(\xi)\widetilde{\xi}\langle \cdot | \xi \rangle .$$

The final assertion follows from

$$\|u\| = \sup_{\iota \in I} \|u_\iota\| = \sup_{\iota \in I} \|f|(A \cap G_\iota)\|_\infty = \|f\|_\infty$$

(Proposition 6.3.10.2. g), Theorem 6.3.9.7 a)). ■

7. C^*–algebras Generated by Groups

Many examples of C^*–algebras are constructed by projective representations of groups, which are presented in the first section of this chapter. The Clifford algebras are special cases of such representations and are treated in the second section.

7.1 Projective Representations of Groups

7.1.1 Schur functions

Proposition 7.1.1.1 (7) *Let T be a group,*

$$f : T \times T \longrightarrow \{\alpha \in \mathbb{K} || \alpha| = 1\},$$

and $p, q \in [1, \infty]$ such that

$$\frac{1}{p} + \frac{1}{q} \geq 1 .$$

We set

$$x*y : T \longrightarrow \mathbb{K}, \quad t \longmapsto \sum_{s \in T} f(ts^{-1}, s)x(ts^{-1})y(s) = \sum_{r \in T} f(r, r^{-1}t)x(r)y(r^{-1}t)$$

for all $(x, y) \in \ell^p(T) \times \ell^q(T)$ (Proposition 1.1.2.6 b), Example 1.2.2.3 a)) and denote by E the set of $x \in \ell^p(T)$ such that $x*y \in \ell^q(T)$ for every $y \in \ell^q(T)$ and such that the map

$$\vec{x} : \ell^q(T) \longrightarrow \ell^q(T), \quad y \longmapsto x*y$$

is continuous. Moreover, we put

$$\vec{E} := \{\vec{x} | x \in E\} .$$

a) *If* $x \in \ell^p(T)$, $y \in \ell^q(T)$, *and* $s,t \in T$, *then*

$$(x*e_t)(s) = f(st^{-1},t)x(st^{-1}),$$

$$(e_t*y)(s) = f(t,t^{-1}s)y(t^{-1}s).$$

In particular, if $x \in E$ *then* $x \in \ell^q(T)$

b) $s,t \in T \Rightarrow e_s*e_t = f(s,t)e_{st}$.

c) *The following are equivalent for all* $r,s,t \in T$:

c_1) $(e_r*e_s)*e_t = e_r*(e_s*e_t)$.

c_2) $f(r,s)f(rs,t) = f(r,st)f(s,t)$.

d) $x \in \ell^1(T) \Rightarrow x \in E$, $\|\vec{x}\| \leq \|x\|_1$.

e) $x \in E$, $p \geq q \Rightarrow \|x\|_p \leq \|\vec{x}\|$.

f) *Let* $\mathfrak{F}$ *be a filter on* E *converging weakly to an* $x_0 \in \ell^p(T)$ *and let* $u \in \mathcal{L}(\ell^q(T))$ *such that*

$$\lim_{x,\mathfrak{F}} \vec{x} = u$$

in $\mathcal{L}(\ell^q(T))_{\ell^q(T)}$ *(Definition 1.2.6.1). Then* $x_0 \in E$ *and* $\vec{x_0} = u$.

g) *If* $p \geq q$ *then the map*

$$E \longrightarrow \mathcal{L}(\ell^q(T)), \quad x \longmapsto \vec{x}$$

is injective, the map

$$\vec{E} \longrightarrow E, \quad \vec{x} \longmapsto x$$

is continuous with norm 1, *where* E *is endowed with the norm induced by* $\ell^p(T)$, *and* $\vec{E}$ *is closed in* $\mathcal{L}(\ell^q(T))$.

h) *Assume* $p \geq q$ *and* $1 < p < \infty$. *If* $u \in \mathcal{L}(\ell^q(T))$ *belongs to the closure in* $\mathcal{L}(\ell^q(T))_{\ell^q(T)}$ *of a bounded set of* $\vec{E}$, *then* $u \in \vec{E}$.

i) *If* S *is a subgroup of* T, $x \in \ell^p(T)$, *and* $y \in \ell^q(T)$, *then*

$$(xe_S)*(ye_S) = ((xe_S)*(ye_S))e_S.$$

j) *If* S *is a subgroup of* T, R *is a subgroup of* S, *and* $x \in E$, *then* xe_R, $xe_S \in E$ *and*

$$\|\overrightarrow{xe_R}\| \leq \|\overrightarrow{xe_S}\| .$$

k) *If* f *is real then*

$$x \in E \Longrightarrow \overline{x} \in E , \|\vec{\overline{x}}\| = \|\vec{x}\| .$$

l) *If* T *is infinite,* $q \neq \infty$, *and* $x \in E\backslash\{0\}$, *then* $\operatorname{Card} T$ *is the topological cardinality of* $\operatorname{Im} \vec{x}$.

a) We have

$$(x*e_t)(s) = \sum_{r\in T} f(sr^{-1}, r)x(sr^{-1})e_t(r) = f(st^{-1}, t)x(st^{-1}) ,$$

$$(e_t*y)(s) = \sum_{r\in T} f(sr^{-1}, r)e_t(sr^{-1})y(r) = f(t, t^{-1}s)y(t^{-1}s) .$$

b) By a),

$$(e_s*e_t)(r) = f(rt^{-1}, t)e_s(rt^{-1}) = f(s, t)\delta_{s,rt^{-1}} =$$

$$= f(s, t)\delta_{st,r} = f(s, t)e_{st}(r)$$

for every $r \in T$, so that

$$e_s*e_t = f(s, t)e_{st} .$$

c) By b),

$$(e_r*e_s)*e_t = f(r, s)e_{rs}*e_t = f(r, s)f(rs, t)e_{rst} ,$$

$$e_r*(e_s*e_t) = e_r*f(s, t)e_{st} = f(s, t)f(r, st)e_{rst} .$$

d) Take $y \in \ell^q(T)$. For $t \in T$,

$$|(x*y)(t)| = \Big|\sum_{s\in T} f(ts^{-1}, s)x(ts^{-1})y(s)\Big| \leq \sum_{s\in T} |x(ts^{-1})|\,|y(s)| .$$

First assume $q = \infty$. Then

$$|(x*y)(t)| \leq \|y\|_\infty \sum_{s\in T} |x(ts^{-1})| = \|x\|_1 \|y\|_\infty$$

for every $t \in T$, so that $x*y \in \ell^\infty(T)$ and

$$\|x*y\|_\infty \leq \|y\|_\infty \|x\|_1 .$$

Now assume $q \neq \infty$. Then for every $t \in T$,

$$|(x*y)(t)| \leq \sum_{s\in T} |x(ts^{-1})|^{\frac{q-1}{q}} (|x(ts^{-1})|^{\frac{1}{q}} |y(s)|) \leq$$

$$\leq \Big(\sum_{s\in T} |x(ts^{-1})|\Big)^{\frac{q-1}{q}} \Big(\sum_{s\in T} |x(ts^{-1})|\, |y(s)|^q\Big)^{\frac{1}{q}} =$$

$$= \|x\|_1^{\frac{q-1}{q}} \Big(\sum_{s\in T} |x(ts^{-1})|\, |y(s)|^q\Big)^{\frac{1}{q}}$$

(Example 1.2.2.3 a)). It follows that

$$\sum_{t\in T} |(x*y)(t)|^q \leq \|x\|_1^{q-1} \sum_{t\in T}\sum_{s\in T} |x(ts^{-1})|\, |y(s)|^q =$$

$$= \|x\|_1^{q-1} \sum_{s\in T} |y(s)|^q \Big(\sum_{t\in T} |x(ts^{-1})|\Big) =$$

$$= \|x\|_1^{q-1} \|x\|_1 \|y\|_q^q = (\|x\|_1 \|y\|_q)^q .$$

Hence $x*y \in \ell^q(T)$ and

$$\|x*y\|_q \leq \|x\|_1 \|y\|_q .$$

Since y is arbitrary, it follows that $x \in E$ and

$$\|\vec{x}\| \leq \|x\|_1 .$$

e) By a),

$$\|\vec{x} e_1\|_q = \|x\|_q \geq \|x\|_p ,$$

(Proposition 1.1.2.6 b),e)) so that

$$\|x\|_p \leq \|\vec{x} e_1\|_q \leq \|\vec{x}\|\, \|e_1\|_q = \|\vec{x}\| .$$

f) Take $y \in \ell^q(T)$ and $t \in T$. Then

$$(x_0 * y)(t) = \sum_{s \in T} f(ts^{-1}, s) x_0(ts^{-1}) y(s) =$$

$$= \lim_{x, \mathfrak{F}} \sum_{s \in T} f(ts^{-1}, s) x(ts^{-1}) y(s) =$$

$$= \lim_{x, \mathfrak{F}} (\vec{x} y)(t) = (uy)(t) .$$

Hence

$$x_0 * y = uy \in \ell^q(T) ,$$

$x_0 \in E$, $\vec{x_0} = u$.

g) By e), the map

$$E \longrightarrow \mathcal{L}(\ell^q(T)) , \quad x \longmapsto \vec{x}$$

is injective and the map

$$\vec{E} \longrightarrow E , \quad \vec{x} \longmapsto x$$

is continuous with norm at most 1. By b), this norm is equal to 1.

Let u be an element of the closure of $\vec{E}$ in $\mathcal{L}(\ell^q(T))$. Then there is a seuqence $(x_n)_{n \in \mathbb{N}}$ in E such that $\vec{x_n}$ converges to u. By e), $(x_n)_{n \in \mathbb{N}}$ converges to an x in $\ell^p(T)$. By f), $x \in E$ and $u = \vec{x} \in \vec{E}$. Hence $\vec{E}$ is closed in $\mathcal{L}(\ell^q(T))$.

h) Let A be a bounded set of $\vec{E}$ and $\mathfrak{F}$ an ultrafilter on A converging to $u \in \mathcal{L}(\ell^q(T))$ in the topology of pointwise convergence. By g), the map

$$\varphi : \vec{E} \longrightarrow E , \quad \vec{x} \longmapsto x$$

is continuous. Hence $\varphi(A)$ is a bounded set of $\ell^p(T)$. By Example 1.3.8.9, $\ell^p(T)$ is reflexive, so that $\varphi(A)$ is weakly relatively compact (Proposition 1.3.8.3). Let x_0 be the weak limit of $\varphi(\mathfrak{F})$ in $\ell^p(T)$. By f), $x_0 \in E$ and

$$u = \vec{x_0} \in \vec{E} .$$

i) For $t \in T \backslash S$,

$$((xe_S) * (ye_S))(t) = \sum_{s \in T} f(ts^{-1}, s) x(ts^{-1}) e_S(ts^{-1}) y(s) e_S(s) =$$

$$= \sum_{s \in S} f(ts^{-1}, s) x(ts^{-1}) e_S(ts^{-1}) y(s) = 0 ,$$

so that

$$(xe_S) * (ye_S) = ((xe_S) * (ye_S)) e_S .$$

j) We set

$$s \sim t :\Longleftrightarrow st^{-1} \in S$$

for all $s, t \in T$. $\sim$ is an equivalence relation on T. Let $\mathfrak{Q}$ be the set of equivalence classes with respect to $\sim$.

Let $y \in \ell^q(T)$. Take $Q \in \mathfrak{Q}$ and $t \in Q$. Then

$$((xe_S) * y)(t) = \sum_{s \in T} f(ts^{-1}, s) x(ts^{-1}) e_S(ts^{-1}) y(s) =$$

$$= \sum_{s \in T} f(ts^{-1}, s) x(ts^{-1}) e_Q(s) y(s) = (\vec{x}(ye_Q))(t) .$$

First assume $q < \infty$. Then for $Q \in \mathfrak{Q}$,

$$\sum_{t \in Q} |((xe_S) * y)(t)|^q = \sum_{t \in Q} |(\vec{x}(ye_Q))(t)|^q \leq$$

$$\leq \sum_{t \in T} |(\vec{x}(ye_Q))(t)|^q = \|\vec{x}(ye_Q)\|_q^q \leq \|\vec{x}\|^q \|ye_Q\|_q^q .$$

It follows that

$$\sum_{t \in T} |((xe_S) * y)(t)|^q = \sum_{Q \in \mathfrak{Q}} \sum_{t \in Q} |((xe_S) * y)(t)|^q \leq$$

$$\leq \|\vec{x}\|^q \sum_{Q \in \mathfrak{Q}} \|ye_Q\|_q^q = \|\vec{x}\|^q \|y\|_q^q .$$

Hence $(xe_S) * y \in \ell^q(T)$ and

$$\|(xe_S) * y\|_q \leq \|\vec{x}\| \, \|y\|_q .$$

Now assume $q = \infty$. Then for $Q \in \mathfrak{Q}$,

$$\sup_{t \in Q} |((xe_S) * y)(t)| = \sup_{t \in Q} |(\vec{x}(ye_Q))(t)| \leq$$

$$\leq \sup_{t\in T} |(\vec{x}(ye_Q))(t)| = \|\vec{x}(ye_Q)\|_\infty \leq \|\vec{x}\|\,\|ye_Q\|_\infty .$$

We deduce that $(xe_S)*y \in \ell^\infty(T)$ and

$$\|(xe_S)*y\|_\infty \leq \|\vec{x}\|\,\|y\|_\infty .$$

Thus $xe_S \in E$ and

$$\|\overrightarrow{xe_S}\| \leq \|\vec{x}\|$$

in both cases. It follows that $xe_R \in E$ and

$$\|\overrightarrow{xe_R}\| = \|\overrightarrow{(xe_S)e_R}\| \leq \|\overrightarrow{xe_S}\| .$$

k) Take $y \in \ell^q(T)$. For $t \in T$,

$$(\overline{x}*y)(t) = \sum_{s\in T} f(ts^{-1}, s)\overline{x(ts^{-1})}y(s) =$$

$$= \overline{\sum_{s\in T} f(ts^{-1}, s)x(ts^{-1})\overline{y(s)}} = \overline{(x*\overline{y})(t)} ,$$

so that

$$\overline{x}*y = \overline{x*\overline{y}} .$$

Hence $\overline{x} \in E$ and

$$\|\vec{\overline{x}}\| = \|\vec{x}\| .$$

l) Let $\aleph$ be the topological cardinality of $\operatorname{Im}\vec{x}$. By Example 1.1.2.5,

$$\aleph \leq \operatorname{Card} T .$$

There is an $r \in T$ with $x(r) \neq 0$. Let $s, t \in T$. By a),

$$|(\vec{x}e_t)(rt)| = |x(r)| ,$$

$$|(\vec{x}e_s)(rt)| = |x(rts^{-1})|$$

so that

$$\|\vec{x}e_t - \vec{x}e_s\| \geq |x(r)| - |x(rts^{-1})| .$$

Since $q \neq \infty$, there is a finite subset S_t of T such that

$$\|\vec{x}e_t - \vec{x}e_s\| \geq \frac{1}{2}|x(r)|$$

for all $s \in T\backslash S_t$. Let $\mathfrak{R}$ be the set of subsets R of T such that

$$\|\vec{x}e_s - \vec{x}e_t\| \geq \frac{1}{2}|x(r)|$$

for all distinct $s, t \in R$. Let $\mathfrak{R}$ be ordered by inclusion and let R be a maximal element of $\mathfrak{R}$ (Zorn's Lemma). By the above

$$\bigcup_{t \in R} S_t = T\,.$$

Hence

$$\operatorname{Card} R = \operatorname{Card} T\,.$$

Since the cardinality of every dense set of $\operatorname{Im} \vec{x}$ is at least equal to $\operatorname{Card} R$ it follows

$$\operatorname{Card} T \leq \aleph\,,$$

$$\operatorname{Card} T = \aleph\,. \qquad \blacksquare$$

Definition 7.1.1.2 (7) *Let T be a group. We denote by $\mathfrak{S}_T$, the set of finite subgroups of T and call T **locally finite** if $\mathfrak{S}_T$ is upward directed and*

$$\bigcup_{S \in \mathfrak{S}_T} S = T\,.$$

In this case, $\mathfrak{G}_T$ denotes the upper section filter of $\mathfrak{S}_T$.

T is locally finite iff every finite subset of T is contained in a finite subgroup of T.

Proposition 7.1.1.3 (7) *We use the notations of Proposition 7.1.1.1 and assume that T is locally finite.*

a) *For every $x \in \ell^p(T)$, $x \in E$ iff $\{\|\overrightarrow{xe_S}\| \mid S \in \mathfrak{S}_T\}$ is bounded and in this case*

$$\|\vec{x}\| = \sup_{S \in \mathfrak{S}_T} \|\overrightarrow{xe_S}\| = \lim_{S, \mathfrak{S}_S} \|\overrightarrow{xe_S}\|\,,$$

$$y \in \ell^q(T) \Longrightarrow \lim_{S, \mathfrak{S}_T} \overrightarrow{xe_S}y = \vec{x}y\,.$$

b) *The following are equivalent for every* $x \in E$ *:*

b$_1$) $\vec{x} = \lim\limits_{S,\mathfrak{S}_T} \overrightarrow{xe_S}$.

b$_2$) $\vec{x}$ *belongs to the closure of* $\{\vec{z} | z \in \mathbb{K}^{(T)}\}$ *in* $\mathcal{L}(\ell^q(T))$.

b$_3$) $\vec{x}$ *belongs to the closure of* $\{\vec{z} | z \in \ell^1(T)\}$ *in* $\mathcal{L}(\ell^q(T))$.

b$_4$) *For every* $\varepsilon > 0$, *there is an* $S_0 \in \mathfrak{S}_T$ *such that*

$$\|\overrightarrow{xe_S} - \overrightarrow{xe_{S_0}}\| < \varepsilon$$

for every $S \in \mathfrak{S}_T$, $S_0 \subset S$.

a) If $x \in E$, then by Proposition 7.1.1.1 j), $\{\|\overrightarrow{xe_S}\| \mid S \in \mathfrak{S}_T\}$ is bounded. Now assume

$$\alpha := \sup_{S \in \mathfrak{S}_T} \|\overrightarrow{xe_S}\| < \infty .$$

If $p = 1$, then $x \in E$ by Proposition 7.1.1.1 d), so that we may assume $p \neq 1$. This implies $q \neq \infty$. Take $y \in \ell^q(T)$. For $S \in \mathfrak{S}$,

$$\sum_{t \in T} |((xe_S) * y)(t)|^q \leq \|\overrightarrow{xe_S}\|^q \|y\|_q^q \leq \alpha^q \|y\|_q^q .$$

Hence

$$\sum_{t \in T} |(x*y)(t)|^q \leq \liminf_{S,\mathfrak{S}_T} \sum_{t \in T} |((xe_S)*y)(t)|^q \leq \alpha^q \|y\|_q^q .$$

Thus $x \in E$ and

$$\|\vec{x}\| \leq \alpha .$$

By Proposition 7.1.1.1 j),

$$\|\vec{x}\| = \sup_{S \in \mathfrak{S}_T} \|\overrightarrow{xe_S}\| = \lim_{S,\mathfrak{S}_T} \|\overrightarrow{xe_S}\| .$$

Take $y \in \ell^q(T)$. If $q = \infty$, then $p = 1$ and

$$\lim_{S,\mathfrak{S}_T} \|xe_S - x\|_1 = 0 ,$$

so that, by Proposition 7.1.1.1 d),

$$\lim_{S,\mathfrak{S}_T} \overrightarrow{xe_S} = \vec{x} ,$$

$$\lim_{S,\mathfrak{G}_T}(xe_S)*y = \lim_{S,\mathfrak{G}_T}\overrightarrow{xe_S}y = \vec{x}y = x*y\,.$$

Hence we may assume $q \neq \infty$. By Proposition 7.1.1.1 a),

$$\lim_{S,\mathfrak{G}_T}\overrightarrow{xe_S}e_t = \vec{x}e_t$$

for every $t \in T$. It follows

$$\lim_{S,\mathfrak{G}_T}\overrightarrow{xe_S}(ye_A) = \vec{x}(ye_A)$$

for every $A \in \mathfrak{P}_f(T)$. Let $\varepsilon > 0$. There is an $A \in \mathfrak{P}_f(T)$ such that $\|ye_{T\setminus A}\|_q < \varepsilon$. By Proposition 7.1.1.1 j),

$$\|\overrightarrow{xe_S}y - \vec{x}y\| \leq \|\overrightarrow{xe_S}(ye_A) - \vec{x}(ye_A)\| +$$

$$+\|\overrightarrow{xe_S}(ye_{T\setminus A})\| + \|\vec{x}(ye_{T\setminus A})\| \leq$$

$$\leq \|\overrightarrow{xe_S}(ye_A) - \vec{x}(ye_A)\| + 2\|\vec{x}\|\varepsilon\,.$$

Hence

$$\limsup_{S,\mathfrak{G}_T}\|\overrightarrow{xe_S}y - \vec{x}y\| \leq 2\|\vec{x}\|\varepsilon\,.$$

ε being arbitrary, it follows

$$\lim_{S,\mathfrak{G}_T}\overrightarrow{xe_S}y = \vec{x}y\,.$$

$b_1 \Rightarrow b_2 \Rightarrow b_3$ and $b_1 \Rightarrow b_4$ are trivial.

$b_3 \Rightarrow b_1$. Let $\varepsilon > 0$. There is a $z \in \ell^1(T)$ such that

$$\|\vec{z} - \vec{x}\| < \frac{\varepsilon}{3}\,.$$

There is an $S_0 \in \mathfrak{S}$ such that

$$\|ze_{S_0} - z\|_1 < \frac{\varepsilon}{3}\,.$$

By Proposition 7.1.1.1 d),j), for every $S \in \mathfrak{S}_T$, $S_0 \subset S$,

$$\|\overrightarrow{xe_S} - \vec{x}\| \leq \|\overrightarrow{xe_S} - \overrightarrow{ze_S}\| + \|\overrightarrow{ze_S} - \vec{z}\| + \|\vec{z} - \vec{x}\| \leq$$

$$\leq \|\overrightarrow{(x-z)e_S}\| + \|ze_S - z\|_1 + \frac{\varepsilon}{3} \leq \|\overrightarrow{x-z}\| + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} < \varepsilon\,.$$

Hence

$$\lim_{S,\mathfrak{G}_T} \overrightarrow{xe_S} = \vec{x}\,.$$

$b_4 \Rightarrow b_1$. Take $\varepsilon > 0$. By b_4), there is an S_0 in $\mathfrak{G}_T$ such that

$$\|\overrightarrow{xe_S} - \overrightarrow{xe_{S_0}}\| < \frac{\varepsilon}{2}$$

for every $S \in \mathfrak{G}_T$, $S_0 \subset S$. By a),

$$x*y = \lim_{S,\mathfrak{G}_T} (xe_S)*y$$

for every $y \in \ell^q(T)$. Take $S \in \mathfrak{G}_T$ with $S_0 \subset S$. For $R \in \mathfrak{G}_T$, $S \subset R$,

$$\|\overrightarrow{xe_R} - \overrightarrow{xe_S}\| \leq \|\overrightarrow{xe_R} - \overrightarrow{xe_{S_0}}\| + \|\overrightarrow{xe_{S0}} - \overrightarrow{xe_S}\| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon\,.$$

Hence, by Proposition 1.2.1.7,

$$\|\vec{x} - \overrightarrow{xe_S}\| \leq \liminf_{R,\mathfrak{G}_T} \|\overrightarrow{xe_R} - \overrightarrow{xe_S}\| \leq \varepsilon\,.$$

Since ε is arbitrary

$$\vec{x} = \lim_{S,\mathfrak{G}_T} \overrightarrow{xe_S}\,. \qquad \blacksquare$$

Proposition 7.1.1.4 (7) *We use the notations of Proposition 7.1.1.1 and suppose that T is locally finite and that $p \geq q$. Let G be the set of $a \in c_0(T)$ such that*

$$\lim_{S,\mathfrak{G}_T} \sum_{t\in S} a(t)x(t)$$

exists for every $x \in E$ and such that the map

$$\widetilde{a} : E \longrightarrow \mathbb{K}\,, \quad x \longmapsto \lim_{S,\mathfrak{G}_T} \sum_{t\in S} a(t)x(t)$$

is continuous with respect to the norm

$$\|\cdot\|_E : E \longrightarrow \mathbb{R}_+\,, \quad x \longmapsto \|\vec{x}\|$$

(Proposition 7.1.1.1 g)).

a) *G is a vector subspace of $c_0(T)$,*

$$a \in G \Longrightarrow \widetilde{a} \in E'\,,$$

and

$$G \longrightarrow \mathbb{R}_+\,, \quad a \longmapsto \|\widetilde{a}\|$$

is a norm. We endow G with this norm and denote the closure of $\mathbb{K}^{(T)}$ in G by H.

b) $a \in \ell^q(T) \Rightarrow a \in G$, $\|\widetilde{a}\| \leq \|a\|_q$.

c) *H is the closure of $\ell^q(T)$ in G.*

d) $a \in G \Rightarrow \|a\|_\infty \leq \|\widetilde{a}\|$.

e) *If R, S are subgroups of T such that $R \subset S$ then for $a \in G$,*

$$ae_R,\ ae_S \in G, \quad \|\widetilde{ae_R}\| \leq \|\widetilde{ae_S}\|.$$

f) $a \in G \Rightarrow \|\widetilde{a}\| = \sup_{S \in \mathfrak{S}} \|\widetilde{ae_S}\|$.

g) *$\{\widetilde{a} \mid a \in G\}$ is a closed vector subspace of E'. In particular, G and H are complete.*

h) *For every $x \in E$, the map*

$$\dot{x} : H \longrightarrow \mathbb{K}, \quad a \longmapsto \widetilde{a}(x)$$

is continuous and the map

$$E \longrightarrow H', \quad x \longmapsto \dot{x}$$

is an isometry.

i) *If $u : H \to c_0(T)$ denotes the inclusion map, then*

$$u' : \ell^1(T) \longrightarrow E$$

is the inclusion map, where we identified E with H' using the isometry of h).

j) *For every $a \in H$,*

$$\widetilde{a} = \lim_{S, \mathfrak{S}_T} \widetilde{ae_S}.$$

k) *Assume $p = q = 2$. There is a unique operator*

$$\psi : \mathcal{L}^1(\ell^2(T)) \longrightarrow c_0(T)$$

such that

$$(\psi(\langle \cdot | \xi \rangle \eta))(t) = \langle e_t * \eta | \xi \rangle = \sum_{s \in T} f(t, t^{-1}s)\eta(t^{-1}s)\overline{\xi(s)}$$

for all $\xi, \eta \in \ell^2(T)$ and $t \in T$. ψ has norm 1, $\operatorname{Im}\psi \subset H$, the map

$$\psi_0 : \mathcal{L}^1(\ell^2(T)) \longrightarrow H\,, \quad y \longmapsto \psi y$$

is an operator of norm 1, *and*

$$\psi_0' x = \vec{x}$$

for every $x \in E$, *where we identified* E *with* H' *using the isometry of* h).

a) is easy to see.

b) For $x \in E$, $ax \in \ell^1(T)$ and

$$\left|\sum_{t\in T} a(t)x(t)\right| \leq \|a\|_q \|x\|_p \leq \|a\|_q \|x\|_E$$

(Proposition 7.1.1.1 e)). Hence $a \in G$ and $\|\widetilde{a}\| \leq \|a\|_q$.

c) follows from b).

d) Take $x \in \ell^1(T)^\#$. Then $x \in E^\#$ (Proposition 7.1.1.1 d)), so that

$$|\langle a, x\rangle| = \left|\sum_{t\in T} a(t)x(t)\right| = |\widetilde{a}(x)| \leq \|\widetilde{a}\|\,.$$

Thus

$$\|a\|_\infty = \sup\{|\langle a, x\rangle| \mid x \in \ell^1(T)^\#\} \leq \|\widetilde{a}\|\,.$$

e) For $x \in E$, xe_R, $xe_S \in E$ so that ae_R, $ae_S \in G$ and

$$|\widetilde{ae_R}(x)| = |\widetilde{ae_S}(xe_R)| \leq \|\widetilde{ae_S}\|\,\|xe_R\|_E \leq \|\widetilde{ae_S}\|\,\|x\|_E$$

(Proposition 7.1.1.1 j)). Hence

$$\|\widetilde{ae_R}\| \leq \|\widetilde{ae_S}\|\,.$$

f) We may assume $\widetilde{a} \neq 0$. Take $\alpha \in]0, \|\widetilde{a}\|[$. Then there is an $x \in E^\#$ with

$$\alpha < \widetilde{a}(x)\,.$$

Furthermore, there is an $S \in \mathfrak{S}_T$ such that

$$\alpha < |\widetilde{ae_S}(x)|\,.$$

Hence

$$\alpha < \|\widetilde{ae_S}\|\,,$$

$$\alpha < \sup_{S \in \mathfrak{S}_T} \|\widetilde{ae_S}\| ,$$

$$\|\widetilde{a}\| \leq \sup_{S \in \mathfrak{S}_T} \|\widetilde{ae_S}\| .$$

The reverse inequality follows from e).

g) Let $x' \in E'$ and let $(a_n)_{n \in \mathbb{N}}$ be a sequence in G with

$$\lim_{n \to \infty} \widetilde{a}_n = x' .$$

By d), $(a_n)_{n \in \mathbb{N}}$ converges to an $a \in c_0(T)$. For every $\varepsilon > 0$, there is an $m_\varepsilon \in \mathbb{N}$ such that

$$\|\widetilde{a}_n - \widetilde{a}_p\| < \varepsilon$$

for all $n, p \in \mathbb{N}$, $n \geq m_\varepsilon$, $p \geq m_\varepsilon$. Take $x \in E$. By e),

$$|\widetilde{a_n e_S}(x) - \widetilde{a_p e_S}(x)| \leq \|\widetilde{(a_n - a_p)e_S}\| \, \|x\|_E \leq \|\widetilde{a}_n - \widetilde{a}_p\| \, \|x\|_E$$

for all $n, p \in \mathbb{N}$ and $S \in \mathfrak{S}_T$. It follows that

$$|\widetilde{a_n e_S}(x) - \widetilde{ae_S}(x)| \leq \varepsilon \|x\|_E$$

for all $\varepsilon > 0$, $S \in \mathfrak{S}_T$, and $n \in \mathbb{N}$, $n \geq m_\varepsilon$. Since ε is arbitrary,

$$\lim_{S, \mathfrak{S}_T} \sum_{t \in S} a(t)x(t)$$

exists, $a \in G$, and

$$\lim_{n \to \infty} \widetilde{a_n} = \widetilde{a} .$$

Hence $x' = \widetilde{a}$.

h) Take $x \in E$. It is obvious that x is continuous and $\|x\| \leq \|x\|_E$.

Let $a' \in H'$. By b) and c), there is an $x \in \ell^{\frac{q}{q-1}}(T)$ such that

$$a' = \langle \cdot, x \rangle$$

on $\ell^q(T)$. For $S \in \mathfrak{S}_T$, there is an $a \in H^\#$ such that

$$a = ae_S , \quad \widetilde{a}(xe_S) = \|xe_S\|_E .$$

Hence

$$\|xe_S\|_E = \sum_{t\in S} a(t)x(t) = \langle a, x\rangle = a'(a) \leq \|a'\| .$$

By Proposition 7.1.1.3 a), $x \in E$ and

$$\|x\|_E = \sup_{S\in\mathfrak{S}_T} \|xe_S\|_E \leq \|a'\| .$$

It follows that $x = a'$,

$$\|\dot{x}\| = \|a'\| \geq \|x\|_E ,$$

so that

$$\|\dot{x}\| = \|x\|_E .$$

Since the map

$$E \longrightarrow H' , \quad x \longmapsto \dot{x}$$

is obviously injective, it is an isometry.

i) Let $v : \ell^1(T) \to E$ denote the inclusion map. For $x \in \ell^1(T)$ and $t \in T$,

$$(vx)(t) = x(t) = \langle ue_t, x\rangle = \langle e_t, u'x\rangle = (u'x)(t) .$$

Hence

$$vx = u'x , \quad v = u' .$$

j) Take $\varepsilon > 0$. There is a $b \in \mathbb{K}^{(T)}$ such that

$$\|\widetilde{a} - \widetilde{b}\| < \frac{\varepsilon}{2} .$$

Let $S \in \mathfrak{S}$ such that

$$b = be_S .$$

By e),

$$\|\widetilde{ae_S} - \widetilde{be_S}\| = \|\widetilde{(a-b)e_S}\| \leq \|\widetilde{a} - \widetilde{b}\| < \frac{\varepsilon}{2} ,$$

so that

$$\|\widetilde{ae_S} - \widetilde{a}\| \leq \|\widetilde{ae_S} - \widetilde{be_S}\| + \|\widetilde{b} - \widetilde{a}\| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon .$$

Hence

$$\widetilde{a} = \lim_{S,\mathfrak{S}_T} \widetilde{ae_S}\,.$$

k) Let $\xi, \eta \in \ell^2(T)$. By Proposition 7.1.1.1 a),

$$\langle e_t * \eta | \xi \rangle = \sum_{s \in T} f(t, t^{-1}s)\eta(t^{-1}s)\overline{\xi(s)}$$

for every $t \in T$. Take $\varepsilon > 0$. There is an $A \in \mathfrak{P}_f(T)$ with

$$\sum_{s \in T \setminus A} |\xi(s)|^2 < \frac{\varepsilon}{2(1 + \|\eta\|_2)}\,.$$

For every $s \in A$, there is a $B_s \in \mathfrak{P}_f(T)$ such that

$$|\eta(t^{-1}s)| \mathrm{Card}\, A < \frac{\varepsilon}{2(1 + \|\xi\|_2)}$$

for every $t \in T \setminus B_s$. Put

$$B := \bigcup_{s \in A} B_s \in \mathfrak{P}_f(T)\,.$$

For $t \in T \setminus B$,

$$|\langle e_t * \eta | \xi \rangle| = \left| \sum_{s \in T} f(t, t^{-1}s)\eta(t^{-1}s)\overline{\xi(s)} \right| \leq$$

$$\leq \sum_{s \in A} |\eta(t^{-1}s)|\,|\xi(s)| + \sum_{s \in T \setminus A} |\eta(t^{-1}s)|\,|\xi(s)| \leq$$

$$\leq \left(\sum_{s \in A} |\eta(t^{-1}s)|^2 \right)^{\frac{1}{2}} \|\xi\|_2 + \|\eta\|_2 \left(\sum_{s \in T \setminus A} |\xi(s)|^2 \right)^{\frac{1}{2}} \leq$$

$$\leq \frac{\varepsilon}{2(1 + \|\xi\|_2)} \|\xi\|_2 + \|\eta\|_2 \frac{\varepsilon}{2(1 + \|\eta\|_2)} < \varepsilon\,.$$

Hence $\langle e_\cdot * \eta | \xi \rangle \in c_0(T)$.

Let $y \in \mathcal{L}^1(\ell^2(T))$ and let

$$\sum_{n \in N} \theta_n \langle \cdot | \xi_n \rangle \eta_n$$

be a Schatten decomposition of y. Then for any $t \in T$,

$$\sum_{n \in N} |\theta_n|\,|\langle e_t * \eta_n | \xi_n \rangle| \leq \sum_{n \in N} |\theta_n| \sum_{s \in T} |\eta_n(t^{-1}s)|\,|\xi_n(s)| \leq$$

$$\leq \sum_{n\in N} |\theta_n| \|\eta_n\|_2 \|\xi_n\|_2 = \sum_{n\in N} |\theta_n| = \|y\|_1 .$$

Hence the map

$$T \longrightarrow \mathbb{K}, \quad t \longmapsto \sum_{n\in N} \theta_n \langle e_t * \eta_n | \xi_n \rangle$$

is well–defined, belongs to $c_0(T)$, and its norm is at most $\|y\|_1$. We define ψy to be the above map. Then ψ is an operator with norm at most 1. Let 1 be the neutral element of T. Since

$$|\langle e_1 * e_1 | e_1 \rangle| = 1 ,$$

the norm of ψ is 1.

Take $s, t \in T$. Then

$$\langle e_t * e_1 | e_s \rangle = f(t, t^{-1}s) e_1(t^{-1}s) = f(s,1)\delta_{st}$$

so that

$$\psi(\langle \cdot | e_1 \rangle e_s) = f(s,1) e_s .$$

Hence $\mathbb{K}^{(T)} \subset \operatorname{Im} \psi$.

Let $\xi, \eta \in \ell^2(T)$, $S \in \mathfrak{S}_T$, and $x \in E$. For $t \in T \backslash S$,

$$\langle e_t * (\eta e_S) | \xi e_S \rangle = \sum_{s\in T} f(t, t^{-1}s) \eta(t^{-1}s) e_S(t^{-1}s) \overline{\xi(s)} e_S(s) = 0 ,$$

so that

$$a := \psi(\langle \cdot | \xi e_S \rangle \eta e_S) \subset \mathbb{K}^{(T)} \subset H .$$

Moreover,

$$\langle \widetilde{a}, x \rangle = \sum_{t\in T} \langle e_t * (\eta e_S) | \xi e_S \rangle x(t) =$$

$$= \sum_{t\in T} \left(\sum_{s\in T} f(t, t^{-1}s) \eta(t^{-1}s) e_S(t^{-1}s) \overline{\xi(s)} e_S(s) \right) x(t) =$$

$$= \sum_{s\in T} \left(\sum_{t\in T} f(t, t^{-1}s) \eta(t^{-1}s) e_S(t^{-1}s) x(t) \right) \overline{\xi(s)} e_S(s) =$$

$$= \sum_{s \in T} (x*(\eta e_S))(s)\overline{\xi(s)} e_S(s) = \langle \vec{x}(\eta e_S) | \xi e_S \rangle =$$

$$= \langle \langle \cdot | \xi e_S \rangle \eta e_S , \vec{x} \rangle .$$

It follows

$$\|\tilde{a}\| = \sup\{|\langle \tilde{a}, x \rangle| \mid x \in E, \|\vec{x}\| \leq 1\} =$$

$$= \sup\{|\langle \langle \cdot | \xi e_S \rangle \eta e_S, \vec{x} \rangle| \mid x \in E, \|\vec{x}\| \leq 1\} \leq$$

$$\leq \|\langle \cdot | \xi e_S \rangle \eta e_S\|_1$$

(Theorem 6.1.6.6). Since

$$\lim_{S, \mathfrak{S}_T} \langle \cdot | \xi e_S \rangle \eta e_S = \langle \cdot | \xi \rangle \eta$$

and since, by g), H is complete, we get

$$\psi(\langle \cdot | \xi \rangle \eta) \in H ,$$

$$\widetilde{\langle \psi(\langle \cdot | \xi \rangle \eta), x \rangle} = \langle \langle \cdot | \xi \rangle \eta, \vec{x} \rangle .$$

Let $y \in \mathcal{L}^1(\ell^2(T))$ and let

$$\sum_{n \in N} \theta_n \langle \cdot | \xi_n \rangle \eta_n$$

be a Schatten decomposition of y. Then by the above,

$$\psi y = \sum_{n \in N} \theta_n \psi(\langle \cdot | \xi_n \rangle \eta_n) \in H$$

and

$$\widetilde{\langle \psi_0 y, x \rangle} = \sum_{n \in N} \theta_n \widetilde{\langle \psi \langle \cdot | \xi_n \rangle \eta_n, x \rangle} =$$

$$= \sum_{n \in N} \theta_n \langle \langle \cdot | \xi_n \rangle \eta_n, \vec{x} \rangle = \langle y, \vec{x} \rangle$$

for every $x \in E$. Hence $\operatorname{Im} \psi \subset H$ and

$$\|\widetilde{\psi_0 y}\| = \sup\{|\langle \widetilde{\psi_0 y}, x\rangle| \ | x \in E, \|\vec{x}\| \leq 1\} =$$

$$= \sup\{|\langle y, \vec{x}\rangle| \ | x \in E, \|\vec{x}\| \leq 1\} \leq \|y\|_1 .$$

Thus ψ_0 is continuous with norm 1. Moreover,

$$\psi_o' x = \vec{x}$$

for every $x \in E$. ■

Theorem 7.1.1.5 (7) *Let T be a group, 1 its neutral element, $p \in [1,2]$, and*

$$f : T \times T \longrightarrow \{\alpha \in \mathbb{K} | \ |\alpha| = 1\}$$

such that

$$f(1,1) = 1$$

and

$$f(r,s)f(rs,t) = f(r,st)f(s,t)$$

for all $r,s,t \in T$. We set

$$x*y : T \longrightarrow \mathbb{K}, \quad t \longmapsto \sum_{s \in T} f(ts^{-1}, s)x(ts^{-1})y(s)$$

for all $x, y \in \ell^p(T)$ and denote by E the set of $x \in \ell^p(T)$ such that $x*y \in \ell^p(T)$ for every $y \in \ell^p(T)$ and such that the map

$$\vec{x} : \ell^p(T) \longrightarrow \ell^p(T), \quad y \longmapsto x*y$$

is continuous. Put

$$\vec{E} := \left\{ \vec{x} \,\middle|\, x \in E \right\} .$$

a) $t \in T \Rightarrow f(t,1) = f(1,t) = 1$, $f(t,t^{-1}) = f(t^{-1},t)$.

b) $x \in \ell^p(T) \Rightarrow e_1 * x = x * e_1 = x$.

c) $x, y \in E$, $z \in \ell^p(T) \Rightarrow (x*y) * z = x*(y*z)$.

d) $x, y \in E \Rightarrow x*y \in E$, $\overrightarrow{x*y} = \vec{x}\vec{y}$.

e) *E endowed with the bilinear map*

$$E \times E \longrightarrow E\,, \quad (x,y) \longmapsto x*y$$

is a unital algebra with e_1 as unit.

f) *The following are equivalent:*

f_1) $s,t \in T \Rightarrow e_s*e_t = e_t*e_s$.

f_2) *T is commutative and*

$$s,t \in T \Longrightarrow f(s,t) = f(t,s)\,.$$

f_3) $x,y \in \ell^p(T) \Rightarrow x*y = y*x$.

f_4) *The algebra E is commutative.*

g) *$\vec{E}$ is a closed unital subalgebra of $\mathcal{L}(\ell^p(T))$ (and therefore a unital Banach algebra), the map*

$$E \longrightarrow \vec{E}\,, \quad x \longmapsto \vec{x}$$

is an isomorphism of unital algebras, and its inverse is continuous with norm 1, where E is endowed with the norm induced by $\ell^p(T)$.

h) *If $p = 2$ and T is infinite, then*

$$\vec{E} \cap \mathcal{K}(\ell^2(T)) = \{0\}\,.$$

i) *We denote by F the set of $x \in \ell^p(T)$ such that $y*x \in \ell^p(T)$ for every $y \in \ell^p(T)$ and such that the map*

$$\overleftarrow{x} : \ell^p(T) \longrightarrow \ell^p(T)\,, \quad y \longmapsto y*x$$

is continuous.

i_1) *If $u \in \{\vec{e_t} | t \in T\}^c$ and $x := ue_1$ then $x \in F$ and $\overleftarrow{x} = u$.*

i_2) $\{\vec{e_t} | t \in T\}^c = (\vec{E})^c = \{\overleftarrow{y} | y \in F\}$.

i_3) $\{\vec{e_t} | t \in T\}^{cc} = (\vec{E})^{cc} = \vec{E}$.

i_4) *If T is commutative and $f(s,t) = f(t,s)$ for all $s,t \in T$, then $E = F$, $\vec{x} = \overleftarrow{x}$ for all $x \in E$ and $(\vec{E})^c = \overleftarrow{E}$.*

i_5) *If* T *is finite then the map*

$$v : \vec{E} \longrightarrow (\vec{E})^c, \quad \vec{x} \longmapsto \overleftarrow{x}$$

is linear and bijective and

$$v(\vec{x}\vec{y}) = (v\vec{y})(v\vec{x})$$

for all $x, y \in E$.

j) *The following are equivalent for every* $x \in E$:

j_1) $x \in E^c$.

j_2) $t \in T \Rightarrow x*e_t = e_t*x$.

j_3) $s, t \in T \Rightarrow f(ts^{-1}, s)x(ts^{-1}) = f(s, s^{-1}t)x(s^{-1}t)$.

j_4) $s, t \in T \Rightarrow x(s^{-1}ts) = \overline{f(s, s^{-1}ts)}f(t, s)x(t)$.

j_5) $y \in \ell^q(T) \Rightarrow x*y = y*x$.

k) *Let* $\mathfrak{R}$ *be the set of conjugacy classes of* T *and* $\mathfrak{Q}$ *the set of finite* $Q \in \mathfrak{R}$ *such that*

$$\overline{f(r, r^{-1}tr)}f(t, r) = \overline{f(s, s^{-1}ts)}f(t, s)$$

for every $t \in Q$ *and for all* $r, s \in T$ *with*

$$r^{-1}tr = s^{-1}ts.$$

An element $x \in E$ *belongs to* E^c *iff* $x = 0$ *on* $T \setminus \bigcup_{Q \in \mathfrak{Q}}$ *and*

$$(s, t) \in T \times Q \Longrightarrow x(s^{-1}ts) = \overline{f(s, s^{-1}ts)}f(t, s)x(t)$$

for every $Q \in \mathfrak{Q}$. *If* $\mathfrak{Q}$ *is finite, then*

$$\operatorname{Dim} E^c = \operatorname{Card} \mathfrak{Q}.$$

If $\{t\} \in \mathfrak{R}$ *for a* $t \in T$, *then* $\{t\} \in \mathfrak{Q}$ *iff*

$$f(s, t) = f(t, s)$$

for all $s \in T$. *If* T *is commutative and*

$$S := \{s \in T | t \in T \Longrightarrow f(s, t) = f(t, s)\}$$

then

$$\operatorname{Dim} E^c = \operatorname{Card} S.$$

l) *For* $t \in T$ *and* $m, n \in \mathbb{N}$,

$$f(t^m, t^n) = f(t^n, t^m).$$

m) *If*

$$\varphi : T \longrightarrow \{\alpha \in \mathbb{K} \,|\, |\alpha| = 1\}$$

is a group homomorphism then

$$(x\varphi) * (y\varphi) = (x * y)\varphi$$

for all $x, y \in \ell^p(T)$. *In particular*

$$x \in E \Longrightarrow x\varphi \in E\,,\ \|\overrightarrow{x\varphi}\| = \|\vec{x}\|\,.$$

a) If we set $s = 1$ in the equality for f, then we obtain

$$f(r,1)f(r,t) = f(r,t)f(1,t)\,,$$

so that

$$f(r,1) = f(1,t)$$

for all $r, t \in T$. It follows that

$$f(r,1) = f(1,1) = 1\,,$$

$$f(1,t) = f(1,1) = 1$$

for all $r, t \in T$.

Putting $r = t$, $s = t^{-1}$ in the equality for f we deduce that

$$f(t,t^{-1})f(1,t) = f(t,1)f(t^{-1},t)\,.$$

By the above,

$$f(t,t^{-1}) = f(t^{-1},t)\,.$$

b) By a) and Proposition 7.1.1.1 a),

$$(e_1 * x)(t) = f(1,t)x(t) = x(t)\,,$$

$$(x * e_1)(t) = f(t,1)x(t) = x(t)$$

for every $t \in T$, so that

$$e_1 * x = x * e_1 = x\,.$$

c) Let $t \in T$. By Proposition 7.1.1.1 a), for $s \in T$,

$$((x*y)*e_t)(s) = f(st^{-1}, t)(x*y)(st^{-1}) =$$

$$= \sum_{r \in T} f(st^{-1}, t) f(st^{-1}r^{-1}, r) x(st^{-1}r^{-1}) y(r) =$$

$$= \sum_{r \in T} f(st^{-1}r^{-1}, rt) f(r, t) x(st^{-1}r^{-1}) y(r) =$$

$$= \sum_{q \in T} f(sq^{-1}, q) f(qt^{-1}, t) x(sq^{-1}) y(qt^{-1}) =$$

$$= \sum_{q \in T} f(sq^{-1}, q) x(sq^{-1})(y*e_t)(q) = (x * (y*e_t))(s)\,.$$

Hence

$$(x*y)*e_t = x*(y*e_t)\,.$$

This implies that

$$(x*y)*(ze_A) = x*(y*(ze_A))$$

for every $A \in \mathfrak{P}_f(T)$. Let $\mathfrak{F}$ be the upper section filter of $\mathfrak{P}_f(T)$. Since

$$\|x*(y*(ze_A)) - x*(y*z)\|_p = \|x*(y*(ze_A - z))\|_p \leq$$

$$\leq \|\vec{x}\|\, \|y*(ze_A - z)\|_p \leq \|\vec{x}\|\, \|\vec{y}\|\, \|ze_A - z\|_p$$

for every $A \in \mathfrak{P}_f(T)$, it follows that

$$\lim_{A, \mathfrak{F}} x*(y*(ze_A)) = x*(y*z)$$

in $\ell^p(T)$. Thus for $t \in T$,

$$((x*y)*z)(t) = \sum_{s \in T} f(ts^{-1}, s)(x * y)(ts^{-1}) z(s) =$$

$$= \lim_{A,\mathfrak{F}}((x*y)*(ze_A))(t) = \lim_{A,\mathfrak{F}}(x*(y*(ze_A)))(t) =$$

$$= (x*(y*z))(t)\,.$$

Hence

$$(x*y)*z = x*(y*z)\,.$$

d) follows from c).

e) follows from b) and c).

$f_1 \Rightarrow f_2$. Take $s,t \in T$. By Proposition 7.1.1.1 b),

$$f(s,t)e_{st} = e_s*e_t = e_t*e_s = f(t,s)e_{ts}\,.$$

Hence T is commutative and

$$f(s,t) = f(t,s)\,.$$

$f_2 \Rightarrow f_3$. For $t \in T$,

$$(x*y)(t) = \sum_{s\in T} f(ts^{-1},s)x(ts^{-1})y(s) =$$

$$= \sum_{r\in T} f(r,r^{-1}t)x(r)y(r^{-1}t) =$$

$$= \sum_{r\in T} f(tr^{-1},r)y(tr^{-1})x(r) = (y*x)(t)\,,$$

so that

$$x*y = y*x\,.$$

$f_3 \Rightarrow f_4 \Rightarrow f_1$ is trivial.

g) By b),d), and e), the map

$$E \longrightarrow \mathcal{L}(\ell^p(T))\,, \quad x \longmapsto \vec{x}$$

is a unital algebra homomorphism. By Proposition 7.1.1.1 g), this map is injective, $\vec{E}$ is closed in $\mathcal{L}(\ell^p(T))$, and the map

$$\vec{E} \longrightarrow E\,, \quad \vec{x} \longmapsto x$$

is continuous with norm 1. In particular, $\vec{E}$ is a closed unital subalgebra of $\mathcal{L}(\ell^p(T))$.

h) follows from Proposition 3.1.2.18 b).

i_1) For $t \in T$,

$$e_t * x = \vec{e_t} x = \vec{e_t} u e_1 = u \vec{e_t} e_1 = u e_t .$$

If $y \in \ell^2(T)$ and $s \in T$, then

$$uy = u\Big(\sum_{t\in T} y(t)e_t\Big) = \sum_{t\in T} y(t) u e_t = \sum_{t\in T} y(t) e_t * x ,$$

$$(uy)(s) = \Big(\sum_{t\in T} y(t) e_t * x\Big)(s) = \sum_{t\in T} y(t)(e_t * x)(s) =$$

$$= \sum_{t\in T} y(t) f(t, t^{-1}s) x(t^{-1}s) = \sum_{r\in T} f(sr^{-1}, r) y(sr^{-1}) x(r) = (y * x)(s)$$

(Proposition 7.1.1.1 a)). We deduce that $uy = y * x$, $x \in F$, and $\overleftarrow{x} = u$.

i_2) Take $x \in E$ and $y \in F$. By c), for $t \in T$,

$$\vec{x}\overleftarrow{y} e_t = \vec{x}(e_t * y) = x * (e_t * y) = (x * e_t) * y =$$

$$= \overleftarrow{y}(x * e_t) = \overleftarrow{y}(\vec{x} e_t) = \overleftarrow{y}\vec{x} e_t ,$$

so that

$$\vec{x}\overleftarrow{y} = \overleftarrow{y}\vec{x} ,$$

$$\overleftarrow{y} \in (\vec{E})^c ,$$

$$\{\overleftarrow{y} | y \in F\} \subset (\vec{E})^c \subset \{\vec{e_t} | t \in T\}^c .$$

By i_1),

$$\{\vec{e_t} | t \in T\}^c \subset \{\overleftarrow{y} | y \in F\} ,$$

so that

$$\{\vec{e_t} | t \in T\}^c = (\vec{E})^c = \{\overleftarrow{y} | y \in F\} .$$

i_3) Let $u \in \{\vec{e_t} | t \in T\}^{cc}$. Put

$$x := ue_1 .$$

By b) and i_2), for $y \in F$,

$$x*y = \overleftarrow{y}x = \overleftarrow{y}ue_1 = u\overleftarrow{y}e_1 = u(e_1*y) = uy .$$

Hence $x \in E$ and $\vec{x} = u$. Thus

$$\{\vec{e_t} \,|t \in T\}^{cc} \subset \vec{E} .$$

By i_2),

$$\{\vec{e_t}|t \in T\}^{cc} = (\vec{E})^{cc} = \vec{E} .$$

i_4) By $f_2 \Rightarrow f_3$, $E = F$ and $\vec{x} = \overleftarrow{x}$ for every $x \in E$, so that the assertion follows from i_2).

i_5) By g) and i_2), the map v is well-defined. It is obviously bijective and linear. By c), for $z \in \ell^p(T)$,

$$\overleftarrow{x*y}z = z*(x*y) = \overleftarrow{y}(z*x) = \overleftarrow{y}\overleftarrow{x}z ,$$

so that by g),

$$v(\vec{x}\,\vec{y}) = v(\overrightarrow{x*y}) = \overleftarrow{x*y} = \overleftarrow{y}\,\overleftarrow{x} = (v\vec{y})(v\vec{x}) .$$

$j_5 \Rightarrow j_1 \Rightarrow j_2$ and $j_3 \Leftrightarrow j_4$ are trivial.

$j_2 \Rightarrow j_3$ follows from Proposition 7.1.1.1 a).

$j_3 \Rightarrow j_5$. For $t \in T$,

$$(x*y)(t) = \sum_{s \in T} f(ts^{-1}, s)x(ts^{-1})y(s) =$$

$$= \sum_{s \in T} f(s, s^{-1}t)x(s^{-1}t)y(s) = (y*x)(t) .$$

k) Take $Q \in \mathfrak{R}$ and $t \in Q$. Let $x \in E^c$. By $j_1 \Rightarrow j_4$),

$$x(s^{-1}ts) = \overline{f(s, s^{-1}ts)}f(t, s)x(t) ,$$

for every $s \in T$, so that

$$|x(s^{-1}ts)| = |x(t)| .$$

If Q is infinite, then $x \in \ell^p(T)$ implies $x = 0$ on Q. Assume Q finite and let $r, s \in T$ such that

$$r^{-1}tr = s^{-1}ts\,.$$

We obtain

$$\overline{f(r,r^{-1}tr)}f(t,r)x(t) = x(r^{-1}tr) =$$

$$= x(s^{-1}ts) = \overline{f(s,s^{-1}ts)}f(t,s)x(t)\,.$$

If $Q \notin \mathfrak{Q}$, it follows that $x = 0$ on Q. Hence

$$\mathrm{Dim}\, E^c \leq \mathrm{Card}\, \mathfrak{Q}\,.$$

If $Q \in \mathfrak{Q}$, then the map

$$Q \longrightarrow \mathbb{K}\,, \quad s^{-1}ts \longmapsto \overline{f(s,s^{-1}ts)}f(t,s)$$

is well–defined. Extending it by 0 on $T\backslash Q$, we get, by $\mathrm{j}_4 \Rightarrow \mathrm{j}_1$, an element of E^c. Hence

$$\mathrm{Dim}\, E^c \geq \mathrm{Card}\, \mathfrak{Q}\,,$$

$$\mathrm{Dim}\, E^c = \mathrm{Card}\, \mathfrak{Q}\,,$$

$\{t\} \in \mathfrak{R}$ implies $s^{-1}ts = t$ for all $s \in T$ and the final assertions follow.

l) We may assume $m < n$. Since

$$f(t^m, t^{n-m})f(t^n, t^m) = f(t^m, t^n)f(t^{n-m}, t^m)\,,$$

the relation follows from a) by induction.

m) For $t \in T$,

$$((x\varphi) * (y\varphi))(t) = \sum_{s\in T} f(ts^{-1}, s)x(ts^{-1})\varphi(ts^{-1})y(s)\varphi(s) =$$

$$= \varphi(t)\sum_{s\in T} f(ts^{-1}, s)x(ts^{-1})y(s) = ((x * y)\varphi)(t)\,.$$

Since

$$y \in \ell^p(T) \Longrightarrow y\varphi \in \ell^p(T)\,,\ \|y\varphi\|_p = \|y\|_p$$

it follows

$$x \in E \Longrightarrow x\varphi \in E\,,\ \|\overrightarrow{x\varphi}\| = \|\vec{x}\|\,. \qquad \blacksquare$$

Proposition 7.1.1.6 (7) *Let T be a group. Take*

$$f : T \times T \longrightarrow \{\alpha \in \mathbb{K} |\ |\alpha| = 1\},$$

$$g : T \longrightarrow \{\alpha \in \mathbb{K} |\ |\alpha| = 1\},$$

and $p \in [1,2]$. We define

$$x*y : T \longrightarrow \mathbb{K}, \quad t \longmapsto \sum_{s \in T} f(ts^{-1}, s)x(ts^{-1})y(s),$$

$$x^* : T \longrightarrow \mathbb{K}, \quad t \longmapsto g(t)\,\overline{x(t^{-1})}$$

for all $x, y \in \ell^p(T)$.

a) *The following are equivalent:*

- a_1) $x, y \in \ell^p(T)$, $x*y \in \ell^p(T) \Rightarrow (x*y)^* = y^**x^*$.
- a_2) $s, t \in T \Rightarrow (e_s*e_t)^* = e_t^**e_s^*$.
- a_3) $s, t \in T \Rightarrow f(s,t)g(s)g(t) = \overline{f(t^{-1}, s^{-1})}g(st)$.

b) $t \in T \Rightarrow e_t^* = g(t^{-1})e_{t^{-1}}$.

c) *The following are equivalent:*

- c_1) $x \in \ell^p(T) \Rightarrow x^{**} = x$.
- c_2) $t \in T \Rightarrow e_t^{**} = e_t$.
- c_3) $t \in T \Rightarrow g(t) = g(t^{-1})$.
- c_4) *$\ell^p(T)$ endowed with the map*

$$\ell^p(T) \longrightarrow \ell^p(T), \quad x \longmapsto x^*$$

is an involutive Banach space.

d) *Assume the equivalent assertions of c) are fulfilled, define*

$$\langle x|y\rangle = \sum_{t \in T} x(t)\overline{y(t)}$$

for all $x, y \in \ell^p(T)$, and let F denote the set of $x \in \ell^p(T)$ such that $x*y \in \ell^p(T)$ for all $y \in \ell^p(T)$. Then $x^* \in F$ for every $x \in F$ and the following are equivalent:

$d_1)$ $x \in F$, $y, z \in \ell^p(T) \Rightarrow \langle x*y|z\rangle = \langle y|x^**z\rangle$.

$d_2)$ $r, s, t \in T \Rightarrow \langle e_r*e_s|e_t\rangle = \langle e_s|e_r^**e_t\rangle$.

$d_3)$ $s, t \in T \Rightarrow f(s,t)g(s) = \overline{f(s^{-1}, st)}$.

e) *The following are equivalent:*

$e_1)$ $x \in \ell^p(T) \cap \mathbb{R}^T \Rightarrow x^* = x$.

$e_2)$ $t \in T \Rightarrow e_t^* = e_t$.

$e_3)$ $t \in T \Rightarrow t = t^{-1}$, $g(t) = 1$.

f) $x, y \in \ell^p(T) \Rightarrow \langle x^*|y^*\rangle = \langle y|x\rangle$.

$a_1 \Rightarrow a_2$ is trivial.

$a_2 \Rightarrow a_3$. By Proposition 7.1.1.1 b), for $r \in T$,

$$(e_s*e_t)^*(r^{-1}) = g(r^{-1})\overline{(e_s*e_t)(r)} =$$

$$= g(r^{-1})\overline{f(s,t)}\,\overline{e_{st}(r)} = g(t^{-1}s^{-1})\overline{f(s,t)}\delta_{r,st},$$

$$(e_t^**e_s^*)(r^{-1}) = \sum_{q\in T} f(r^{-1}q^{-1}, q)e_t^*(r^{-1}q^{-1})e_s^*(q) =$$

$$= \sum_{q\in T} f(r^{-1}q^{-1}, q)g(r^{-1}q^{-1})\overline{e_t(qr)}g(q)\overline{e_s(q^{-1})} =$$

$$= f(r^{-1}s, s^{-1})g(r^{-1}s)g(s^{-1})\overline{e_t(s^{-1}r)} =$$

$$= f(t^{-1}, s^{-1})g(t^{-1})g(s^{-1})\delta_{r,st},$$

so that

$$f(t^{-1}, s^{-1})g(t^{-1})g(s^{-1}) = \overline{f(s,t)}\,g(t^{-1}s^{-1})$$

$$f(s,t)g(s)g(t) = \overline{f(t^{-1}, s^{-1})}\,g(st).$$

$a_3 \Rightarrow a_1$. For $t \in T$,

$$(y^**x^*)(t) = \sum_{s\in T} f(ts^{-1}, s)y^*(ts^{-1})x^*(s) =$$

$$= \sum_{s\in T} f(ts^{-1}, s)g(ts^{-1})\overline{y(st^{-1})}\, g(s)\overline{x(s^{-1})} =$$

$$= \sum_{s\in T} \overline{f(s^{-1}, st^{-1})}g(t)\,\overline{y(st^{-1})}\,\overline{x(s^{-1})} =$$

$$= g(t)\sum_{r\in T} \overline{f(t^{-1}r^{-1}, r)}\,\overline{x(t^{-1}r^{-1})}\,\overline{y(r)} =$$

$$= g(t)\,\overline{(x*y)(t^{-1})} = (x*y)^*(t)\,,$$

and so

$$(x*y)^* = y^**x^*\,.$$

b) Given $s \in T$,

$$e_t^*(s) = g(s)\,\overline{e_t(s^{-1})} = g(t^{-1})\delta_{t,s^{-1}} =$$

$$= g(t^{-1})\delta_{t^{-1},s} = g(t^{-1})e_{t^{-1}}(s)\,,$$

so that

$$e_t^* = g(t^{-1})e_{t^{-1}}\,.$$

$c_1 \Rightarrow c_2$ is trivial.

$c_2 \Rightarrow c_3$. By b),

$$e_t^{**} = \overline{g(t^{-1})}e_{t^{-1}}^* = \overline{g(t^{-1})}\,g(t)e_t\,,$$

so that

$$g(t) = g(t^{-1})\,.$$

$c_3 \Rightarrow c_1$. For $t \in T$,

$$x^{**}(t) = g(t)\overline{x^*(t^{-1})} = g(t)\,\overline{g(t^{-1})}\,x(t) = x(t)\,.$$

Hence

$$x^{**} = x\,.$$

$c_1 \Leftrightarrow c_4$ is obvious.

$d_1 \Rightarrow d_2$ is trivial.

$d_2 \Rightarrow d_3$. By b) and Proposition 7.1.1.1 b), for $r \in T$,

$$f(r,s)\delta_{rs,t} = \langle f(r,s)e_{rs}|e_t\rangle = \langle e_r * e_s|e_t\rangle =$$

$$= \langle e_s|e_r^* * e_t\rangle = \overline{(e_r^* * e_t)(s)} =$$

$$= \sum_{q\in T} \overline{f(sq^{-1},q)}\,\overline{e_r^*(sq^{-1})}\,\overline{e_t(q)} = \overline{f(st^{-1},t)}\,\overline{e_r^*(st^{-1})} =$$

$$= \overline{f(st^{-1},t)}\,\overline{g(st^{-1})}\,e_r(ts^{-1}) = \overline{f(st^{-1},t)}\,\overline{g(st^{-1})}\delta_{rs,t},$$

so that

$$f(ts^{-1},s) = \overline{f(st^{-1},t)}\,\overline{g(st^{-1})},$$

and

$$f(s,t)g(s) = \overline{f(s^{-1},st)}.$$

$d_3 \Rightarrow d_1$. $\langle y|x^* * z\rangle = \sum_{t\in T} y(t)\,\overline{(x^* * z)(t)} =$

$$= \sum_{t\in T} y(t)\Big(\sum_{s\in T} \overline{f(ts^{-1},s)}\,\overline{x^*(ts^{-1})}\,\overline{z(s)}\Big) =$$

$$= \sum_{t\in T} y(t)\Big(\sum_{s\in T} \overline{f(ts^{-1},s)}\,\overline{g(ts^{-1})}x(st^{-1})\,\overline{z(s)}\Big) =$$

$$= \sum_{t\in T} y(t)\Big(\sum_{s\in T} f(st^{-1},t)x(st^{-1})\,\overline{z(s)}\Big) =$$

$$= \sum_{s\in T} \left(\sum_{t\in T} f(st^{-1},t)x(st^{-1})y(t)\right)\overline{z(s)} =$$

$$= \sum_{s\in T} (x*y)(s)\,\overline{z(s)} = \langle x*y|z\rangle.$$

e) follows from b).

f) $\langle x^*|y^*\rangle = \sum_{t\in T} x^*(t)\,\overline{y^*(t)} = \sum_{t\in T} g(t)\,\overline{x(t^{-1})}\,\overline{g(t)\overline{y(t^{-1})}} =$

$$= \sum_{t\in T} \overline{x(t^{-1})}y(t^{-1}) = \sum_{t\in T} y(t)\overline{x(t)} = \langle y|x\rangle. \qquad \blacksquare$$

Definition 7.1.1.7 (7) *Let T be a group with neutral element denoted by 1. A **Schur function for** T (also called normalized factor set in the literature) is a function*

$$f : T \times T \longrightarrow \{\alpha \in \mathbb{K} |\ |\alpha| = 1\}$$

such that

$$f(1,1) = 1$$

and

$$f(r,s)f(rs,t) = f(r,st)f(s,t)$$

for all $r,s,t \in T$ (Schur, 1904). *We put*

$$\widetilde{f} : T \longrightarrow \{\alpha \in \mathbb{K} |\ |\alpha| = 1\}, \quad t \longmapsto \overline{f(t,t^{-1})}.$$

For all $x, y \in \ell^2(T)$, define

$$x*y : T \longrightarrow \mathbb{K}, \quad t \longmapsto \sum_{s \in T} f(ts^{-1}, s)x(ts^{-1})y(s),$$

$$x^* : T \longrightarrow \mathbb{K}, \quad t \longmapsto \widetilde{f}(t)\,\overline{x(t^{-1})}$$

and denote by $\mathcal{S}_w^{\mathbb{K}}(f)$ (or simply $\mathcal{S}_w(f)$) the set of $x \in \ell^2(T)$ such that $x*y \in \ell^2(T)$ for every $y \in \ell^2(T)$ and such that the map

$$\vec{x}\ : \ell^2(T) \longrightarrow \ell^2(T), \quad y \longmapsto x*y$$

is continuous. We put

$$\vec{A} := \{\vec{x} | x \in A\}$$

for every $A \subset \mathcal{S}_w(f)$. $\mathcal{S}_w(f)$ is a vector subspace of $\ell^2(T)$ which we endow with the norm

$$\mathcal{S}_w(f) \longrightarrow \mathbb{R}_+, \quad x \longmapsto \|\vec{x}\|$$

(Proposition 7.1.1.1 g)). If T is locally finite, then we denote by $\mathcal{S}_c^{\mathbb{K}}(f)$ (or simply $\mathcal{S}_c(f)$) the closure of $\mathbb{K}^{(T)}$ in $\mathcal{S}_w(f)$.

If T' is a group, f' is a Schur function for T', and $u : T \to T'$ is a group homomorphism then

$$T \times T \longrightarrow \{\alpha \in \mathbb{K} | \, |\alpha| = 1\}, \quad (s,t) \longmapsto f'(u(s), u(t))$$

is a Schur function for T.

If T is finite, then

$$\ell^1(T) = \mathcal{S}_c(f) = \mathcal{S}_w(f) = \ell^2(T).$$

If T is infinite (and locally finite for $\mathcal{S}_c(f)$), then the above four spaces are pairwise distinct (Proposition 7.1.2.4).

Proposition 7.1.1.8 *Let T be a group, 1 its neutral element, $\mathcal{F}$ the set of Schur functions for T, and Λ the set of*

$$\lambda \in \{\alpha \in \mathbb{K} | \, |\alpha| = 1\}^T$$

with $\lambda(1) = 1$. For $f \in \mathcal{F}$ and $\lambda \in \Lambda$, put

$$f^* : T \times T \longrightarrow \{\alpha \in \mathbb{K} | \, |\alpha| = 1\}, \quad (s,t) \longmapsto f(t^{-1}, s^{-1}),$$

$$\lambda' : T \times T \longrightarrow \{\alpha \in \mathbb{K} | \, |\alpha| = 1\}, \quad (s,t) \longmapsto \frac{\lambda(s)\lambda(t)}{\lambda(st)},$$

$$\lambda^* : T \longrightarrow \{\alpha \in \mathbb{K} | \, |\alpha| = 1\}, \quad s \longmapsto \lambda(s^{-1}).$$

a) *$\mathcal{F}$ is a subgroup of $\{\alpha \in \mathbb{K} | \, |\alpha| = 1\}^{T \times T}$ such that $\overline{f}$ is the inverse of f for every $f \in \mathcal{F}$.*

b) *If $f, g \in \mathcal{F}$ then $f^* \in \mathcal{F}$, $f^{**} = f$, and*

$$(fg)^* = f^* g^*.$$

c) *$\{\lambda' | \lambda \in \Lambda\}$ is a subgroup of $\mathcal{F}$ and*

$$(\lambda')^* = (\lambda^*)'$$

for every $\lambda \in \Lambda$.

a) is easy to see.

b) For $r, s, t \in T$,

$$f^*(r,s)f^*(rs,t) = f(s^{-1},r^{-1})f(t^{-1},s^{-1}r^{-1}) =$$

$$= f(t^{-1},s^{-1})f(t^{-1}s^{-1},r^{-1}) = f^*(s,t)f^*(r,st)$$

so that $f^* \in \mathcal{F}$. The other assertions are easy to see.

c) Take $\lambda \in \Lambda$. Then

$$\lambda'(1,1) = \frac{\lambda(1)\lambda(1)}{\lambda(1)} = 1\,,$$

$$\lambda'(r,s)\lambda'(rs,t) = \frac{\lambda(r)\lambda(s)}{\lambda(rs)}\,\frac{\lambda(rs)\lambda(t)}{\lambda(rst)} = \frac{\lambda(r)\lambda(s)\lambda(t)}{\lambda(rst)}\,,$$

$$\lambda'(r,st)\lambda'(s,t) = \frac{\lambda(r)\lambda(st)}{\lambda(rst)}\,\frac{\lambda(s)\lambda(t)}{\lambda(st)} = \frac{\lambda(r)\lambda(s)\lambda(t)}{\lambda(rst)} = \lambda'(r,s)\lambda'(rs,t)$$

for all $r, s, t \in T$ so that $\lambda' \in \mathcal{F}$. Moreover,

$$(\lambda^*)'(s,t) = \frac{\lambda^*(s)\lambda^*(t)}{\lambda^*(st)} = \frac{\lambda(s^{-1})\lambda(t^{-1})}{\lambda(t^{-1}s^{-1})} = \lambda'(t^{-1},s^{-1}) = (\lambda')^*(s,t)$$

for all $s, t \in T$. It is easy to see that $\{\lambda' | \lambda \in \Lambda\}$ is a subgroup of $\mathcal{F}$. ■

Proposition 7.1.1.9 (7) *Let f be a Schur function for a group T and $s, t \in T$.*

a) $\widetilde{f}(t) = \widetilde{f}(t^{-1})$.

b) $f(s^{-1},st)f(t,t^{-1}s^{-1}) = f(st,t^{-1}s^{-1}) = \overline{\widetilde{f}(st)}$.

c) $f(s,t)\widetilde{f}(s) = \overline{f(s^{-1},st)}$.

d) $f(s,t)\widetilde{f}(s)\widetilde{f}(t) = \overline{f(t^{-1},s^{-1})}\,\widetilde{f}(st)$.

e) $f(s,t^{-1})\widetilde{f}(t^{-1}) = \overline{f(st^{-1},t)}$.

a) follows from Theorem 7.1.1.5 a).

b) By Theorem 7.1.1.5 a),

$$f(s^{-1},st)f(t,t^{-1}s^{-1}) = f(s^{-1},1)f(st,t^{-1}s^{-1}) = f(st,t^{-1}s^{-1})\,.$$

c) We have

$$f(s,t)f(st,t^{-1}s^{-1}) = f(s,s^{-1})f(t,t^{-1}s^{-1}),$$

so that, by b),

$$f(s,t)\widetilde{f}(s) = f(s,t)\overline{f(s,s^{-1})} = \overline{f(st,t^{-1}s^{-1})}\, f(t,t^{-1}s^{-1}) =$$

$$= \overline{f(s^{-1},st)}\,\overline{f(t,t^{-1}s^{-1})}\, f(t,t^{-1}s^{-1}) = \overline{f(s^{-1},st)}.$$

d) By a) and c),

$$\overline{f(t^{-1},s^{-1})}\,\widetilde{f}(st) = \overline{f(t^{-1},s^{-1})}\,\overline{f(t^{-1}s^{-1},st)} =$$

$$= \overline{f(t^{-1},t)}\,\overline{f(s^{-1},st)} = f(s,t)\widetilde{f}(s)\widetilde{f}(t).$$

e) By a),c), and d),

$$\overline{f(st^{-1},t)} = f(ts^{-1},s)\widetilde{f}(ts^{-1}) =$$

$$= \overline{f(s^{-1},st^{-1})}\;\overline{\widetilde{f}(s^{-1})}\;\widetilde{f}(t) = f(s,t^{-1})\widetilde{f}(t^{-1}).$$

■

7.1.2 Projective Representations

Theorem 7.1.2.1 (7) *Let f be a Schur function for a group T.*

a) $s,t \in T \Rightarrow e_s^* * e_t = \overline{f(s,s^{-1}t)}\, e_{s^{-1}t}\,, \quad e_s * e_t^* = \overline{f(st^{-1},t)} e_{st^{-1}}$.

b) $t \in T \Rightarrow e_t^* * e_t = e_t * e_t^* = e_1$.

c) $\mathcal{S}_w(f)$ *endowed with the bilinear map*

$$\mathcal{S}_w(f) \times \mathcal{S}_w(f) \longrightarrow \mathcal{S}_w(f)\,, \quad (x,y) \longmapsto x*y$$

(Theorem 7.1.1.5 e)) and with the involution

$$\mathcal{S}_w(f) \longrightarrow \mathcal{S}_w(f)\,, \quad x \longmapsto x^*$$

(Propositions 7.1.1.9 a),d), 7.1.1.6 $a_3 \Rightarrow a_1$, $c_3 \Rightarrow c_1$) is a unital W^*-algebra with e_1 as unit and the map

$$\varphi : \mathcal{S}_w(f) \longrightarrow \mathcal{L}(\ell^2(T))\,, \quad x \longmapsto \vec{x}$$

is an injective unital W^*-homomorphism. $(\ell^2(T)\,,\, \varphi\,,\, e_1)$ is the GNS-triple associated to the positive linear form

$$x' : \mathcal{S}_w(f) \longrightarrow \mathbb{K}\,, \quad x \longmapsto \langle x|e_1\rangle\,.$$

d) *If f is real then the map*

$$\overbrace{\mathcal{S}_w^{\mathbb{R}}(f)}^{\circ} \longrightarrow \mathcal{S}_w^{\mathbb{C}}(f)\,, \quad (x,y) \longmapsto x + iy$$

is an isomorphism of involutive unital complex algebras.

e) *If S is a subgroup of T, then $\{xe_S \mid x \in \mathcal{S}_w(f)\}$ is a unital W^*-subalgebra of $\mathcal{S}_w(f)$.*

f) *If T is infinite then*

$$x \in \mathcal{S}_w(f)\,,\, u \in \mathcal{K}(\ell^2(T)) \Rightarrow \|\vec{x}\| \leq \|\vec{x} + u\|\,.$$

g) *If T is infinite, then the restriction of the quotient map*

$$\mathcal{L}(\ell^2(T)) \longrightarrow \mathcal{L}(\ell^2(T))/\mathcal{K}(\ell^2(T))$$

to $\overbrace{\mathcal{S}_w(f)}^{\rightarrow}$ is injective but not surjective.

h) $\ell^1(T)$ *is an involutive unital subalgebra of* $\mathcal{S}_w(f)$. $\ell^1(T)$ *endowed with the induced algebraic structure and with its usual* $\|\cdot\|_1$*-norm is an involutive unital Banach algebra such that every algebra homomorphism* $\ell^1(T) \to \mathbb{K}$ *is positive.*

i) *If* f *is real then the map*

$$\overset{\circ}{\overbrace{\ell^1(T)^{\mathbb{R}}}} \longrightarrow \ell^1(T)^{\mathbb{C}}, \quad (x,y) \longmapsto x + iy$$

is an isomorphism of involutive unital complex algebras.

j) $\overrightarrow{\mathcal{S}_w(f)}$ *is the (unital) von Neumann algebra on* $\ell^2(T)$ *generated by*

$$\overset{\rightarrow}{\overbrace{\{e_t \mid t \in T\}}}.$$

k) *We denote by* F *the set of* $x \in \ell^2(T)$ *such that* $y*x \in \ell^2(T)$ *for every* $y \in \ell^2(T)$ *and such that the map*

$$\overleftarrow{x} : \ell^2(T) \longrightarrow \ell^2(T), \quad y \longmapsto y*x$$

is continuous. Then $F = \mathcal{S}_w(f)$, $\overleftarrow{x} \in \left(\overrightarrow{\mathcal{S}_w(f)}\right)^c$ *for every* $x \in \mathcal{S}_w(f)$, *and the map*

$$u : \overrightarrow{\mathcal{S}_w(f)} \longrightarrow \left(\overrightarrow{\mathcal{S}_w(f)}\right)^c, \ \vec{x} \longmapsto \overleftarrow{x}$$

is an isometry of involutive Banach spaces such that

$$u(\vec{x}\vec{y}) = (u\vec{y})(u\vec{x})$$

for all $x, y \in \mathcal{S}_w(f)$.

l) *If* $\mathbb{K} = \mathbb{C}$ *and if* $\mathcal{S}_w(f)$ *is a Gelfand* C^**-algebra, then* $\overrightarrow{\mathcal{S}_w(f)}$ *is a maximal Gelfand* C^**-subalgebra of* $\mathcal{L}(\ell^2(T))$.

m) *If* $\mathbb{K} = \mathbb{C}$ *and if* $y \in \ell^1(T)$ *such that*

$$\sum_{t \in T \setminus \{1\}} |y(t)| \leq y(1)$$

then the linear form

$$\ell : \mathcal{S}_w(f) \longrightarrow \mathbb{K}\,, \quad x \longmapsto \sum_{t\in T} x(t)y(t)$$

is positive. In particular, if $T \neq \{1\}$, *the positive linear form*

$$x' : \mathcal{S}_w(f) \longrightarrow \mathbb{K}\,, \quad x \longmapsto \langle x|e_1\rangle$$

from c) is not an extreme point of $\tau(\mathcal{S}_w(f))$ *and the representations of* $\mathcal{S}_w(f)$ *associated to it is not irreducible.*

n) *Let* A *be a set of generators of* T *such that*

$$t^2 = 1\,, \quad \widetilde{f}(t) = 1$$

for every $t \in A$. *If* E *is a real* C^**-algebra with*

$$\operatorname{Dim}\operatorname{Re} E = 1$$

and $u : \mathcal{S}_c^{\mathbb{R}}(f) \to E$ *is an involutive algebra homomorphism then*

$$\operatorname{Im} u \subset \operatorname{Re} E\,.$$

o) $x, y \in \mathcal{S}_w(f) \Longrightarrow x{*}y(1) = y * x(1)\,, \quad x * x^*(1) = \|x\|_2^2\,.$

a) By Propositions 7.1.1.1 b), 7.1.1.6 b), and 7.1.1.9 c),e),

$$e_s^*{*}e_t = \widetilde{f}(s^{-1})e_{s^{-1}}{*}e_t = \widetilde{f}(s^{-1})f(s^{-1},t)e_{s^{-1}t} = \overline{f(s,s^{-1}t)}e_{s^{-1}t}\,,$$

$$e_s{*}e_t^* = e_s{*}\widetilde{f}(t^{-1})e_{t^{-1}} = \widetilde{f}(t^{-1})f(s,t^{-1})e_{st^{-1}} = \overline{f(st^{-1},t)}\; e_{st^{-1}}\,.$$

b) By a) and Theorem 7.1.1.5 a),

$$e_t^*{*}e_t = \overline{f(t,1)}\; e_1 = e_1\,,$$

$$e_t{*}e_t^* = \overline{f(1,t)}\; e_1 = e_1\,.$$

c) By Theorem 7.1.1.5 e),g), $\mathcal{S}_w(f)$ is a unital Banach algebra with e_1 as unit and φ is a norm preserving unital algebra homomorphism. By Proposition 7.1.1.6 $a_3 \Rightarrow a_1\,, c_3 \Rightarrow c_1\,, d_3 \Rightarrow d_1$ and Proposition 7.1.1.9 d),a),c), $\mathcal{S}_w(f)$ is an involutive algebra and φ is involutive. By Theorem 7.1.1.5 i_3) and Corollary 6.3.5.5, $\overrightarrow{\mathcal{S}_w(f)}$ is a von Neumann algebra on $\ell^2(T)$, so that $\mathcal{S}_w(f)$

is a unital W^*–algebra and φ is an injective W^*–homomorphism (Corollary 4.4.4.9 a $\Rightarrow$ b).

For $x \in \mathcal{S}_w(f)$,

$$x'(x^*) = \langle x^*|e_1\rangle = x^*(1) = \widetilde{f}(1)\overline{x(1)} = \overline{x(1)} = \overline{x'(x)}\,,$$

$$x'(x^* * x) = \langle x^* * x|e_1\rangle = \langle x|x * e_1\rangle = \langle x|x\rangle = \|x\|_2^2$$

(Proposition 7.1.1.9 a),c), 7.1.1.6 $c_3 \Rightarrow c_1$, $d_3 \Rightarrow d_1$, Theorem 7.1.1.5 b)). Hence x' is positive, $N_{x'} = \{0\}$, and

$$\mathcal{S}_w(f)/N_{x'} = \mathcal{S}_w(f)\,.$$

Since $\mathcal{S}_w(f)$ is dense in $\ell^2(T)$, $(\ell^2(T), \varphi, e_1)$ is the GNS–triple associated to x' (Theorem 5.4.1.2 k)).

d) Put

$$u : \mathcal{S}_w^{\mathbb{C}}(f) \longrightarrow \mathcal{S}_w^{\mathbb{C}}(f)\,, \quad x \longmapsto \overline{x}\,.$$

For $x, y \in \mathcal{S}_w^{\mathbb{C}}(f)$ and $t \in T$,

$$(u(x*y))(t) = \overline{(x*y)(t)} = \sum_{s\in T} f(ts^{-1}, s)\ \overline{x(ts^{-1})}\ \overline{y(s)} =$$

$$= (\overline{x}*\overline{y})(t) = ((ux)*(uy))(t)\,,$$

$$(ux^*)(t) = \overline{x^*(t)} = \widetilde{f}(t)x(t^{-1}) = \overline{x}^*(t) = (ux)^*(t)$$

(Proposition 7.1.1.9 a)), so that

$$u(x*y) = (ux)*(uy)\,, \quad (ux^*) = (ux)^*\,.$$

Hence u is a conjugate involution and the assertion follows from Proposition 2.3.1.43 a).

e) By Proposition 7.1.1.1 i), $\{xe_S|x \in \mathcal{S}_w(f)\}$ is a subalgebra of $\mathcal{S}_w(f)$ and it is easy to see that it is unital and involutive. It remains to show, that

$$\{\overrightarrow{xe_S}|x \in \mathcal{S}_w(f)\}$$

is closed in $\{\vec{x}|x \in \mathcal{S}_w(f)\}_{\ell^2(T)}$ (c), Proposition 6.3.4.2 d $\Leftrightarrow$ a, and Corollary 4.4.4.10). Let $x_0 \in \mathcal{S}_w(f)$ and $\mathfrak{F}$ a filter on $\{\overrightarrow{xe_S}|x \in \mathcal{S}_w(f)\}$ converging to $\overrightarrow{x_0}$ in the topology of pointwise convergence. Take $t \in T\backslash S$ and $x \in \mathcal{S}_w(f)$. Then

$$(\overrightarrow{xe_S}\, e_{t^{-1}})(1) = \sum_{s\in T} f(s^{-1},s)x(s^{-1})e_S(s^{-1})e_{t^{-1}}(s) =$$

$$= f(t,t^{-1})x(t)e_S(t) = 0\,.$$

It follows that

$$f(t,t^{-1})x_0(t) = (\overrightarrow{x_0}e_{t^{-1}})(1) =$$

$$= \lim_{x,\mathfrak{F}}(\overrightarrow{xe_S}e_{t^{-1}})(1) = 0\,,$$

$$x_0(t) = 0\,.$$

Hence $x_0 = x_0e_S$ and $\{\overrightarrow{xe_S}|x\in\mathcal{S}_w(f)\}$ is closed in $\{\overrightarrow{x}|x\in\mathcal{S}_w(f)\}_{\ell^2(T)}$.

f) follows from Theorem 7.1.1.5 h) and Corollary 4.2.6.7.

g) The injectivity of the map follows from Theorem 7.1.1.5 h). By Proposition 5.5.1.27, $\mathcal{L}(\ell^2(T))/\mathcal{K}(\ell^2(T))$ is not a W^*-algebra. By c), $\mathcal{S}_w(f)$ is a W^*-algebra, so that the map cannot be surjective.

h) By Proposition 7.1.1.1 d), $\ell^1(T)\subset\mathcal{S}_w(f)$. Take $x,y\in\ell^1(T)$. For $t\in T$,

$$|x*y)(t)| = \left|\sum_{s\in T} f(ts^{-1},s)x(ts^{-1})y(s)\right| \leq$$

$$\leq \sum_{s\in T}|x(ts^{-1})|\,|y(s)|\,,$$

$$|x^*(t)| = |\widetilde{f}(t)\overline{x(t^{-1})}| = |x(t^{-1})|\,,$$

so that

$$\sum_{t\in T}|(x*y)(t)| \leq \sum_{t\in T}\sum_{s\in T}|x(ts^{-1})|\,|y(s)| =$$

$$= \sum_{s\in T}\left(\sum_{t\in T}|x(ts^{-1})|\right)|y(s)| = \|x\|_1\|y\|_1\,,$$

$$\sum_{t\in T}|x^*(t)| = \sum_{t\in T}|x(t^{-1})| = \|x\|_1\,.$$

Hence $x*y$, $x^*\in\ell^1(T)$ and

$$\|x*y\|_1 \leq \|x\|_1 \|y\|_1 , \quad \|x^*\|_1 = \|x\|_1 .$$

Thus $\ell^1(T)$ is an involutive unital subalgebra of $\mathcal{S}_w(f)$ and $\ell^1(T)$ endowed with the induced algebraic structure and with its usual $\|\cdot\|_1$–norm is an involutive unital Banach algebra.

Let $x' : \ell^1(T) \to \mathbb{K}$ be an algebra homomorphism. By Proposition 2.2.4.19, x' is continuous with $\|x'\| = 0$ or $\|x'\| = 1$. We may assume $\|x'\| = 1$. Take $t \in T$. By b),

$$1 = x'(1) = x'(e_t^* * e_t) = x'(e_t^*) x'(e_t) .$$

Since

$$|x'(e_t^*)| \leq \|x'\| \, \|e_t^*\| = 1 ,$$

$$|x'(e_t)| \leq \|x'\| \, \|e_t\| = 1 ,$$

it follows that

$$x'(e_t^*) = \overline{x'(e_t)} .$$

Hence x' is involutive . By Proposition 2.3.4.5, x' is positive.

i) follows from h) and d).

j) By Theorem 7.1.1.5 i_3),

$$\overrightarrow{\mathcal{S}_w(f)} = \{\vec{e}_t | t \in T\}^{cc}$$

and the assertion follows from Corollary 6.3.5.6.

k) Take $x \in \mathcal{S}_w(f)$. By c), $x^* \in \mathcal{S}_w(f)$. By c), Proposition 7.1.1.9 d), and Proposition 7.1.1.6 $\mathrm{a}_3 \Rightarrow \mathrm{a}_1$, for $y \in \ell^2(T)$,

$$(y*x)^* = x^* * y^* \in \ell^2(T) ,$$

$$y*x \in \ell^2(T) ,$$

$$\|y*x\|_2 = \|(y*x)^*\|_2 = \|x^* * y^*\|_2 \leq \|\vec{x^*}\| \, \|y^*\|_2 = \|\vec{x}\| \, \|y\|_2 .$$

Hence $\overleftarrow{x}$ is continuous and

$$\|\overleftarrow{x}\| \leq \|\vec{x}\| .$$

By symmetry, $F = \mathcal{S}_w(f)$ and $\|\vec{x}\| = \|\overleftarrow{x}\|$. By Theorem 7.1.1.5 i_2), $\overleftarrow{x} \in \left(\overrightarrow{\mathcal{S}_w(f)}\right)^c$.

By the above, u is an isometry of Banach spaces. By Theorem 7.1.1.5 c) and symmetry, for $x, y \in \mathcal{S}_w(f)$ and $z \in \ell^2(T)$,

$$\overleftarrow{x*yz} = z*(x*y) = (z*x)*y = \overleftarrow{y}(z*x) = \overleftarrow{y}(\overleftarrow{x}z) = \overleftarrow{y}\overleftarrow{x}z$$

so that

$$u(\vec{x}\vec{y}) = u(\overrightarrow{x*y}) = \overleftarrow{x*y} = \overleftarrow{y}\overleftarrow{x} = (u\vec{y})(u\vec{x}).$$

Take $x \in \mathcal{S}_w(f)$. By Propositions 7.1.1.9 a),c),d) and 7.1.1.6 f), $a_3 \Rightarrow a_1$, c_3 & $d_3 \Rightarrow c_1$ & d_1, for $y, z \in \ell^2(T)$,

$$\langle \overleftarrow{x}^* y|z\rangle = \langle y|\overleftarrow{x}z\rangle = \langle y|z*x\rangle = \langle x^**z^*|y^*\rangle =$$

$$= \langle z^*|x*y^*\rangle = \langle y*x^*|z\rangle = \langle \overleftarrow{x^*}y|z\rangle.$$

Hence

$$u(\vec{x}^*) = u\vec{x^*} = \overleftarrow{x^*} = \overleftarrow{x}^* = (u\vec{x})^*$$

and u is involutive.

l) By Theorem 7.1.1.5 $f_4 \Rightarrow f_3$,

$$x, y \in \ell^2(T) \Longrightarrow x*y = y*x.$$

Hence

$$\vec{x} = \overleftarrow{x}$$

for every $x \in \mathcal{S}_w(f)$. It follows by Theorem 7.1.1.5 i_2),

$$\overrightarrow{\mathcal{S}_w(f)} = \left(\overrightarrow{\mathcal{S}_w(f)}\right)^c$$

and the assertion follows from Proposition 4.2.2.14 b $\Rightarrow$ a.

m) Take $x \in \mathcal{S}_w(f)$. By Proposition 7.1.1.9 c) and Theorem 7.1.1.5 a),

$$(x^**x)(1) = \sum_{s\in T} f(s^{-1}, s)\widetilde{f}(s^{-1})|x(s)|^2 =$$

$$= \sum_{s\in T} \overline{f(s,1)}|x(s)|^2 = \|x\|_2^2 .$$

For $t \in T$,

$$|(x^**x)(t)| = \left|\sum_{s\in T} f(ts^{-1},s)\widetilde{f}(ts^{-1})\overline{x(st^{-1})}x(s)\right| \leq$$

$$\leq \sum_{s\in T} |f(ts^{-1},s)|\,|\widetilde{f}(ts^{-1})|\,|x(st^{-1})|\,|x(s)| \leq$$

$$\leq \left(\sum_{s\in T} |x(st^{-1})|^2\right)^{\frac{1}{2}} \left(\sum_{s\in T} |x(s)|^2\right)^{\frac{1}{2}} = \|x\|_2^2 .$$

It follows

$$\ell(x^**x) = \sum_{t\in T} (x^**x)(t)y(t) \geq$$

$$\geq (x^**x)(1)y(1) - \sum_{t\in T\setminus\{0\}} |(x^**x)(t)|\,|y(t)| \geq$$

$$\geq \|x\|_2^2 \left(y(1) - \sum_{t\in T\setminus\{1\}} |y(t)|\right) \geq 0 .$$

Hence ℓ is positive.

Assume now $T \neq \{1\}$ and take $t_0 \in T\setminus\{1\}$. By the above, the maps

$$\ell_\pm : \mathcal{S}_w(f) \longrightarrow \mathbb{K}, \quad x \longmapsto x(1) \pm x(t_0)$$

are positive. They obviously belong to $\tau(\mathcal{S}_w(f))$ (Corollary 2.3.4.7). Since

$$\frac{1}{2}(\ell_+ + \ell_-) = x'$$

it follows that x' is not an extreme point of $\tau(\mathcal{S}_w(f))$. By Theorem 6.3.6.2 b $\Rightarrow$ a, the representation of $\mathcal{S}_w(f)$ associated to x' is not irreducible.

n) For every $t \in A$,

$$e_t^* = \widetilde{f}(t)e_{t^{-1}} = e_t ,$$

so that $ue_t \in \operatorname{Re} E$. It follows successively,

$$t \in T \Longrightarrow ue_t \in \operatorname{Re} E\,,$$

$$u(\mathbb{R}^{(T)}) \subset \operatorname{Re} E\,,$$

$$\operatorname{Im} u \subset \operatorname{Re} E\,.$$

o) By Theorem 7.1.1.5 a),

$$x * y(1) = \sum_{t\in T} f(t^{-1},t)x(t^{-1})y(t) = \sum_{t\in T} f(t,t^{-1})x(t)y(t^{-1}) =$$

$$= \sum_{t\in T} f(t^{-1},t)y(t^{-1})x(t) = y * x(1)\,,$$

$$x * x^*(1) = \sum_{t\in T} f(t^{-1},t)x(t^{-1})x^*(t) =$$

$$= \sum_{t\in T} f(t^{-1},t)x(t^{-1})\overline{f(t,t^{-1})}\;\overline{x(t^{-1})} =$$

$$= \sum_{t\in T} |x(t)|^2 = \|x\|_2^2\,. \qquad \blacksquare$$

Proposition 7.1.2.2 (7) *Assume* $\mathbb{K} = \mathbb{R}$. *Let* f *be a Schur function for the group* T *and let* $\mathfrak{Q}$ *be the set of finite conjugacy classes* Q *of* T *such that*

$$f(r,r^{-1}tr)f(t,r) = f(s,s^{-1}ts)f(t,s)$$

for every $t \in Q$ *and every* $r,s \in T$ *with*

$$r^{-1}tr = s^{-1}ts\,.$$

Put

$$\mathfrak{Q}_1 := \{Q \in \mathfrak{Q} | Q = \{t^{-1} | t \in Q\}\}\,,$$

$$\mathfrak{Q}_2 := \{Q \in \mathfrak{Q} | Q \cap \{t^{-1} | t \in Q\} = \emptyset\}$$

and denote by $\mathfrak{Q}'$ *(by* $\mathfrak{Q}''$*) the set of* $Q \in \mathfrak{Q}_1$ $(Q \in \mathfrak{Q}_2)$ *such that*

$$(s,t) \in T \times Q\,,\ s^{-1}ts = t^{-1} \Longrightarrow f(s,s^{-1}t^{-1}s)f(t^{-1},s)\widetilde{f}(t) = 1$$

$$((s,t) \in T \times Q \Longrightarrow f(s,s^{-1}t^{-1}s)f(t^{-1},s)\widetilde{f}(t) = f(s,s^{-1}t^{-1}s)f(t^{-1},s)\widetilde{f}(s^{-1}ts))\,.$$

a) *The following are equivalent for every* $x \in (\mathcal{S}_w(f))^c$ *:*

a_1) x *is selfadjoint.*

a_2) $x = 0$ *on* $T \setminus \bigcup_{Q \in \mathfrak{Q}' \cup \mathfrak{Q}''} Q$ *and*

$$x(t) = \tilde{f}(t)x(t^{-1})$$

for every $Q \in \mathfrak{Q}''$ *and* $t \in Q$.

b) *If* $\mathfrak{Q}' \cup \mathfrak{Q}''$ *is finite then*

$$\operatorname{Dim}\operatorname{Re}(\mathcal{S}_w(f))^c = \operatorname{Card}\mathfrak{Q}' + \frac{1}{2}\operatorname{Card}\mathfrak{Q}''.$$

$a_1 \Rightarrow a_2$. By Theorem 7.1.1.5 k), $x = 0$ on $T \setminus \bigcup_{Q \in \mathfrak{Q}} Q$. Take $Q \in \mathfrak{Q} \setminus (\mathfrak{Q}' \cup \mathfrak{Q}'')$. By Theorem 2.2.2.7 d_5), $\mathfrak{Q} = \mathfrak{Q}_1 \cup \mathfrak{Q}_2$.

First assume $Q \in \mathfrak{Q}_1$. Since $Q \notin \mathfrak{Q}'$, there is an $(s,t) \in T \times Q$ with $s^{-1}ts = t^{-1}$ and

$$f(s, s^{-1}ts)f(t,s)\tilde{f}(t) \neq 1.$$

By Theorem 7.1.1.5 $j_1 \Rightarrow j_4$,

$$x(t) = x^*(t) = \tilde{f}(t)x(t^{-1}) = \tilde{f}(t)x(s^{-1}ts) =$$

$$= \tilde{f}(t)f(s, s^{-1}ts)f(t,s)x(t)$$

so that $x(t) = 0$. Again, by Theorem 7.1.1.5 $j_1 \Rightarrow j_4$, it follows $x = 0$ on Q.

Now assume $Q \in \mathfrak{Q}_2$. Since $Q \notin \mathfrak{Q}''$, there is an $(s,t) \in T \times Q$ such that

$$f(s, s^{-1}t^{-1}s)f(t^{-1}, s)\tilde{f}(t) \neq f(s, s^{-1}ts)f(t,s)\tilde{f}(s^{-1}ts).$$

By Theorem 7.1.1.5 $j_1 \Rightarrow j_4$ (and Proposition 7.1.1.9 a)),

$$x(t^{-1}) = x^*(t^{-1}) = \tilde{f}(t)x(t) = \tilde{f}(t)f(s, s^{-1}ts)f(t,s)x(s^{-1}ts),$$

$$x(s^{-1}ts) = x^*(s^{-1}ts) = \tilde{f}(s^{-1}ts)x(s^{-1}t^{-1}s) =$$

$$= \tilde{f}(s^{-1}ts)f(s, s^{-1}t^{-1}s)f(t^{-1}, s)x(t^{-1}) =$$

$$= \tilde{f}(s^{-1}ts)f(s, s^{-1}t^{-1}s)f(t^{-1}, s)\tilde{f}(t)f(s, s^{-1}ts)f(t,s)x(s^{-1}ts)),$$

so that $x(s^{-1}ts) = 0$. By Theorem 7.1.1.5 $j_1 \Rightarrow j_4$, again, it follows $x = 0$ on Q.

$a_2 \Rightarrow a_1$. Take $Q \in \mathfrak{Q}'$ and $t \in Q$. There is an $s \in T$ such that $s^{-1}ts = t^{-1}$. By Theorem 7.1.1.5 $j_1 \Rightarrow j_4$,

$$x^*(t) = \tilde{f}(t)x(t^{-1}) = \tilde{f}(t)x(s^{-1}ts) =$$

$$= \tilde{f}(t)f(s, s^{-1}ts)f(t, s)x(t) = x(t).$$

Take $Q \in \mathfrak{Q}''$ and $t \in Q$. Then, by a_2),

$$x^*(t) = \tilde{f}(t)x(t^{-1}) = x(t).$$

Hence x is selfadjoint.

b) Take $Q \in \mathfrak{Q}'$ and $t \in Q$. The function on T equal to 0 on $T\backslash Q$ and equal to

$$s^{-1}ts \longmapsto f(s, s^{-1}ts)f(t, s)$$

on Q is well-defined. By Theorem 7.1.1.5 $j_4 \Rightarrow j_1$, this function belongs to $(\mathcal{S}_w(f))^c$ and by $a_2 \Rightarrow a_1$, it is selfadjoint.

Take $Q \in \mathfrak{Q}''$ and $t \in Q$. Put

$$Q' := \{s^{-1} | s \in Q\}.$$

The function x on T equal to 0 on $T\backslash(Q \cup Q')$ and equal to

$$s^{-1}ts \longmapsto f(s, s^{-1}ts)f(t, s)$$

$$\left(s^{-1}t^{-1}s \longmapsto f(s, s^{-1}t^{-1}s)f(t^{-1}, s)\tilde{f}(t)\right)$$

on Q (on Q') is well-defined. By Theorem 7.1.1.5 $j_4 \Rightarrow j_1$, x belongs to $(\mathcal{S}_w(f))^c$. Moreover, for every $s \in T$,

$$x(s^{-1}ts) = f(s, s^{-1}ts)f(t, s) =$$

$$= \tilde{f}(s^{-1}ts)f(s, s^{-1}t^{-1}s)f(t^{-1}, s)\tilde{f}(t) = \tilde{f}(s^{-1}ts)x(s^{-1}t^{-1}s),$$

$$x(s^{-1}t^{-1}s) = \tilde{f}(s^{-1}t^{-1}s)x(s^{-1}ts)$$

(Proposition 7.1.1.9 a)). By $a_2 \Rightarrow a_1$, x is selfadjoint.

By the above,

$$\operatorname{Dim}\operatorname{Re}(\mathcal{S}_w(f))^c \geq \operatorname{Card}\mathfrak{Q}' + \frac{1}{2}\operatorname{Card}\mathfrak{Q}''.$$

Since the reverse inequality follows from $a_2 \Rightarrow a_1$ and Theorem 7.1.1.5 $j_1 \Rightarrow j_3$ & j_4, we obtain

$$\operatorname{Dim}\operatorname{Re}(\mathcal{S}_w(f))^c = \operatorname{Card}\mathfrak{Q}' + \frac{1}{2}\operatorname{Card}\mathfrak{Q}''. \qquad \blacksquare$$

Proposition 7.1.2.3 (7) *Let f be a Schur function for the locally finite group T.*

a) $\mathbb{K}^{(T)}$ *is an involutive unital subalgebra of $\mathcal{S}_c(f)$, which will be denoted by $\mathcal{S}^{\mathbb{K}}(f)$ (or simply, by $\mathcal{S}(f)$).*

b) $\mathcal{S}_c(f)$ *is the closure of $\ell^1(T)$ in $\mathcal{S}_w(f)$ and, in the complex case, the C^*-hull of $\ell^1(T)$ (Theorem 7.1.2.1 h)).*

c) *If f is real then the map*

$$\overbrace{\mathcal{S}_c^{\mathbb{R}}(f)}^{\circ} \longrightarrow \mathcal{S}_c^{\mathbb{C}}(f), \quad (x,y) \longmapsto x + iy$$

is an isomorphism of unital complex C^*-algebras.

d) *If f is real then the map*

$$\overbrace{\mathcal{S}^{\mathbb{R}}(f)}^{\circ} \longrightarrow \mathcal{S}^{\mathbb{C}}(f), \quad (x,y) \longmapsto x + iy$$

is an isomorphism of involutive unital complex algebras.

e) *The predual of $\mathcal{S}_w(f)$ (Theorem 7.1.2.1 c)) is the Banach space H defined in Proposition 7.1.1.4 a) and the map ψ_0 from Proposition 7.1.1.4 k) is the pretranspose of the W^*-homomorphism*

$$\mathcal{S}_w(f) \longrightarrow \mathcal{L}(\ell^2(T)), \quad x \longmapsto \overrightarrow{x}$$

(Theorem 7.1.2.1 c)). ψ_0 is surjective. For $a \in H$, $x \in \mathcal{S}_w(f)$, and $t \in T$,

$$\widetilde{ax}(t) = \langle \widetilde{a}, x{*}e_t \rangle, \quad \widetilde{xa}(t) = \langle \overline{a}, e_t{*}x \rangle.$$

f) *If u and v denote the evaluations $\ell^1(T) \to \ell^1(T)''$ and $\mathcal{S}_w(f) \to \mathcal{S}_w(f)''$, respectively, and $w : \ell^1(T) \to \mathcal{S}_w(f)$ denotes the inclusion map, then*

$$(\operatorname{Im} v) \cap (\operatorname{Im} w'') = \operatorname{Im}(v \circ w) = \operatorname{Im}(w'' \circ u).$$

g) *If $u : \ell^1(T) \to \mathcal{S}_c(f)$, $v : \mathcal{S}_c(f) \longrightarrow \ell^2(T)$, and $w : \ell^1(T) \to \mathcal{S}_w(f)$ denote the inclusion maps, then:*

g_1) *u' and v' are injective.*

g_2) $x' \in (\mathcal{S}_c(f))'_+ \Rightarrow u'x' \in \ell^1(T)'_+$, $\|u'x'\| = \|x'\|$.

g_3) $u'(\mathcal{S}_c(f))'_+ = \ell^1(T)'_+$.

g_4) *If $x \in \mathcal{S}_w(f)$, $x' \in (\mathcal{S}_w(f))'$, and $t \in T$, then*

$$\begin{cases} (w'(xx'))(t) = \sum_{s \in T} f(t, t^{-1}s)(w'x')(s)x(t^{-1}s) \\ (w'(x'x))(t) = \sum_{s \in T} f(st^{-1}, t)(w'x')(s)x(st^{-1}), \end{cases}$$

where $\ell^\infty(T)$ and $(\ell^1(T))'$ are canonically identified.

h) $\operatorname{Card} T$ *is the topological cardinality of $\mathcal{S}_c(f)$. In particular, $\mathcal{S}_c(f)$ is separable iff T is countable.*

i) *$\mathcal{S}_w(f)$ is the W^*-subalgebra of $\mathcal{S}_w(f)$ generated by $\mathcal{S}_c(f)$.*

a) follows from Theorem 7.1.2.1 e).

b) $\mathbb{K}^{(T)}$ is dense in $\mathcal{S}_c(f)$, so that $\ell^1(T)$ is dense in $\mathcal{S}_c(f)$.

Let E be the C^*-hull of $\ell^1(T)$, $q : \ell^1(T) \to E$ the canonical map, and u the factorization of the inclusion map $\ell^1(T) \to \mathcal{S}_c(f)$ through E (Proposition 4.1.1.22 f)). Then q is injective (Proposition 7.1.1.1 d),g)) and

$$\ell^1(T) = u(q(\ell^1(T))) \subset u(E).$$

Since $u(E)$ is closed (Theorem 4.2.6.6) it is equal to $\mathcal{S}_c(f)$. By Theorem 4.2.6.6, again, u preserves the norms.

c) and d) follow from Theorem 7.1.2.1 d).

e) The first assertion follows from Proposition 7.1.1.4 a),g),h),k). By Corollary 4.4.4.9, ψ_0 is surjective.

Take $s \in T$ and $y \in \mathcal{S}_w(f)$. Then

$$\langle \widetilde{e_s x}, y \rangle = \langle \widetilde{e_s}, x*y \rangle = (x*y)(s) =$$

$$= \sum_{t \in T} f(st^{-1}, t)x(st^{-1})y(t)\,,$$

$$\langle \widetilde{xe_s}, y \rangle = \langle \widetilde{e}_s, y*x \rangle = (y*x)(s) = \sum_{t \in T} f(t, t^{-1}s)y(t)x(t^{-1}s),$$

so that (Proposition 7.1.1.1 a))

$$\widetilde{e_s x}(t) = f(st^{-1}, t)x(st^{-1}) = (x*e_t)(s) = \langle \widetilde{e}_s, x*e_t \rangle\,,$$

$$\widetilde{xe_s}(t) = f(t, t^{-1}s)x(t^{-1}s) = (e_t*x)(s) = \langle \widetilde{e}_s, e_t*x \rangle\,.$$

It follows

$$\widetilde{ax}(t) = \langle \widetilde{a}, x*e_t \rangle\,, \quad \widetilde{xa}(t) = \langle \widetilde{a}, e_t*x \rangle$$

for every $a \in \mathbb{K}^{(T)}$. By continuity, these relations hold for every $a \in H$.

f) Let H be the predual of $\mathcal{S}_w(f)$ and $w_0 : H \to c_0(T)$ the inclusion map. Then $w = w_0'$ and the assertion follows from Proposition 1.3.6.25.

g_1) follows from b) and Corollary 1.3.5.9.

g_2) follows from Proposition 2.3.4.20.

g_3) For $S \in \mathfrak{S}_T$, $\{xe_S | x \in \ell^1(T)\}$ is a finite-dimensional unital involutive subalgebra of $\ell^1(T)$ (Theorem 7.1.2.1 e)), so that the assertion follows from Proposition 2.3.4.21.

g_4) By Proposition 7.1.1.1 a),

$$(w'(xx'))(t) = \langle xx', e_t \rangle = \langle x', e_t*x \rangle =$$

$$= \sum_{s \in T} (w'x')(s)(e_t*x)(s) = \sum_{s \in T} (w'x')(s)f(t, t^{-1}s)x(t^{-1}s)\,,$$

$$(w'(x'x))(t) = \langle x'x, e_t \rangle = \langle x', x*e_t \rangle =$$

$$= \sum_{s \in T} (w'x')(s)(x*e_t)(s) = \sum_{s \in T} (w'x')(s)f(st^{-1}, t)x(st^{-1})\,.$$

h) By Example 1.1.2.5, there is a dense set A of $\ell^1(T)$ with

$$\operatorname{Card} A = \operatorname{Card} T\,.$$

By b) and Proposition 7.1.1.1 d), A is a dense set of $\mathcal{S}_c(f)$.

Let B be a dense set of $\mathcal{S}_c(f)$. By Proposition 7.1.1.1 e), B is a dense set of $\ell^2(T)$. By Example 1.1.2.5,

$$\operatorname{Card} B \geq \operatorname{Card} T .$$

i) Since

$$\{\vec{e}_t | t \in T\}^{cc} \subset \overrightarrow{(\mathcal{S}_c(f))}^{cc} \subset \overrightarrow{(\mathcal{S}_w(f))}^{cc},$$

it follows from Theorem 7.1.1.5 i_3),

$$\overrightarrow{(\mathcal{S}_c(f))}^{cc} = \overrightarrow{\mathcal{S}_w(f)} .$$

The assertion now follows from the Corollaries 6.3.5.6 and 4.4.4.10. ■

Proposition 7.1.2.4 *Let f be a Schur function for the infinite group T.*

a) $\mathcal{S}_w(f) \neq \ell^2(T)$.

b) $\mathcal{S}_w(f) \neq \ell^1(T)$.

c) *If in addition T is locally finite,*

$$\mathcal{S}_c(f) \neq \ell^1(T), \quad \mathcal{S}_c(f) \neq \mathcal{S}_w(f), \quad G \neq c_0(T), \quad H \neq \ell^2(T),$$

where G and H are the sets defined in Proposition 7.1.1.4.

a) Assume

$$\mathcal{S}_w(f) = \ell^2(T) .$$

Since the identity map

$$\mathcal{S}_w(f) \longrightarrow \ell^2(T)$$

is continuous (Proposition 7.1.1.1 e)), it is an isomorphism of Banach spaces (Principle of Inverse Operators). Since $\ell^2(T)$ is reflexive, $\mathcal{S}_w(f)$ is also reflexive (Proposition 1.3.8.8). By Theorem 6.3.6.15 (and Theorem 7.1.2.1 c)), $\mathcal{S}_w(f)$ is finite–dimensional, i.e. T is finite.

b) By Corollary 6.3.6.16 a), there is a strictly increasing sequence $(p_n)_{n\in\mathbb{N}}$ in $\operatorname{Pr}\mathcal{S}_w(f)$. For every $n \in \mathbb{N}$,

$$p_{n+1} - p_n \in \operatorname{Pr}\mathcal{S}_w(f)$$

(Corollary 4.2.7.6 a $\Rightarrow$ g), so that

$$\|p_{n+1} - p_n\| = 1$$

(Proposition 4.1.2.21). Beside $(p_n)_{n\in\mathbb{N}}$ is weakly Cauchy (Corollary 4.1.2.7 d), Proposition 4.2.1.22, Theorem 4.2.2.1 a $\Rightarrow$ d).

Now assume

$$\mathcal{S}_w(f) = \ell^1(T).$$

Since the identity map

$$\ell^1(T) \longrightarrow \mathcal{S}_w(f)$$

is continuous (Proposition 7.1.1.1 d)), it is an isomorphism of Banach spaces (Principle of Inverse Operators). Hence $(p_n)_{n\in\mathbb{N}}$ is a weak Cauchy sequence in $\ell^1(T)$. By Theorem 1.3.6.11, $(p_n)_{n\in\mathbb{N}}$ is norm convergent in $\ell^1(T)$. Thus $(p_n)_{n\in\mathbb{N}}$ converges in norm in $\mathcal{S}_w(f)$, which contradicts the above result.

c) Let S be a countable infinite subgroup of T. Put

$$E := \{xe_S | x \in \mathcal{S}_w(f)\}, \quad F := \{xe_S | x \in \mathcal{S}_c(f)\}.$$

By Theorem 7.1.2.1 e), E is a W^*–algebra. Since it is infinite–dimensional, it is not separabel (Corollary 6.3.6.16 b)). By Proposition 7.1.2.3 h), F is separable. Hence $E \neq F$ and

$$\mathcal{S}_w(f) \neq \mathcal{S}_c(f).$$

Now assume

$$\mathcal{S}_c(f) = \ell^1(T).$$

By Proposition 7.1.1.1 d) and the Principle of Inverse Operators, there is an $\alpha > 0$ such that

$$\|x\|_1 \leq \alpha \|\vec{x}\|$$

for every $x \in \ell^1(T)$.

Let H be the predual of $\mathcal{S}_w(f)$. By Theorem 7.1.2.3 i), $\mathcal{S}_c(f)$ generates $\mathcal{S}_w(f)$ as W^*–algebra. Let $x \in \mathcal{S}_w(f)^\#$. By Corollary 6.3.8.7, there is a filter on $\mathcal{S}_w(f)$ converging to x in $\mathcal{S}_w(f)^\#_H$ with $\mathcal{S}_c(f)^\# \in \mathfrak{F}$. In particular,

$$x(t) = \lim_{y,\mathfrak{F}} y(t)$$

for every $t \in T$ (Proposition 7.1.1.4 h)). By the above $\alpha \ell^1(T)^\# \in \mathfrak{F}$,. Hence

$$\sum_{t \in T} |x(t)| \leq \liminf_{y, \mathfrak{F}} \sum_{t \in T} |y(t)| \leq \alpha ,$$

$$x \in \ell^1(T) ,$$

$$\ell^1(T) = \mathcal{S}_w(f) .$$

This contradicts b).

Now assume $G = c_0(T)$. By Proposition 7.1.1.4 d),g), and the Principle of Inverse Operators, the identity map $G \to c_0(T)$ is an isomorphism of Banach spaces. Since $\mathbb{K}^{(T)}$ is dense in $c_0(T)$ it follows $H = c_0(T)$. By Proposition 7.1.1.4 i) (and Corollary 1.3.4.7), $\ell^1(T) = \mathcal{S}_w(f)$ and this contradicts b).

Finally assume $\ell^2(T) = H$. By Proposition 7.1.1.4 b),c),g) and the Principle of Inverse Operators the identity map $\ell^2(T) \to H$ is an isomorphism of Banach spaces. By Proposition 7.1.1.4 h) and Corollary 1.3.4.7, $\ell^2(T) = \mathcal{S}_w(f)$ which contradicts a). ■

Proposition 7.1.2.5 (7) *If f is a Schur function for the group T, then the following are equivalent:*

a) *$\mathcal{S}_w(f)$ is a Gelfand C^*-algebra.*

b) *$\ell^1(T)$ is an involutive Gelfand algebra.*

c) *$\ell^1(T)$ is a symmetric involutive Gelfand algebra.*

d) *T is commutative,*

$$s, t \in T \Longrightarrow f(s,t) = f(t,s) ,$$

and, in the real case,

$$t \in T \Longrightarrow t = t^{-1} , \widetilde{f}(t) = 1 .$$

If these conditions are fulfilled then the Gelfand transform of $\ell^1(T)$ is injective, involutive, and with dense image. Moreover, if $\mathbb{K} = \mathbb{C}$ then $\overrightarrow{\mathcal{S}_w(f)}$ is a maximal Gelfand C^*-subalgebra and a faithful set of $\mathcal{L}(\ell^2(T))$.

a $\Rightarrow$ b and c $\Rightarrow$ b are trivial.

b $\Rightarrow$ c. Case 1 $\mathbb{K} = \mathbb{C}$

By Theorem 7.1.2.1 h), every character of $\ell^1(T)$ is involutive so that by Proposition 2.4.2.3 d $\Rightarrow$ a, $\ell^1(T)$ is symmetric.

Case 2 $\mathbb{K} = \mathbb{R}$

By Case 1 and Theorem 7.1.2.1 i), the complexification of $\ell^1(T)$ is a symmetric involutive Gelfand algebra . By Proposition 2.4.2.3, $\ell^1(T)$ is a symmetric involutive Gelfand algebra.

c $\Rightarrow$ d $\Rightarrow$ a follows from Theorem 7.1.1.5 $f_3 \Rightarrow f_2 \Rightarrow f_4$ and Proposition 7.1.1.6 $e_2 \Rightarrow e_3 \Rightarrow e_1$.

We now prove the final assertion. By c) and Proposition 2.4.2.3 a $\Rightarrow$ c, the Gelfand transform of $\ell^1(T)$ is involutive and by c) and Corollary 2.4.2.6, its image is dense. By Theorem 7.1.2.1 c), h) and Proposition 2.3.2.26), $\ell^1(T)$ is semi–simple, so that the Gelfand transform of $\ell^1(T)$ is injective (Corollary 2.4.1.14). If $\mathbb{K} = \mathbb{C}$ then by Theorem 7.1.2.1 h),l) and Corollary 4.2.2.20, $\overrightarrow{\mathcal{S}_w(f)}$ is a maximal Gelfand C^*–subalgebra and a faithful set of $\mathcal{L}(\ell^2(T))$. ■

Proposition 7.1.2.6 *Let f be the constant Schur function on the commutative locally finite group T such that, in the real case,*

$$t = t^{-1}$$

for all $t \in T$.

a) *$\sigma(\ell^1(T))$ (Proposition 7.1.2.5 d $\Rightarrow$ c) is a totally disconnected compact commutative group. Let λ denote its Haar measure with $\lambda(\sigma(\ell^1(T))) = 1$.*

b) *If $u : \ell^1(T) \to \mathcal{S}_c(f)$ denotes the inclusion map (Proposition 7.1.2.3 b)), then the map*

$$\sigma(\mathcal{S}_c(f)) \longrightarrow \sigma(\ell^1(T)), \quad x' \longmapsto u'x'$$

(Proposition 7.1.2.5 d $\Rightarrow$ a) is a homeomorphism. We identify $\sigma(\mathcal{S}_c(f))$ and $\sigma(\ell^1(T))$ using this homeomorphism.

c) *$(\widehat{e}_t)_{t\in T}$ is an orthonormal basis of $L^2(\lambda)$. Denote by v the isomorphism of Hilbert spaces*

$$v : \ell^2(T) \longrightarrow L^2(\lambda), \quad x \longmapsto \sum_{t\in T} x(t)\widehat{e}_t .$$

d) $x \in \mathcal{S}_c(f) \Rightarrow vx = \widehat{x}$.

e) *If H denotes the Banach space defined in Proposition 7.1.1.4 a), then there is a unique isometry $w : L^1(\lambda) \to H$ such that*

$$w\widehat{e}_t = e_{t^{-1}}$$

for every $t \in T$.

f) *For every $x \in \mathcal{S}_w(f)$,*

$$vx \in L^\infty(\lambda), \quad \|vx\|_\infty = \|\vec{x}\|,$$

$$y \in \ell^2(T) \Longrightarrow v(\vec{x}y) = (vx)(vy).$$

g) *If we canonically identify H with the predual of $\mathcal{S}_w(f)$ (Proposition 7.1.2.3 e), Proposition 7.1.1.4 h)), then*

$$vx = w'x$$

for every $x \in \mathcal{S}_w(f)$, where w is the isometry of e).

h) *The map*

$$\widetilde{v} : \mathcal{S}_w(f) \longrightarrow L^\infty(\lambda), \quad x \longmapsto vx$$

is an isomorphism of W^*-algebras and w is its pretranspose.

i) *If u_0 and v_0 denote the evaluations of $\mathcal{S}_c(f)$ and $\mathcal{S}_w(f)$, respectively, and if $w_0 : \mathcal{S}_c(f) \to \mathcal{S}_w(f)$ denotes the inclusion map, then*

$$(\operatorname{Im} v_0) \cap (\operatorname{Im} w_0'') = \operatorname{Im}(v_0 \circ w_0) = \operatorname{Im}(w_0'' \circ u_0).$$

a) follows from Theorem 7.1.2.1 e), Corollary 2.4.4.6, and Corollary 4.1.2.17 d) (and Theorem 2.4.4.4).

b) By Proposition 2.4.1.17 a),c),d), the map is well-defined, continuous, and injective. Thus we have to prove only, that this map is surjective. Let $y' \in \sigma(\ell^1(T))$. By Proposition 2.4.2.2 c), $y' \in \ell^1(T)'_+$. By Proposition 2.3.4.21 (and Theorem 7.1.2.1 e)), there is an $x' \in (\mathcal{S}_c(f))'_+$ such that $u'x' = y'$. Since $\ell^1(T)$ is dense in $\mathcal{S}_c(f)$, it follows that $x' \in \sigma(\mathcal{S}_c(f))$.

c) By a), Proposition 7.1.2.5, and Corollary 2.4.4.6, for $s, t \in T$,

$$\langle \widehat{e}_s | \widehat{e}_t \rangle = \int \widehat{e}_s \overline{\widehat{e}}_t d\lambda = \int \widehat{e}_s \widehat{e}_t^* d\lambda =$$

$$= \int \widehat{e}_s \widehat{e}_{t^{-1}} d\lambda = \int \widehat{e}_{st^{-1}} d\lambda = \delta_{s,t} .$$

Hence $(\widehat{e}_t)_{t\in H}$ is an orthonormal family in $L^2(\lambda)$. By Proposition 7.1.2.5, the image of the Gelfand transform of $\ell^1(T)$ is dense, so that $(\widehat{e}_t)_{t\in T}$ is an orthonormal basis of $L^2(\lambda)$.

d) $\mathbb{K}^{(T)}$ is dense in $\mathcal{S}_c(f)$ and v and the Gelfand transform of $\mathcal{S}_c(f)$ coincide on $\mathbb{K}^{(T)}$. Hence $vx = \widehat{x}$ for every $x \in \mathcal{S}_c(f)$.

e) Put

$$w_0 : \mathbb{K}^{(T)} \longrightarrow L^1(\lambda), \quad a \longmapsto \sum_{t\in T} a(t)\widehat{e}_{t^{-1}} .$$

Take $a \in \mathbb{K}^{(T)}$ and $x \in \mathcal{S}_c(f)$. By d), for $t \in T$,

$$x(t) = \langle vx|\widehat{e}_t\rangle = \int \overline{\widehat{e}}_t(vx)d\lambda = \int \widehat{e}_{t^{-1}}\widehat{x}d\lambda =$$

$$= \langle \widehat{e}_{t^{-1}}, \widehat{x}\rangle ,$$

so that

$$\sum_{t\in T} a(t)x(t) = \sum_{t\in T} a(t)\langle \widehat{e}_{t^{-1}}, \widehat{x}\rangle = \langle w_0 a, \widehat{x}\rangle .$$

By Proposition 7.1.1.4 h),j),

$$\|a\| = \sup\{|\langle a, x\rangle| | x \in \mathcal{S}_w(f)^{\#}\} =$$

$$= \sup\{|\langle a, x\rangle| | x \in \mathcal{S}_c(f)^{\#}\} = \sup\{|\langle w_0 a, \widehat{x}\rangle| | x \in \mathcal{S}_c(f)^{\#}\} =$$

$$= \sup\{|\langle w_0 a, y\rangle| | y \in \mathcal{C}(\sigma(\ell^1(T)))^{\#}\} = \|w_0 a\|_1 .$$

w_0 preserves the norms, so that it extends to a linear map

$$w_1 : H \longrightarrow L^1(\lambda)$$

preserving the norms. Since $w_0(\mathbb{K}^{(T)})$ is dense in $L^1(\lambda)$, w_0 is an isometry. $w = w_1^{-1}$ has the required properties.

f) For $r, s \in T$,

$$\widehat{e}_s \widehat{e}_t = \widehat{e_s * e_t} = \widehat{e_{st}} .$$

Thus for $t \in T$,

$$(vx)(ve_t) = \left(\sum_{s\in T} x(s)\widehat{e}_s\right)\widehat{e}_t = \sum_{s\in T} x(s)\widehat{e}_s\widehat{e}_t = \sum_{s\in T} x(s)\widehat{e}_{st} =$$

$$= \sum_{r\in T} x(rt^{-1})\widehat{e}_r = \sum_{r\in T}(x*e_t)(r)\widehat{e}_r = v(\vec{x}e_t)\,.$$

Hence

$$(vx)(vy) = v(\vec{x}y)\,,$$

$$\|(vx)(vy)\|_2 = \|v(\vec{x}y)\|_2 = \|\vec{x}y\|_2 \le \|\vec{x}\|\,\|y\|_2$$

for every $y \in \mathbb{K}^{(T)}$. Since $\mathbb{K}^{(T)}$ is dense in $\ell^2(T)$, it follows

$$vx \in L^\infty(\lambda)\,,\ \|vx\|_\infty = \|\vec{x}\|\,,$$

and

$$(vx)(vy) = v(\vec{x}y)$$

for every $y \in \ell^2(T)$.

g) For $t \in T$,

$$\langle\widehat{e}_t, w'x\rangle = \langle w\widehat{e}_t, x\rangle = \langle e_{t^{-1}}, x\rangle = x(t^{-1}) =$$

$$\langle vx|\widehat{e}_{t^{-1}}\rangle = \langle vx|\overline{\widehat{e}_t}\rangle = \langle\widehat{e}_t, vx\rangle\,.$$

It follows that $w'x = vx$.

h) For $x \in \mathcal{S}_w(f)$,

$$vx^* = \sum_{t\in T}\overline{x(t)}\widehat{e}_{t^{-1}} =$$

$$= \sum_{t\in T}\overline{x(t)}\,\overline{\widehat{e}_t} = \overline{\sum_{t\in T} x(t)\widehat{e}_t} = \overline{vx}\,.$$

Hence v is involutive. By f), $\widetilde{v}$ is an algebra homomorphism and by g), the given map is an isometry of W^*–algebras with w as its pretranspose.

i) follows from d),h), and Proposition 1.3.6.24. ■

Remark. This proposition is a concrete realization of Theorem 6.3.7.4.

Proposition 7.1.2.7 *Let f be the constant Schur function on the locally finite group T. We put*

$$E := \mathcal{S}_c(f)\,,$$

and

$$p_S := \frac{1}{\operatorname{Card} S}\, e_S$$

for every $S \in \mathfrak{S}_T$.

a) *If S is a subgroup of T, then $\{xe_S | x \in \ell^1(T)\}$ is a closed involutive subalgebra of $\ell^1(T)$ and the map*

$$\{xe_S | x \in \ell^1(S)\} \longrightarrow \mathbb{K}\,, \quad x \longmapsto \sum_{s \in S} x(s)$$

is an involutive algebra homomorphism and therefore positive.

b) b$_1$) $R, S \in \mathfrak{S}_T$, $R \subset S \Rightarrow p_R$, $p_S \in \operatorname{Pr} E$, $p_S \leq p_R$.

b$_2$) *The map*

$$E \longrightarrow p_S E p_S\,, \quad x \longmapsto \Big(\sum_{s \in S} x(s)\Big) p_S$$

is a homomorphism of unital C^*-algebras for every $S \in \mathfrak{S}_T$.

c) *If $p \in E^c \cap \operatorname{Pr} E$, then:*

c$_1$) $pE(1-p) = (1-p)Ep = \{0\}$.

c$_2$) *E is the C^*-direct sum of pEp and $(1-p)E(1-p)$.*

c$_3$) *If T is finite, then*

$$\operatorname{Dim}(pEp) = \text{rang of } [p(st^{-1})]_{s,t \in T}\,.$$

d) *If $S \in \mathfrak{S}_T$ is a normal subgroup of T then:*

d$_1$) $p_S \in E^c \cap \operatorname{Pr} E$.

d$_2$) *If T is finite, then*

$$\operatorname{Dim}(p_S E p_S) = \frac{\operatorname{Card} T}{\operatorname{Card} S}\,.$$

d$_3$) *If T is the permutation group of a finite set containing at least two points, then E is isomorphic to the C^*-direct product of $\mathbb{K}^2$ and of a C^*-subalgebra of E.*

e) *If T is finite, then:*

e$_1$) $x \in E \Rightarrow x * p_T = \Big(\sum_{t\in T} x(t)\Big)p_T$.

e$_2$) *For $p \in \Pr E$,*

$$\sum_{t\in T} p(t) = 1 \text{ and } p_T \leq p$$

or

$$\sum_{t\in T} p(t) = 0 \text{ and } p_T \leq 1 - p.$$

e$_3$) *p_T is a minimal element of $\Pr E\backslash\{0\}$.*

e$_4$) *The map*

$$E \longrightarrow p_T E p_T, \quad x \longmapsto \Big(\sum_{t\in T} x(t)\Big)p_T$$

is a surjective homomorphism of unital C^*-algebras.

e$_5$) $p_T E p_T = \mathbb{K} p_T$.

e$_6$) *If $\operatorname{Card} T \neq 1$ then E is not simple.*

f) *If $(S_n)_{n\in\mathbb{N}}$ is a strictly increasing sequence in $\mathfrak{S}_T$, then the infimum of $(p_{S_n})_{n\in\mathbb{N}}$ (of $(\overrightarrow{p_{S_n}})_{n\in\mathbb{N}}$) in $\mathcal{S}_w(f)$ (in $\mathcal{L}(\ell^2(T))$) is 0.*

a) By Theorem 7.1.2.1 e),h), $\{xe_S | x \in \ell^1(T)\}$ is a closed involutive subalgebra of $\ell^1(T)$. For $x \in \ell^1(T)$,

$$\sum_{s\in S} x^*(s) = \sum_{s\in S} \overline{x(s^{-1})} = \overline{\sum_{s\in S} x(s)},$$

so that the map

$$\{xe_S | x \in \ell^1(T)\} \longrightarrow \mathbb{K}, \quad x \longmapsto \sum_{s\in S} x(s)$$

is involutive. Take $x, y \in \ell^1(T)$. For $t \in S$,

$$((xe_S)*(ye_S))(t) = \sum_{s\in T} (xe_S)(ts^{-1})(ye_S)(s) = \sum_{s\in S} x(ts^{-1})y(s),$$

so that

$$\sum_{t\in S} ((xe_S) * (ye_S))(t) = \sum_{t\in S}\sum_{s\in S} x(ts^{-1})y(s) =$$

$$= \sum_{s\in S} y(s)\Big(\sum_{t\in S} x(ts^{-1})\Big) = \Big(\sum_{s\in S} y(s)\Big)\Big(\sum_{t\in S} x(t)\Big)\,.$$

Hence the map

$$\{xe_S | x \in \ell^1(T)\} \longrightarrow \mathbb{K}\,, \quad x \longmapsto \sum_{s\in S} x(s)$$

is an algebra homomorphism. By Theorem 7.1.2.1 h), this map is positive.

b_1) We have

$$p_S^* = \frac{1}{\operatorname{Card} S} e_S^* = \frac{1}{\operatorname{Card} S} \sum_{t\in S} e_t^* = \frac{1}{\operatorname{Card} S} \sum_{t\in S} e_{t^{-1}} =$$

$$= \frac{1}{\operatorname{Card} S} \sum_{t\in S} e_t = \frac{1}{\operatorname{Card} S} e_S = p_S\,,$$

$$p_R * p_S = \frac{1}{(\operatorname{Card} R)(\operatorname{Card} S)} \sum_{r\in R} \sum_{s\in S} e_r * e_S =$$

$$= \frac{1}{(\operatorname{Card} R)(\operatorname{Card} S)} \sum_{r\in R} \sum_{s\in S} e_{rs} = \frac{1}{(\operatorname{Card} R)(\operatorname{Card} S)} \sum_{r\in R} \sum_{t\in S} e_t =$$

$$= \frac{1}{\operatorname{Card} S} \sum_{t\in S} e_t = p_S\,.$$

Hence $p_R, p_S \in \operatorname{Pr}\mathcal{S}_c(f)$ and by Corollary 4.2.7.6 d $\Rightarrow$ a,

$$p_S \leq p_R\,.$$

b_2) follows from a) and b_1).

c_1) $pE(1-p) = Ep(1-p) = \{0\}$,

$(1-p)Ep = (1-p)pE = \{0\}$.

c_2) follows from c_1) (and Corollary 4.1.1.21).

c_3) For $x \in E$ and $s \in T$,

$$(p * x)(s) = \sum_{t\in T} p(st^{-1})x(t)\,.$$

Hence

$$\operatorname{Dim}(1-p)E(1-p) = \operatorname{Dim}\{x \in E | p * x = 0\} =$$

$$= \operatorname{Card} T - \text{rang of } [p(st^{-1})]_{s,t\in T}\,.$$

By c_2),

$$\operatorname{Dim}(pEp) = \text{rang of } [p(st^{-1})]_{s,t\in T}\,.$$

d_1) For $t \in T$,

$$e_{t^{-1}} * e_S * e_t = \sum_{s\in S} e_{t^{-1}} * e_s * e_t = \sum_{s\in S} e_{t^{-1}st} = \sum_{s\in S} e_s = e_S\,,$$

so that

$$p_S * e_t = e_t * p_S\,.$$

Hence $p_S \in E^c$. By b_1), $p_S \in \operatorname{Pr} E$.

d_2) follows from c_3).

d_3) Let S be the normal subgroup of T of even permutations. By b_1) and d_1),

$$p_S\,,\ p_T \in E^c \cap \operatorname{Pr} E\,, \quad p_T \le p_S\,.$$

By d_2)

$$\operatorname{Dim}(p_S E p_S) = 2\,, \quad \operatorname{Dim}(p_T E p_T) = 1\,,$$

so that

$$\operatorname{Dim}((p_S - p_T)E(p_S - p_T)) = 1\,.$$

By Proposition 4.2.7.19, E is the C^*-direct product of

$$p_T E p_T\,,\ (p_S - p_T)E(p_S - p_T)\,,\ \text{and}\ (1 - p_S)E(1 - p_S)\,.$$

e_1) For $t \in T$,

$$(x * p_T)(t) = \sum_{s\in T} x(ts^{-1})p_T(s) = \frac{1}{\operatorname{Card} T}\sum_{s\in T} x(ts^{-1}) = \frac{1}{\operatorname{Card} T}\sum_{s\in T} x(s) =$$

$$= \Big(\sum_{s\in T} x(s)\Big)p_T(t)\,,$$

so that

$$x * p_T = \Big(\sum_{s\in T} x(s)\Big) p_T .$$

e_2) By a),

$$\sum_{t\in T} p(t) \in \{0,1\},$$

and by e_1),

$$p * p_T = \Big(\sum_{t\in T} p(t)\Big) p_T .$$

Hence

$$p * p_T = 0 \text{ or } p * p_T = p_T .$$

By Corollary 4.2.7.6 c $\Rightarrow$ a and d $\Rightarrow$ a,

$$p \le 1 - p_T \text{ or } p_T \le p .$$

e_3) follows from e_2) and b_1).

e_4) follows from b_2) and d_2).

e_5) follows from e_4).

e_6) follows from c_2) and d_1).

f) By b_1), $(p_{S_n})_{n\in\mathbb{N}}$ is a decreasing sequence in $\Pr E$. By Theorem 7.1.2.1 c), $\mathcal{S}_w(f)$ is a W^*-algebra and by Proposition 7.1.2.3 e), H is its predual. By Theorem 4.4.1.8 b),c), $(p_{S_n})_{n\in\mathbb{N}}$ has an infimum p in $\mathcal{S}_w(f)$ and $(p_{S_n})_{n\in\mathbb{N}}$ converges to p in $(\mathcal{S}_w(f))_H$. Let $a \in H$ and let $\varepsilon > 0$. Put

$$A := \{|a| > \varepsilon\} .$$

For $n \in \mathbb{N}$,

$$|\langle a, p_{S_n}\rangle| \le \sum_{t\in A} p_{S_n}(t)|a(t)| + \sum_{t\in T\setminus A} p_{S_n}(t)|a(t)| \le$$

$$\le \frac{\|a\|_\infty \operatorname{Card} A}{\operatorname{Card} S_n} + \varepsilon ,$$

so that

$$\limsup_{n\to\infty} |\langle a, p_{S_n}\rangle| \le \varepsilon .$$

Since ε is arbitrary,

$$\lim_{n\to\infty} \langle a, p_{S_n} \rangle = 0 .$$

Since a is arbitrary, $p = 0$.

By Theorem 4.4.1.8 d) (and Theorem 7.1.2.1 c)), the infimum of $(\vec{p}_{S_n})_{n\in\mathbb{N}}$ in $\mathcal{L}(\ell^2(T))$ is 0. ■

7.1.3 Supplementary Results

Proposition 7.1.3.1 (7) *Let T_1, T_2 be two groups and let f_1, f_2 be Schur functions for T_1 and T_2, respectively. Take*

$$g : T_1 \times T_2 \longrightarrow \{\alpha \in \mathbb{K} |\ |\alpha| = 1\},$$

$$h : T_1 \times T_2 \longrightarrow \{\alpha \in \mathbb{K} |\ |\alpha| = 1\}$$

and put

$$T := T_1 \times T_2,$$

$$f : T \times T \longrightarrow \{\alpha \in \mathbb{K} |\ |\alpha| = 1\}, \quad ((s_1, s_2), (t_1, t_2)) \longmapsto$$

$$\longmapsto \overline{g(s_1, s_2)}\ \overline{g(t_1, s_2 t_2)}\ g(s_1 t_1, s_2 t_2)\ \overline{h(t_1, t_2)}\ h(t_1, s_2 t_2) f_1(s_1, t_1) f_2(s_2, t_2).$$

Then the following are equivalent:

a) *For $(s_1, s_2), (t_1, t_2) \in T$,*

$$\begin{cases} g(t_1, 1) = h(t_1, 1) = g(1, t_2) = h(1, t_2) = 1 \\ g(s_1, t_2) g(t_1, t_2)\ \overline{g(s_1 t_1, t_2)} = h(s_1, t_2) h(t_1, t_2)\ \overline{h(s_1 t_1, t_2)} \\ g(t_1, s_2) g(t_1, t_2)\ \overline{g(t_1, s_2 t_2)} = h(t_1, s_2) h(t_1, t_2)\ \overline{h(t_1, s_2 t_2)}. \end{cases}$$

b) *(resp. c)) f is the (unique) Schur function for T such that*

$$f((s_1, 1), (t_1, 1)) = f_1(s_1, t_1), \quad f((1, s_2), (1, t_2)) = f_2(s_2, t_2),$$

$$f((t_1, 1), (1, t_2)) = g(t_1, t_2), \quad f((1, t_2), (t_1, 1)) = h(t_1, t_2),$$

for all $(s_1, s_2), (t_1, t_2) \in T$.

The above conditions imply:

d) *For $(t_1, t_2) \in T$,*

$$e_{(t_1,1)} * e_{(1,t_2)} = g(t_1, t_2)\overline{h(t_1, t_2)}\ e_{(1,t_2)} * e_{(t_1,1)}.$$

a $\Rightarrow$ b. Take (r_1, r_2), (s_1, s_2), $(t_1, t_2) \in T$. Then

$$f((r_1,r_2),(s_1,s_2))f((r_1s_1,r_2s_2),(t_1,t_2))\ \overline{f((r_1,r_2),(s_1t_1,s_2t_2))}\ \overline{f((s_1,s_2),(t_1,t_2))} =$$

$$= \overline{g(r_1,r_2)}\ \overline{g(s_1,r_2s_2)}\ g(r_1s_1,r_2s_2)\ \overline{h(s_1,s_2)}\ h(s_1,r_2s_2)\ f_1(r_1,s_1)\ f_2(r_2,s_2)\times$$

$$\times\overline{g(r_1s_1,r_2s_2)}\ \overline{g(t_1,r_2s_2t_2)}\ g(r_1s_1t_1,r_2s_2t_2)\ \overline{h(t_1,t_2)}\ h(t_1,r_2s_2t_2)\times$$

$$\times f_1(r_1s_1,t_1)\ f_2(r_2s_2,t_2)g(r_1,r_2)\ g(s_1t_1,r_2s_2t_2)\ \overline{g(r_1s_1t_1,r_2s_2t_2)}\times$$

$$\times h(s_1t_1,s_2t_2)\ \overline{h(s_1t_1,r_2s_2t_2)}\ \overline{f_1(r_1,s_1t_1)}\ \overline{f_2(r_2,s_2t_2)}\ g(s_1,s_2)\times$$

$$\times g(t_1,s_2t_2)\ \overline{g(s_1t_1,s_2t_2)}\ h(t_1,t_2)\ \overline{h(t_1,s_2t_2)}\ \overline{f_1(s_1,t_1)}\ \overline{f_2(s_2,t_2)} =$$

$$= \overline{g(s_1,r_2)}\ h(s_1,r_2)\ g(s_1,r_2s_2t_2)\ \overline{h(s_1,r_2s_2t_2)}\ h(s_1,s_2t_2)\ \overline{g(s_1,s_2t_2)} = 1\,.$$

Moreover

$$f((1,1),(1,1)) = 1\,.$$

Hence f is a Schur function for T.

We have

$$f((s_1,1)(t_1,1)) = \overline{g(s_1,1)}\ \overline{g(t_1,1)}\ g(s_1t_1,1)\ \overline{h(t_1,1)}\ h(t_1,1)\ f_1(s_1,t_1)\ f_2(1,1) =$$

$$= f_1(s_1,t_1)\,,$$

$$f((1,s_2),(1,t_2)) =$$

$$= \overline{g(1,s_2)}\ \overline{g(1,s_2t_2)}\ g(1,s_2t_2)\ \overline{h(1,t_2)}\ h(1,s_2t_2)\ f_1(1,1)\ f_2(s_2,t_2) =$$

$$= f_2(s_2,t_2)\,,$$

$$f((t_1,1),(1,t_2)) =$$

$$= \overline{g(t_1,1)}\ g(1,t_2)\ g(t_1,t_2)\ \overline{h(1,t_2)}\ h(1,t_2)\ f_1(t_1,1)\ f_2(1,t_2) = g(t_1,t_2)\,,$$

$$f((1,t_2)),(t_1,1)) = \overline{g(1,t_2)}\ \overline{g(t_1,t_2)}\ g(t_1,t_2)\ \overline{h(t_1,1)}\ h(t_1,t_2) = h(t_1,t_2)$$

(Theorem 7.1.1.5 a)).

b $\Rightarrow$ c follows from the multiplicative property of Schur functions.

c $\Rightarrow$ a. By Theorem 7.1.1.5 a),

$$g(t_1,1) = f((t_1,1),(1,1)) = 1\,, \quad g(1,t_2) = f((1,1),(1,t_2)) = 1\,,$$

$$h(t_1,1) = f((1,t_1),(t_1,1)) = 1\,, \quad h(1,t_2) = f((1,t_2),(1,1)) = 1\,.$$

We have

$$f((1,s_2),(s_1,1))\ f((s_1,s_2),(t_1,1)) = f((1,s_2),(s_1t_1,1))\ f((s_1,1),(t_1,1))$$

so that

$$h(s_1,s_2)\ f((s_1,s_2),(t_1,1)) = h(s_1t_1,s_2)\ f_1(s_1,t_1)\,,$$

$$f((s_1,s_2),(t_1,1)) = \overline{h(s_1,s_2)}\ h(s_1t_1,s_2)\ f_1(s_1,t_1)\,.$$

Moreover,

$$\begin{cases} f((s_1,1),(t_1,t_2)) = \overline{g(t_1,t_2)}\ g(s_1t_1,t_2)\ \overline{h(t_1,t_2)}\ h(t_1,t_2)\ f_1(s_1,t_1) \\ f((s_1,s_2),(1,t_2)) = \overline{g(s_1,s_2)}\ g(s_1,s_2t_2)\ f_2(s_2,t_2) \\ f((1,s_2),(t_1,t_2)) = \overline{g(t_1,s_2t_2)}\ g(t_1,s_2t_2)\ \overline{h(t_1,t_2)}\ h(t_1,s_2t_2)\ f_2(s_2,t_2)\,. \end{cases}$$

From

$$\begin{cases} f((s_1,1),(1,t_2))\ f((s_1,t_2),(t_1,1)) = f((s_1,1),(t_1,t_2))\ f((1,t_2),(t_1,1)) \\ f((1,s_2),(t_1,1))f((t_1,s_2),(1,t_2)) = f((1,s_2),(t_1,t_2))f((t_1,1),(1,t_2))\,, \end{cases}$$

we deduce

$$\begin{cases} g(s_1,t_2)\ \overline{h(s_1,t_2)}\ h(s_1t_1,t_2)\ f_1(s_1,t_1) = \overline{g(t_1,t_2)}\ g(s_1t_1,t_2)\ f_1(s_1,t_1)\ h(t_1,t_2)\,, \\ h(t_1,s_2)\ \overline{g(t_1,s_2)}\ g(t_1,s_2t_2)\ f_2(s_2,t_2) = \overline{h(t_1,t_2)}\ h(t_1,s_2t_2)\ f_2(s_2,t_2)\ g(t_1,t_2)\,, \end{cases}$$

$$\begin{cases} g(s_1,t_2)\ g(t_1,t_2)\ \overline{g(s_1t_1,t_2)} = h(s_1,t_2)\ h(t_1,t_2)\ \overline{h(s_1t_1,t_2)} \\ g(t_1,s_2)\ g(t_1,t_2)\ \overline{g(t_1,s_2t_2)} = h(t_1,s_2)\ h(t_1,t_2)\ \overline{h(t_1,s_2t_2)}\,. \end{cases}$$

b $\Rightarrow$ d. We have

$$e_{(t_1,1)} * e_{(1,t_2)} = f((t_1,1),(1,t_2))e_{(t_1,1)(1,t_2)} =$$

$$= g(t_1,t_2)e_{(1,t_2)(t_1,1)} = g(t_1,t_2)\ \overline{f((1,t_2),(t_1,1))}\ e_{(1,t_2)} * e_{(t_1,1)} =$$

$$= g(t_1,t_2)\ \overline{h(t_1,t_2)}\ e_{(1,t_2)} * e_{(t_1,1)}\,. \qquad \blacksquare$$

Corollary 7.1.3.2 (7) *Let T_1, T_2 be two groups and let f_1, f_2 be Schur functions for T_1 and T_2, respectively. For $j \in \mathbb{N}_2$, let T_j^+ be a subgroup of T_j such that*

$$T_j^- T_j^- \subset T_j^+, \quad (T_j^+ T_j^-) \cup (T_j^- T_j^+) \subset T_j^-,$$

where

$$T_j^- := T_j \backslash T_j^+ .$$

Put

$$T := T_1 \times T_2,$$

$$\varepsilon : T \longrightarrow \{\alpha \in \mathbb{K} |\ |\alpha| = 1\}, \quad (t_1, t_2) \longmapsto \begin{cases} 1 & \text{if } t_1 \in T_1^+ \text{ or } t_2 \in T_2^+ \\ -1 & \text{if } t_1 \in T_1^- \text{ and } t_2 \in T_2^-, \end{cases}$$

$$f : T \times T \longrightarrow \{\alpha \in \mathbb{K} |\ |\alpha| = 1\}, \ ((s_1, s_2), (t_1, t_2)) \longmapsto \varepsilon(t_1, s_2) f_1(s_1, t_1) f_2(s_2, t_2) .$$

Then f is the unique Schur function for T such that

$$f((s_1, 1), (t_1, 1)) = f_1(s_1, t_1), \quad f((1, s_2), (1, t_2)) = f_2(s_2, t_2),$$

$$f((t_1, 1), (1, t_2)) = 1, \quad f((1, t_2), (t_1, 1)) = \varepsilon(t_1, t_2)$$

for all $(s_1, s_2), (t_1, t_2) \in T$. Moreover,

$$e_{(t_1,1)} * e_{(1,t_2)} = \varepsilon(t_1, t_2)\ e_{(1,t_2)} * e_{(t_1,1)}$$

for all $(t_1, t_2) \in T$.

Put

$$\begin{cases} g := e_T \\ h := \varepsilon . \end{cases}$$

g und h fufill the condition a) of Proposition 7.1.3.1. By the implication a $\Rightarrow$ c of this proposition, it follows the assertion about f. The final relation follows from Proposition 7.1.3.1 d). ■

Theorem 7.1.3.3 (7) *We use the notation of Corollary 7.1.3.2 and assume T_1 is locally finite, $T_2 := \mathbb{Z}_2^{2n}$ for some $n \in \mathbb{N}$,*

$$T_2^+ := \left\{ t_2 \in \mathbb{Z}_2^{2n} \,\middle|\, \sum_{j=1}^{2n} t_2(j) \equiv 0 \pmod 2 \right\},$$

and f_2 real. Put

$$\theta : \mathbb{N}_{2n} \longrightarrow \mathbb{Z}_2^{2n}, \quad j \longmapsto e_j^{\mathbb{N}_{2n}},$$

$$q := \operatorname{Card}\{j \in \mathbb{N}_{2n} | \tilde{f}_2(\theta(j)) = 1\}$$

and suppose

$$f_2(\theta(j), \theta(k)) = -f_2(\theta(k), \theta(j))$$

for all distinct $j, k \in \mathbb{N}_{2n}$. Let $\psi_0 : \mathcal{S}(f_2) \to \mathbb{K}_{2^n,2^n}$ be an isomorphism of unital C^*-algebras, $\psi : \mathcal{S}(f_2) \to (\mathcal{S}_c(f_1))_{2^n,2^n}$ the map obtained from ψ_0 by replacing the unit of $\mathbb{K}$ with the unit of $\mathcal{S}_c(f_1)$, and

$$\prod : T_1 \longrightarrow (\mathcal{S}_c(f_1))_{2^n,2^n}, \quad t_1 \longmapsto \begin{cases} i^{n-q} \prod_{j=1}^{2n} \psi(e_{\theta(j)}) & \text{if } t_1 \in T_1^- \\ 1 & \text{if } t_1 \in T_1^+ . \end{cases}$$

a) *For $(t_1, t_2) \in T$,*

$$e_{t_1} * \psi(e_{t_2}) = \psi(e_{t_2}) * e_{t_1}$$

(Theorem 5.6.6.1 f)),

$$\prod(t_1) * \psi(e_{t_2}) = \varepsilon(t_1, t_2)\psi(e_{t_2}) * \prod(t_1).$$

b) *For $s_1, t_1 \in T_1$,*

$$\prod(s_1 t_1) = \prod(s_1) * \prod(t_1), \quad \left(\prod(t_1)\right)^* = \prod(t_1).$$

c) *There is a unique isomorphism of unital C^*-algebras*

$$\varphi : \mathcal{S}_c(f) \longrightarrow (\mathcal{S}_c(f_1))_{2^n,2^n}$$

such that

$$\varphi(e_{(t_1,t_2)}) = e_{t_1} * \prod(t_1) * \psi(e_{t_2})$$

for all $(t_1, t_2) \in T$ (Theorem 5.6.6.1 f)).

a) The relation

$$e_{t_1} * \psi(e_{t_2}) = \psi(e_{t_2}) * e_{t_1}$$

follows form the fact that ψ was defined by using ψ_0. In order to verify the second equality we may assume $t_1 \in T_1^-$ and $t_2 = \theta(k)$ for a $k \in \mathbb{N}_{2n}$. Then

$$\prod(t_1) * \psi(e_{t_2}) = i^{n-q}\left(\prod_{j=1}^{2n}\psi(e_{\theta(j)})\right) * \psi(e_{\theta(k)}) =$$

$$= -i^{n-q}\psi(e_{\theta(k)}) * \prod_{j=1}^{2n}\psi(e_{\theta(j)}) = \varepsilon(t_1,t_2)\psi(e_{t_2}) * \prod(t_1)\,.$$

b) We may assume $s_1, t_1 \in T_1^-$. Then $s_1t_1 \in T_1^+$ and by a),

$$\prod(s_1) * \prod(t_1) = (-1)^{n-q}\left(\prod_{j=1}^{2n}\psi(e_{\theta(j)})\right)^2 =$$

$$= (-1)^{n-q}(-1)^{2n-q}(-1)^{(2n-1)+(2n-2)+\cdots+2+1} =$$

$$= (-1)^{3n-2q+n(2n-1)} = 1 = \prod(s_1t_1)\,,$$

$$\left(\prod(t_1)\right)^* = (-1)^{n-q}i^{n-q}\left(\prod_{j=1}^{2n}\psi(e_{\theta(j)})\right)^* =$$

$$= (-1)^{n-q}i^{n-q}(-1)^{2n-q}(-1)^{(2n-1)+(2n-2)+\cdots+2+1}\prod_{j=1}^{2n}\psi(e_{\theta(j)}) =$$

$$= (-1)^{3n-2q+n(2n-1)}\prod(t_1) = \prod(t_1)\,.$$

c) We define φ first on $\mathcal{S}(f)$ by extending the map

$$\{e_t | t \in T\} \longrightarrow (\mathcal{S}_c(f_1))_{2^n,2^n}\,, \quad e_{(t_1,t_2)} \longmapsto e_{t_1} * \prod(t_1) * \psi(e_{t_2})$$

linearly. Take (s_1,s_2), $(t_1,t_2) \in T$. By a) and b),

$$\varphi\Big(e_{(s_1,s_2)} * e_{(t_1,t_2)}\Big) = f((s_1,s_2),(t_1,t_2))\varphi(e_{(s_1t_1,s_2t_2)}) =$$

$$= \varepsilon(t_1, s_2) f_1(s_1, t_1) f_2(s_2, t_2) e_{s_1 t_1} * \prod(s_1 t_1) * \psi(e_{s_2 t_2}) =$$

$$= \varepsilon(t_1, s_2) e_{s_1} * e_{t_1} * \prod(s_1) * \prod(t_1) * \psi(e_{s_2}) * \psi(e_{t_2}) =$$

$$= e_{s_1} * \prod(s_1) * \psi(e_{s_2}) * e_{t_1} * \prod(t_1) * \psi(e_{t_2}) = \varphi(e_{(s_1, s_2)}) * \varphi(e_{(t_1, t_2)}),$$

$$\varphi(e^*_{(t_1, t_2)}) = \widetilde{f}((t_1, t_2)) \varphi(e_{(t_1^{-1}, t_2)}) =$$

$$= \varepsilon(t_1, t_2) \widetilde{f}_1(t_1) \widetilde{f}_2(t_2) e_{t_1^{-1}} * \prod(t_1) * \psi(e_{t_2}) =$$

$$= \widetilde{f}_1(t_1) \widetilde{f}_2(t_2) \psi(e_{t_2}) * \prod(t_1) * e_{t_1^{-1}} =$$

$$= \psi(e^*_{t_2}) * \prod(t_1) * e^*_{t_1} = \psi(e_{t_2})^* * \left(\prod(t_1)\right)^* * e^*_{t_1} =$$

$$= (e_{t_1} * \prod(t_1) * \psi(e_{t_2}))^* = \varphi(e_{(t_1, t_2)})^*.$$

Hence φ is an involutive unital algebra homomorphism. It is easy to see that φ is injective.

Case 1 T_1 finite

Since

$$\operatorname{Dim} \mathcal{S}(f) = \operatorname{Card} T = 4^n \operatorname{Card} T_1 = \operatorname{Dim} (\mathcal{S}(f_1))_{2^n, 2^n},$$

φ is bijective. Hence φ is an isomorphism of C^*–algebras.

Case 2 T_1 infinite

Take $S \in \mathfrak{S}_T$. Let $f_{1,S}$ be the restriction of f_1 to $S \times S$ and f_S the restriction of f to $(S \times T_2) \times (S \times T_2)$. By Case 1, the map

$$\mathcal{S}(f_S) \longrightarrow (\mathcal{S}(f_{1,S}))_{2^n, 2^n}$$

defined by φ is an isometry of unital C^*–algebras. Since S is arbitrary, the map

$$\mathcal{S}(f) \longrightarrow (\mathcal{S}(f_1))_{2^n, 2^n}$$

defined by φ is an isometry. It extends therefore to an isometry of unital C^*–algebras

$$\mathcal{S}_c(f) \longrightarrow (\mathcal{S}_c(f_1))_{2^n, 2^n}$$

(Theorem 5.6.6.1 a)). ■

Corollary 7.1.3.4 (7) *We use the notation and assumptions of Theorem 7.1.3.3 with* $n = 1$. *Put*

$$\varepsilon_1 := e_{T_1^+}^{T_1} - e_{T_1^-}^{T_1}$$

and $\delta := 1$ *(resp.* $\delta := -1$ *and* $\mathbb{K} = \mathbb{C}$*).*

a) *The function* f_2 *defined by*

f_2	$(1,0)$	$(0,1)$	$(1,1)$
$(1,0)$	-1	1	-1
$(0,1)$	-1	δ	$-\delta$
$(1,1)$	1	δ	δ

is a Schur function for $\mathbb{Z}_2^2$ *such that*

$$f_2(\theta(j), \theta(k)) = -f_2(\theta(k), \theta(j))$$

for all distinct $j, k \in \mathbb{N}_2$ *and* $q = \frac{1+\delta}{2}$.

b) *There is a unique isomorphism of unital* C^**-algebras* $\psi_0 : \mathcal{S}(f_2) \to \mathbb{K}_{2,2}$ *such that*

$$\psi_0(e_{(1,0)}) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad \psi_0(e_{(0,1)}) = i^{\frac{1-\delta}{2}} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

c) *For* $t_1 \in T_1^-$,

$$\prod(t_1) = (-1)^{\frac{1-\delta}{2}} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

d) *There is a unique isomorphism of unital* C^**-algebras*

$$\varphi : \mathcal{S}_c(f) \longrightarrow (\mathcal{S}_c(f_1))_{2,2}$$

such that

$$\varphi(e_{(t_1,t_2)}) = (-1)^{\frac{(1-\delta)(1-\varepsilon_1(t_1))}{4}} \begin{bmatrix} e_{t_1} & 0 \\ 0 & \varepsilon_1(t_1) e_{t_1} \end{bmatrix} * \psi(e_{t_2})$$

for every $(t_1, t_2) \in T$.

e) *Assume* $\mathbb{K} = \mathbb{C}$, $\delta = -1$, S *is a set, and* $\mathcal{S}_c(f_1)$ *is identified with* $\mathcal{L}(\ell^2(S))$ *via an isomorphism of unital* C^**-algebras. Take* $\tau \in \{-1, +1\}$ *and* $t_1 \in T_1$ *such that*

$$\overline{e}_{t_1} = \tau e_{t_1}$$

(Definition 5.5.7.1). Then

$$\overline{\varphi(e_{(t_1,0)})} = \tau\varphi(e_{(t_1,0)})$$

and

$$\overline{\varphi(e_{(0,(1,0))})} = \varphi(e_{(0,(1,0))}), \quad \overline{\varphi(e_{(0,(0,1))})} = -\varphi(e_{(0,(0,1))}).$$

a) is a straightforward verification.

b) The uniqueness of ψ_0 follows from the fact that $e_{(1,0)}$ and $e_{(0,1)}$ generate $\mathcal{S}(f_2)$.

Put

$$a := \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad b := i^{\frac{1-\delta}{2}} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad c := ab = i^{\frac{1-\delta}{2}} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Then

$$a^2 = -1, \quad b^2 = \delta 1, \quad c^2 = \delta 1,$$

$$a^* = -a, \quad b^* = \delta b, \quad c^* = \delta c,$$

$$ac = -ca = -b, \quad bc = -cb = -\delta a, \quad ba = -c.$$

First define ψ_0 on $\{e_{t_2} | t_2 \in \mathbb{Z}_2^2\}$ by putting

$$\psi_0(e_{(1,0)}) := a, \quad \psi_0(e_{(0,1)}) := b,$$

$$\psi_0(e_{(0,0)}) := 1, \quad \psi_0(e_{(1,1)}) := c,$$

and then extend ψ_0 linearly to $\mathcal{S}(f_2)$. It is easy to see that ψ_0 is bijective. By the above,

$$\psi_0(e_{s_2} * e_{t_2}) = \psi_0(e_{s_2}) * \psi_0(e_{t_2}),$$

$$\psi_0(e_{t_2}^*) = \psi_0(e_{t_2})^*$$

for all $s_2, t_2 \in \mathbb{Z}_2^2$. Hence ψ_0 is an isomorphism of unital C^*-algebras.

c) By a) and b),

$$\prod(t_1) = i^{1-\frac{1+\delta}{2}} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} * i^{\frac{1-\delta}{2}} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = (-1)^{\frac{1-\delta}{2}} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

d) Take $(t_1, t_2) \in T$. If $t_1 \in T_1^+$ then

$$e_{t_1} * \prod(t_1) = \begin{bmatrix} e_{t_1} & 0 \\ 0 & e_{t_1} \end{bmatrix} = (-1)^{\frac{(1-\delta)(1-\varepsilon_1(t_1))}{4}} \begin{bmatrix} e_{t_1} & 0 \\ 0 & \varepsilon_1(t_1)e_{t_1} \end{bmatrix}.$$

If $t_1 \in T_1^-$ then by c),

$$e_{t_1} * \prod(t_1) = (-1)^{\frac{1-\delta}{2}} \begin{bmatrix} e_{t_1} & 0 \\ 0 & -e_{t_1} \end{bmatrix} = (-1)^{\frac{(1-\delta)(1-\varepsilon_1(t_1))}{4}} \begin{bmatrix} e_t & 0 \\ 0 & \varepsilon_1(t_1)e_{t_1} \end{bmatrix}.$$

Hence the assertion follows from Theorem 7.1.3.3 c).

e) follows immediately from d). ∎

Proposition 7.1.3.5 *Let T be an uncountable group, f a Schur function for T, and $\aleph', \aleph''$ cardinal numbers such that:*

1) $\aleph' < \operatorname{Card} T$.

2) $\aleph'' \leq \operatorname{Card} T$.

3) *The sum of any countable family of cardinal numbers strictly smaller than $\aleph''$ is strictly smaller than $\aleph''$.*

Denote by $\mathcal{F}$ the set of $x \in \mathcal{L}(\ell^2(T))$ for which the Hilbert dimension of $\operatorname{Im} x$ *is at most equal to $\aleph'$ (is strictly smaller than $\aleph''$).*

a) *$\mathcal{F}$ is a closed proper ideal of $\mathcal{L}(\ell^2(T))$.*

b) *$\mathcal{F} + \overrightarrow{\mathcal{S}_w(f)}$ is the C^*-subalgebra of $\mathcal{L}(\ell^2(T))$ generated by $\mathcal{F} \cup \overrightarrow{\mathcal{S}_w(f)}$.*

c) $\mathcal{F} \cap \overrightarrow{\mathcal{S}_w(f)} = \{0\}$.

d) $(x, y) \in \mathcal{F} \times \overrightarrow{\mathcal{S}_w(f)} \Rightarrow \|y\| \leq \|x + y\|$.

a) was proved in Proposition 5.5.3.13 a).

b) follows from Proposition 5.5.3.13 b) (and Theorem 7.1.2.1 j)).

c) follows from Proposition 7.1.1.1 l) and Theorem 5.5.2.5.

d) follows from c) and Proposition 5.5.3.13 c). ∎

Proposition 7.1.3.6 *Let T, T' be groups, f, f' Schur functions for T and T', respectively, $\varphi : T \to T'$ an injective group homomorphism, and*

$$\lambda : T \longrightarrow \{\alpha \in \mathbb{K} |\ |\alpha| = 1\}\,.$$

Then the following are equivalent:

a) $s, t \in T \Rightarrow f'(\varphi(s), \varphi(t)) = \frac{\lambda(s)\lambda(t)}{\lambda(st)} f(s,t)$.

b) *For all $x, y \in \ell^2(T')$ with $y = ye_{\varphi(T)}$,*

$$(\lambda(x \circ \varphi)) * (\lambda(y \circ \varphi)) = \lambda((x * y) \circ \varphi)\,.$$

c) $s, t \in T \Rightarrow (\lambda(e_{\varphi(s)} \circ \varphi)) * (\lambda(e_{\varphi(t)} \circ \varphi)) = \lambda((e_{\varphi(s)} * e_{\varphi(t)}) \circ \varphi)$.

If these conditions are fulfilled then:

d) $\lambda(1) = 1$.

e) $x \in \mathcal{S}_w(f') \Rightarrow \lambda(x \circ \varphi) \in \mathcal{S}_w(f)$, $\|\lambda(x \circ \varphi)\| \leq \|x\|$.

f) *If φ is bijective then the map*

$$\mathcal{S}_w(f') \to \mathcal{S}_w(f)\,, \quad x \longmapsto \lambda(x \circ \varphi)$$

is an isometry of Banach algebras.

g) *The following are equivalent:*

g_1) $x \in \ell^2(T') \Rightarrow (\lambda(x \circ \varphi))^* = \lambda(x^* \circ \varphi)$;

g_2) $t \in T \Rightarrow (\lambda(e_{\varphi(t)} \circ \varphi))^* = \lambda(e^*_{\varphi(t)} \circ \varphi)$;

g_3) $t \in T \Rightarrow \lambda(t)^2 \lambda(t^{-1})^2 = 1$.

h) *If g is a Schur function for T such that the map*

$$\mathcal{S}_w(g) \longrightarrow \mathcal{S}_w(f)\,, \quad x \longmapsto \lambda x$$

is an isometry of Banach algebras then

$$g(s,t) = \frac{\lambda(s)\lambda(t)}{\lambda(st)} f(s,t)$$

for all $s, t \in T$.

a $\Rightarrow$ b. For $t \in T$,

$$((\lambda(x\circ\varphi))*(\lambda(y\circ\varphi)))(t) =$$

$$= \sum_{s\in T} f(ts^{-1}, s)\lambda(ts^{-1})x(\varphi(ts^{-1}))\lambda(s)y(\varphi(s)) =$$
$$= \lambda(t)\sum_{s\in T} f'(\varphi(t)\varphi(s)^{-1}, \varphi(s))x(\varphi(t)\varphi(s)^{-1})y(\varphi(s)) =$$
$$= \lambda(t)\sum_{s'\in T'} f'(\varphi(t)s'^{-1}, s')x(\varphi(t)s'^{-1})y(s') = \lambda(t)(x*y)(\varphi(t)).$$

Hence

$$(\lambda(x\circ\varphi))*(\lambda(y\circ\varphi)) = \lambda((x*y)\circ\varphi).$$

b $\Rightarrow$ c is trivial.

c $\Rightarrow$ a. For $r \in T$,

$$((\lambda(e_{\varphi(s)}\circ\varphi))*(\lambda(e_{\varphi(t)}\circ\varphi)))(r) = ((\lambda e_s)*(\lambda e_t))(r) =$$

$$\sum_{q\in T} f(rq^{-1}, q)\lambda(rq^{-1})e_s(rq^{-1})\lambda(q)e_t(q) =$$
$$= f(rt^{-1}, t)\lambda(rt^{-1})e_s(rt^{-1})\lambda(t) = f(s,t)\lambda(s)\lambda(t)\delta_{r,st},$$

$$(\lambda((e_{\varphi(s)}*e_{\varphi(t)})\circ\varphi))(r) = \lambda(r)(e_{\varphi(s)}*e_{\varphi(t)})(\varphi(r)) =$$

$$= \lambda(r)\sum_{q'\in T'} f'(\varphi(r)q'^{-1}, q')e_{\varphi(s)}(\varphi(r)q'^{-1})e_{\varphi(t)}(q') =$$
$$= \lambda(r)f'(\varphi(r)\varphi(t)^{-1}, \varphi(t))e_{\varphi(s)}(\varphi(r)\varphi(t)^{-1}) =$$

$$= \lambda(st)f'(\varphi(s), \varphi(t))\delta_{r,st}.$$

Putting $r := st$, we obtain

$$f'(\varphi(s), \varphi(t)) = \frac{\lambda(s)\lambda(t)}{\lambda(st)} f(s,t).$$

a $\Rightarrow$ d. Take $s = t = 1$.

b $\Rightarrow$ e. Take $z \in \ell^2(T)$. There is a $y \in \ell^2(T')$ such that

$$y = ye_{\varphi(T)}, \quad \lambda(y\circ\varphi) = z.$$

By b),

$$(\lambda(x\circ\varphi))*z = (\lambda(x\circ\varphi))*(\lambda(y\circ\varphi)) =$$

$$= \lambda((x*y)\circ\varphi) \in \ell^2(T)\,,$$

$$\|(\lambda(x\circ\varphi))*z\|_2 = \|\lambda((x*y)\circ\varphi)\|_2 \leq$$

$$\leq \|x*y\|_2 \leq \|x\|\,\|y\|_2 = \|x\|\,\|z\|_2\,.$$

Hence

$$\lambda(x\circ\varphi) \in \mathcal{S}_w(f)\,, \quad \|\lambda(x\circ\varphi)\| \leq \|x\|\,.$$

b & e $\Rightarrow$ f is obvious.

g) For every $x \in \ell^2(T')$ and $t \in T$,

$$(\lambda(x\circ\varphi))^*(t) = \widetilde{f}(t)\,\overline{(\lambda(x\circ\varphi))(t^{-1})} = f(t,t^{-1})\overline{\lambda(t^{-1})}\,\overline{x(\varphi(t^{-1}))}\,,$$

$$(\lambda(x^*\circ\varphi))(t) = \lambda(t)x^*(\varphi(t)) = \lambda(t)f'(\varphi(t),\varphi(t)^{-1})\overline{x(\varphi(t)^{-1})} =$$

$$= \lambda(t)\frac{\lambda(t)\lambda(t^{-1})}{\lambda(1)}f(t,t^{-1})\overline{x(\varphi(t^{-1}))}\,.$$

$g_3 \Rightarrow g_1$ follows by d).

$g_1 \Rightarrow g_2$ is trivial.

$g_2 \Rightarrow g_3$ follows from the above and from d) by putting $x := e_{\varphi(t^{-1})}$.

h) Put

$$g' : T\times T \longrightarrow \{\alpha \in \mathbb{K} \mid |\alpha| = 1\}\,, \quad (s,t) \longmapsto \frac{\lambda(s)\lambda(t)}{\lambda(st)}f(s,t)\,.$$

By d) and Proposition 7.1.1.8. a),c), g' is a Schur function for T. By a $\Rightarrow$f, the map

$$\mathcal{S}_w(g') \longrightarrow \mathcal{S}_w(f)\,, \quad x \longmapsto \lambda x$$

is an isometry of Banach algebras. Hence the Banach algebras $\mathcal{S}_w(g)$ and $\mathcal{S}_w(g')$ coincide and this implies $g = g'$. ■

Proposition 7.1.3.7 *Let T be a finite group, f a Schur function for T, and $(R,\mathfrak{R},\mu)$ a measure space. For every $p \in [1,\infty]$ denote by H_p the set of*

$$\xi : T\times R \longrightarrow \mathbb{K}$$

such that

$$\xi(t,\cdot) \in L^p(\mu)$$

for every $t \in T$, and for every $\xi \in H_1$ put

$$\int \xi d\mu : T \longrightarrow \mathbb{K}\,, \quad t \longmapsto \int \xi(t,r)d\mu(r)\,.$$

a) *Let* $p, q, r \in [1, \infty]$ *with*

$$\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$$

and $(\xi, \eta) \in H_p \times H_q$. *Then* $\xi * \eta \in H_r$ *and*

$$(\xi * \eta)^* = \eta^* * \xi^*,$$

where

$$\xi * \eta : T \times R \longrightarrow \mathbb{K}, \quad (t, r) \longmapsto \sum_{s \in T} f(ts^{-1}, s)\xi(ts^{-1}, r)\eta(s, r)$$

and

$$\xi^* : T \times R \longrightarrow \mathbb{K}, \quad (t, r) \longmapsto \widetilde{f}(t)\overline{\xi(t^{-1}, r)}.$$

b) *If* $\xi \in H_1$ *and* $x \in \mathcal{S}(f)$, *then*

$$\int (x * \xi) d\mu = x * \int \xi d\mu, \quad \int (\xi * x) d\mu = \left(\int \xi d\mu \right) * x,$$

$$\left(\int \xi d\mu \right)^* = \int \xi^* d\mu,$$

where

$$x * \xi : T \times R \longrightarrow \mathbb{K}, \quad (t, r) \longmapsto \sum_{s \in T} f(ts^{-1}, s)x(ts^{-1})\xi(s, r),$$

$$\xi * x : T \times R \longrightarrow \mathbb{K}, \quad (t, r) \longmapsto \sum_{s \in T} f(ts^{-1}, s)\xi(ts^{-1}, r)x(r).$$

c) *Let* $\xi \in H_1$ *such that* $\xi(\cdot, r) \in \mathcal{S}(f)_+$ *for* μ*-almost all* $r \in R$. *Then*

$$\int \xi d\mu \in \mathcal{S}(f)_+$$

and

$$\int \xi d\mu = 0$$

implies

$$\xi(t, r) = 0$$

for all $t \in T$ *and for* μ*-almost all* $r \in R$.

d) H_2 *endowed with the right and left multiplications*

$$H_2 \times \mathcal{S}(f) \longrightarrow H_2\,, \quad (\xi, x) \longmapsto \xi * x$$

$$\mathcal{S}(f) \times H_2 \longrightarrow H_2\,, \quad (x, \xi) \longmapsto x * \xi$$

and with the inner-product

$$H_2 \times H_2 \longrightarrow \mathcal{S}(f)\,, \quad (\xi, \eta) \longmapsto \int (\eta^* * \xi) d\mu$$

is a unital Hilbert $\mathcal{S}(f)$*-module.*

e) *For every* $t \in T$ *and* $k \in L^2(\mu)$*,*

$$ke_t \in H_2\,, \; \|ke_t\| = \|k\|_2\,.$$

f) *For every* $\xi, \eta \in H_2$*,*

$$\langle \xi | \eta \rangle : T \longrightarrow \mathbb{K}\,, \quad t \longmapsto \sum_{s \in T} \langle \xi(s, \cdot) | f(st^{-1}, t) \eta(st^{-1}, \cdot) \rangle\,,$$

$$\|\xi\| \leq \sum_{t \in T} \|\xi(t, \cdot)\|_2\,.$$

g) *If* $(k_\iota)_{\iota \in I}$ *is an orthonormal basis of* $L^2(\mu)$ *then for every* $\xi \in H_2$*,*

$$\xi = \sum_{\iota \in I} k_\iota \langle \xi | k_\iota 1 \rangle\,.$$

h) *For every* $\xi \in H_\infty$ *define*

$$\overset{\sim}{\xi} : H_2 \longrightarrow H_2\,, \quad \eta \longmapsto \xi * \eta\,.$$

Then $\overset{\sim}{\xi} \in \mathcal{L}_{\mathcal{S}(f)}(H_2)$ *and*

$$\overset{\sim}{\xi}^* = \overset{\sim}{\xi^*}\,, \quad \sup_{t \in T} \|(\xi^* * \xi)(t, \cdot)\|_\infty^{\frac{1}{2}} \leq \|\overset{\sim}{\xi}\| \leq \sum_{t \in T} \|\xi(t, \cdot)\|_\infty$$

for every $\xi \in H_\infty$ *and the map*

$$H_\infty \longrightarrow \mathcal{L}_{\mathcal{S}(f)}(H_2)\,, \quad \xi \longmapsto \overset{\sim}{\xi}$$

is an injective involutive unital algebra homomorphism. Moreover, for every $k \in L^\infty(\mu)$ *and* $t \in T$*,*

$$ke_t \in H_\infty\,, \quad \|\overset{\sim}{ke_t}\| = \|k\|_\infty\,.$$

i) *Assume there is an* $\alpha \in \mathbb{R}_+$ *such that*

$$\sup_{t\in T} \|\xi(t,\cdot)\|_\infty \leq \alpha \|\overset{\sim}{\xi}\|$$

for every $\xi \in H_\infty$ *and that* μ *is* σ*-finite or it is a Radon measure on a locally compact space. Put*

$$\widetilde{\xi} : H_1 \longrightarrow \mathbb{K}, \quad \eta \longmapsto \sum_{t\in T} \int \xi(t,\cdot)\eta(t,\cdot)d\mu$$

for every $\xi \in H_\infty$. *Then* H_∞ *endowed with the norm*

$$H_\infty \longrightarrow \mathbb{R}_+, \quad \xi \longmapsto \|\overset{\sim}{\xi}\|$$

is a W^**-algebra and* H_1 *endowed with the norm*

$$H_1 \longrightarrow \mathbb{R}_+, \quad \eta \longmapsto \sup_{\xi\in H_\infty^\#} |\widetilde{\xi}(\eta)|$$

is its predual and

$$\sup_{t\in T} \|\eta(t,\cdot)\|_1 \leq \|\eta\| \leq \alpha \sum_{t\in T} \|\eta(t,\cdot)\|_1$$

for every $\eta \in H_1$.

a) For every $t \in T$,

$$(\xi * \eta)(t,\cdot) = \sum_{s\in T} f(ts^{-1},s)\xi(ts^{-1},\cdot)\eta(s,\cdot) \in L^r(\mu),$$

$$(\xi * \eta)^*(t,\cdot) = \eta^*(t,\cdot) * \xi^*(t,\cdot) = (\eta^* * \xi^*)(t,\cdot).$$

b) For every $t \in T$,

$$\int (x*\xi)(t,r)d\mu(r) = \int \sum_{s\in T} f(ts^{-1},s)x(ts^{-1})\xi(s,r)d\mu(r) =$$

$$= \sum_{s\in T} f(ts^{-1},s)x(ts^{-1}) \int \xi(s,r)d\mu(r) = (x * \int \xi d\mu)(t),$$

$$\int (\xi*x)(t,r)d\mu(r) = \int \sum_{s\in T} f(ts^{-1},s)\xi(ts^{-1},r)x(s)d\mu(r) =$$

$$= \sum_{s \in T} f(ts^{-1}, s) \left(\int \xi(ts^{-1}, r) d\mu(r) \right) x(s) = \left(\left(\int \xi d\mu \right) * x \right)(t),$$

$$\int \xi^*(t, r) d\mu(r) = \int \widetilde{f}(t) \overline{\xi(t^{-1}, r)} d\mu(r) =$$

$$= \widetilde{f}(t) \overline{\int \xi(t^{-1}, r) d\mu(r)} = \left(\int \xi d\mu \right)^* (t).$$

c) For every $x' \in (\mathcal{S}(f))'_+$,

$$\langle \xi(\cdot, r), x' \rangle \in \mathbb{R}_+$$

for μ–almost all $r \in R$. It follows

$$\left\langle \int \xi d\mu, x' \right\rangle = \int \langle \xi(\cdot, r), x' \rangle d\mu(r) \in \mathbb{R}_+$$

so that

$$\int \xi d\mu \in \mathcal{S}(f)_+.$$

Now suppose

$$\int \xi d\mu = 0.$$

Then for every $x' \in (\mathcal{S}(f))'_+$,

$$0 = \left\langle \int \xi d\mu, x' \right\rangle = \int \langle \xi(\cdot, r), x' \rangle d\mu(r)$$

so that

$$\langle \xi(\cdot, r), x' \rangle = 0$$

for μ–almost all $r \in \mathbb{R}$. It follows

$$\xi(t, r) = 0$$

for all $t \in T$ and for μ–almost all $r \in R$.

d) Take $x, y \in \mathcal{S}(f)$ and $\xi, \eta \in H_2$. The relations

$$\xi * (x * y) = (\xi * x) * y, \quad (x * y) * \xi = x * (y * \xi),$$

$$(x * \xi) * y = x * (\xi * y), \quad 1 * \xi = \xi * 1 = \xi$$

are easy to see. By a) and b),

$$\langle \xi * x | \eta \rangle = \int (\eta^* * (\xi * x)) d\mu = \int ((\eta^* * \xi) * x) d\mu =$$

$$= \left(\int (\eta^* * \xi) d\mu \right) * x = \langle \xi | \eta \rangle * x \,,$$

$$\langle x * \xi | \eta \rangle = \int (\eta^* * (x * \xi)) d\mu = \int ((\eta^* * x) * \xi) d\mu =$$

$$= \int ((x^* * \eta)^* * \xi) d\mu = \langle \xi | x^* * \eta \rangle \,,$$

$$\langle \xi | \eta \rangle = \int (\eta^* * \xi) d\mu = \int (\xi^* * \eta)^* d\mu =$$

$$= \left(\int (\xi^* * \eta) d\mu \right)^* = \langle \eta | \xi \rangle^* \,.$$

By c),

$$\langle \xi | \xi \rangle \in \mathcal{S}(f)_+$$

and

$$\langle \xi | \xi \rangle = 0 \Longrightarrow \xi = 0 \,.$$

By Example 5.6.1.5 (and Theorem 7.1.2.1 c)),

$$(\langle \xi(\cdot, r) | \xi(\cdot, r) \rangle + \langle \eta(\cdot, r) | \eta(\cdot, r) \rangle \,,\, \langle \eta(\cdot, r) | \xi(\cdot, r) \rangle - \langle \xi(\cdot, r) | \eta(\cdot, r) \rangle) \in \overbrace{(\mathcal{S}(f))}^{\circ}{}_+$$

$$\langle x\xi(\cdot, r) | x\xi(\cdot, r) \rangle \leq \|x\|^2 \langle \xi(\cdot, r) | \xi(\cdot, r) \rangle$$

for μ-almost all $r \in R$. By c),

$$(\langle \xi | \xi \rangle + \langle \eta | \eta \rangle \,,\, \langle \eta | \xi \rangle - \langle \xi | \eta \rangle) \in \overbrace{(\mathcal{S}(f))}^{\circ}{}_+$$

$$\langle x\xi | x\xi \rangle \leq \|x\|^2 \langle \xi | \xi \rangle \,.$$

e) For every $s \in T$,

$$(ke_t)(s,\cdot) = \delta_{st}k \in L^2(\mu)$$

so that $ke_t \in H_2$. By Theorem 7.1.2.1 b),

$$\|ke_t\|^2 = \|\langle ke_t|ke_t\rangle\| = \left\|\int \overline{k}ke_t^* e_t d\mu\right\| =$$

$$= \left\|\left(\int |k|^2 d\mu\right) e_1\right\| = \|k\|_2 .$$

f) For every $t \in T$,

$$(\eta^* * \xi)(t,\cdot) = \sum_{s\in T} f(ts^{-1},s)\eta^*(ts^{-1},\cdot)\xi(s,\cdot) =$$

$$= \sum_{s\in T} f(ts^{-1},s)\widetilde{f}(ts^{-1})\overline{\eta(st^{-1},\cdot)}\xi(s,\cdot) =$$

$$= \sum_{s\in T} \overline{f(st^{-1},t)}\,\overline{\eta(st^{-1},\cdot)}\xi(s,\cdot)$$

(Proposition 7.1.1.9 c)), so that

$$\langle\xi|\eta\rangle(t) = \int (\eta^* * \xi)(t,\cdot)d\mu =$$

$$= \sum_{s\in T}\langle\xi(s,\cdot)|f(st^{-1},\cdot)\eta(st^{-1},\cdot)\rangle .$$

By Proposition 7.1.1.1 d),

$$\|\xi\|^2 = \|\langle\xi|\xi\rangle\| \le \sum_{t\in T} |\langle\xi|\xi\rangle(t)| \le$$

$$\le \sum_{s,t\in T} |\langle\xi(s,\cdot)|f(st^{-1},\cdot)\xi(st^{-1},\cdot)| \le$$

$$\le \sum_{s,t\in T} \|\xi(s,\cdot)\|_2\|\xi(st^{-1},\cdot)\|_2 = \left(\sum_{t\in T}\|\xi(t,\cdot)\|_2\right)^2 ,$$

$$\|\xi\| \le \sum_{t\in T}\|\xi(t,\cdot)\|_2 .$$

g) For every $t \in T$ and $\iota \in I$,

$$(\langle \xi | k_\iota 1 \rangle)(t) = \int \overline{k_\iota(r)} \xi(t,r) d\mu(r) = \langle \xi(t,\cdot) | k_\iota \rangle$$

so that

$$\xi(t,\cdot) = \sum_{\iota \in I} \langle \xi(t,\cdot) | k_\iota \rangle k_\iota = \sum_{\iota \in I} k_\iota (\langle \xi | k_\iota 1 \rangle)(t) .$$

Since T is finite

$$\xi = \sum_{\iota \in I} k_\iota \langle \xi | k_\iota 1 \rangle$$

(Proposition 7.1.1.1 d)).

h) Take $\eta, \zeta \in H_2$. By a),

$$\langle \widetilde{\xi} \eta | \zeta \rangle = \langle \xi * \eta | \zeta \rangle = \int (\zeta^* * (\xi * \eta)) d\mu = \int ((\zeta^* * \xi) * \eta) d\mu =$$

$$= \int ((\xi^* * \zeta)^* * \eta) d\mu = \langle \eta | \xi^* * \zeta \rangle = \langle \eta | \widetilde{\xi^*} \zeta \rangle .$$

Hence $\widetilde{\xi} \in \mathcal{L}_{\mathcal{S}(f)}(H_2)$ and $\widetilde{\xi}^* = \widetilde{\xi^*}$. Thus the map

$$H_\infty \longrightarrow \mathcal{L}_{\mathcal{S}(f)}(H_2) , \quad \xi \longmapsto \widetilde{\xi}$$

is involutive . It is obviously an algebra homomorphism.

Now we prove the inequalities. By Proposition 7.1.1.1 d),

$$\|\xi(\cdot, r)\| \leq \sum_{t \in T} |\xi(t,r)| \leq \sum_{t \in T} \|\xi(t,\cdot)\|_\infty$$

for μ–almost all $r \in R$. Take $\eta \in H_2$. Then

$$0 \leq \eta^* * \xi^* * \xi * \eta \leq \|\xi\|^2 \eta^* * \eta \leq \left(\sum_{t \in T} \|\xi(t,\cdot)\|_\infty \right)^2 \eta^* * \eta \quad \mu\text{–a.e.}$$

(Corollary 4.2.2.3), so that by c),

$$0 \leq \langle \xi * \eta | \xi * \eta \rangle \leq \left(\sum_{t \in T} \|\xi(t,\cdot)\|_\infty \right)^2 \langle \eta | \eta \rangle .$$

By Corollary 4.2.1.18,

$$\|\xi * \eta\|^2 \leq \left(\sum_{t\in T} \|\xi(t,\cdot)\|_\infty\right)^2 \|\eta\|^2 .$$

It follows

$$\|\widetilde{\xi}\eta\| = \|\xi * \eta\| \leq \left(\sum_{t\in T} \|\xi(t,\cdot)\|_\infty\right) \|\eta\| ,$$

$$\|\widetilde{\xi}\| \leq \sum_{t\in T} \|\xi(t,\cdot)\|_\infty .$$

Take $t \in T$ and $\alpha \in]0,1[$. There is a $k \in L^2(\mu)^\#$ such that

$$\left|\int |k|^2 (\xi^* * \xi)(t,\cdot) d\mu\right| \geq \alpha \|(\xi * \xi)(t,\cdot)\|_\infty .$$

Since

$$\int |k|^2 (\xi^* * \xi)(t,\cdot) d\mu = \langle k\xi | k\xi\rangle(t)$$

it follows

$$|\langle k\xi | k\xi\rangle(t)| \geq \alpha \|(\xi^* * \xi)(t,\cdot)\|_\infty .$$

By e), $ke_1 \in H_2^\#$ so that

$$\|\widetilde{\xi}\|^2 \geq \|\widetilde{\xi}(ke_1)\|^2 = \|\xi * (ke_1)\|^2 = \|k\xi\|^2 =$$

$$= \|\langle k\xi | k\xi\rangle\| \geq |\langle k\xi | k\xi\rangle(t)| \geq \alpha \|(\xi^* * \xi)(t,\cdot)\|_\infty$$

(Theorem 7.1.1.1 e)). Since α and t are arbitrary,

$$\|\widetilde{\xi}\| \geq \sup_{t\in T} \|(\xi^* * \xi)(t,\cdot)\|_\infty .$$

In particular,

$$\widetilde{\xi} = 0 \Longrightarrow \xi = 0 .$$

We prove now the last assertion. We have

$$(ke_t)(s,\cdot) = \delta_{st} k$$

for every $s \in T$ so that $ke_t \in H_\infty$ and

$$\|\widetilde{\xi e_t}\| \leq \sum_{s \in T} \|(ke_t)(s, \cdot)\|_\infty = \|k\|_\infty .$$

In order to prove the reverse inequality take $\alpha \in]0, \|k\|_\infty[$. There is an $h \in L^2(\mu)^\#$ such that

$$\|kh\|_2 > \alpha .$$

By e),

$$he_{t^{-1}} \in H_2 , \quad \|he_{t^{-1}}\| = \|h\|_2 \leq 1$$

so that, by e), again,

$$\|\widetilde{ke_t}\| \geq \|\widetilde{ke_t}\| \, \|he_{t^{-1}}\| \geq \|\widetilde{ke_t}(he_{t^{-1}})\| =$$

$$= \|(ke_t)*(he_{t^{-1}})\| = \|khe_1\| = \|kh\|_2 > \alpha .$$

Since α is arbitrary,

$$\|\widetilde{ke_t}\| \geq \|k\|_\infty .$$

i) By the inequality of h) and the supplementary hypothesis of i), H_∞ endowed with the norm

$$H_\infty \longrightarrow \mathbb{R}_+ , \quad \xi \longmapsto \|\widetilde{\xi}\|$$

is complete and thus a C^*-algebra.

It is obvious that

$$H_1 \longrightarrow \mathbb{R}_+ , \quad \eta \longmapsto \sup_{\xi \in H_\infty^\#} |\widetilde{\xi}(\eta)|$$

is a norm. For every $\eta \in H_1$ and $\xi \in H_\infty$,

$$|\widetilde{\xi}(\eta)| \leq \sum_{t \in T} \|\xi(t, \cdot)\|_\infty \|\eta(t, \cdot)\|_1 \leq \alpha \|\widetilde{\xi}\| \sum_{t \in T} \|\eta(t, \cdot)\|_1$$

so that

$$\|\eta\| \leq \alpha \sum_{t \in T} \|\eta(t, \cdot)\|_1 .$$

Let $\eta \in H_1$ and $t \in T$. Take $k \in L^\infty(\mu)^\#$. By h),

$$ke_t \in H_\infty , \quad \|\widetilde{ke_t}\| = \|k\|_\infty \leq 1 .$$

Since

$$(ke_t)(s,\cdot) = \delta_{st}k$$

for every $s \in T$,

$$\widetilde{ke_t}(\eta) = \sum_{s\in T} \int (ke_t)(s,\cdot)\eta(s,\cdot)d\mu =$$

$$= \sum_{s\in T} \delta_{st} \int k\eta(s,\cdot)d\mu = \int k\eta(t,\cdot)d\mu\,.$$

It follows

$$\|\eta\| \geq |\widetilde{ke_t}(\eta)| = \left| \int k\eta(t,\cdot)d\mu \right|.$$

Since k is arbitrary, we get

$$\|\eta\| \geq \|\eta(t,\cdot)\|_1\,.$$

Hence

$$\|\eta\| \geq \sup_{t\in T} \|\eta(t,\cdot)\|_1\,.$$

It follows that H_1 is complete.

Let $k \in L^1(\mu)$ and $t \in T$. Then

$$(ke_t)(s,\cdot) = \delta_{st}k$$

for every $s \in T$ so that $ke_t \in H_1$ and by the above,

$$\|k\|_1 = \sup_{s\in T} \|(ke_t)(s,\cdot)\|_1 \leq \|ke_t\|\,.$$

For every $\xi \in H_\infty^\#$,

$$\widetilde{\xi}(ke_t) = \sum_{s\in T} \int \xi(s,\cdot)(ke_t)(s,\cdot)d\mu =$$

$$= \sum_{s\in T} \delta_{st} \int \xi(s,\cdot)kd\mu = \int \xi(t,\cdot)kd\mu$$

so that

$$|\widetilde{\xi}(ke_t)| \leq \|\xi(t,\cdot)\|_\infty \|k\|_1 \leq \alpha \|\widetilde{\xi}\|\,\|k\|_1 \leq \alpha\|k\|_1\,,$$

$$\|ke_t\| = \sup_{\xi \in H_\infty^\#} |\widetilde{\xi}(ke_t)| \leq \alpha \|k\|_1 .$$

Let $\eta' \in H_1'$. Take $t \in T$. Then for every $k \in L^1(\mu)$, the map

$$L^1(\mu) \longrightarrow \mathbb{K}, \quad k \longmapsto \eta'(ke_t)$$

is continuous. Hence there is an $h_t \in L^\infty(\mu)$ such that

$$\int h_t k d\mu = \eta'(ke_t)$$

for every $k \in L^1(\mu)$. Put

$$\xi : T \times R \longrightarrow \mathbb{K}, \quad (t,r) \longmapsto h_t(r) .$$

Then $\xi \in H_\infty$ and for every $\eta \in H_1$,

$$\widetilde{\xi}(\eta) = \sum_{t \in T} \int \xi(t,\cdot)\eta(t,\cdot)d\mu = \sum_{t \in T} \int h_t \eta(t,\cdot) d\mu =$$

$$= \sum_{t \in T} \eta'(\eta(t,\cdot)e_t) = \eta'(\eta) .$$

Hence

$$\widetilde{\xi} = \eta'$$

and the map

$$H_\infty \longrightarrow H_1', \quad \xi \longmapsto \widetilde{\xi}$$

is surjective. Moreover,

$$\|\widetilde{\xi}\| = \sup_{\eta \in H_1^\#} |\widetilde{\xi}(\eta)| \leq \|\overrightarrow{\xi}\| .$$

We want to show that $H_\infty^\#$ is a closed set of $(H_\infty)_{H_1}$. Let ξ_0 be a point of adherence of $H_\infty^\#$ in $(H_\infty)_{H_1}$. There is a filter $\mathfrak{F}$ on H_∞ such that $H_\infty^\# \in \mathfrak{F}$ and

$$\lim_{\xi,\mathfrak{F}} \widetilde{\xi}(\eta) = \widetilde{\xi}_0(\eta)$$

for every $\eta \in H_1$. Take $k \in L^1(\mu)$ and $t \in T$. Then

$$\int \xi_0(t,\cdot)k d\mu = \widetilde{\xi}_0(ke_t) = \lim_{\xi,\mathfrak{F}} \widetilde{\xi}(ke_t) = \lim_{\xi,\mathfrak{F}} \int \xi(t,\cdot)k d\mu .$$

Take $\zeta \in H_2^{\#}$ and put

$$\zeta_0 := \widetilde{\xi_0}\zeta \in H_2 .$$

Then for every $t \in T$,

$$(\zeta_0^* * (\xi_0 * \zeta))(t, \cdot) = \sum_{s \in T} f(ts^{-1}, s)\zeta_0^*(ts^{-1}, \cdot)(\xi_0 * \zeta)(s, \cdot) =$$

$$= \sum_{r,s \in T} f(ts^{-1}, s)\zeta_0^*(ts^{-1}, \cdot) f(sr^{-1}, r)\xi_0(sr^{-1}, \cdot)\zeta(r, \cdot) .$$

Since $\zeta_0^*(ts^{-1}, \cdot)\zeta(r, \cdot) \in L^1(\mu)$ for all $r, s, t \in T$ it follows

$$\langle \xi_0 * \zeta | \xi_0 * \zeta \rangle(t) = \int (\zeta_0^* * (\xi_0 * \zeta))(t, \cdot) d\mu =$$

$$= \lim_{\xi, \mathfrak{F}} \int (\zeta_0^* * (\xi * \zeta))(t, \cdot) d\mu = \lim_{\xi, \mathfrak{F}} \langle \xi * \zeta | \xi_0 * \zeta \rangle(t) .$$

By Theorem 7.1.1.1 d) and Proposition 5.6.1.2 d),

$$\|\widetilde{\xi_0}\zeta\|^2 = \|\langle \xi_0 * \zeta | \xi_0 * \zeta \rangle\| = \lim_{\xi, \mathfrak{F}} \|\langle \xi * \zeta | \xi_0 * \zeta \rangle\| \leq$$

$$\leq \liminf_{\xi, \mathfrak{F}} \|\xi * \zeta\| \, \|\xi_0 * \zeta\| \leq \|\widetilde{\xi_0}\zeta\| ,$$

$$\|\widetilde{\xi_0}\| = \sup_{\zeta \in H_2^{\#}} \|\widetilde{\xi_0}\zeta\| \leq 1 ,$$

$$\widetilde{\xi_0} \in H_\infty^{\#} .$$

Hence $H_\infty^{\#}$ is closed in $(H_\infty)_{H_1}$.

Take $\xi_0 \in H_\infty \backslash H_\infty^{\#}$. Putting $E := H_1$, $F := H_\infty$, $A' := H_\infty^{\#}$, and $x' := \xi_0$ in Proposition 1.3.1.8 we get an $\eta \in H_1$ such that

$$\|\eta\| = \sup_{\xi \in H_\infty^{\#}} |\widetilde{\xi}(\eta)| < \widetilde{\xi_0}(\eta) \leq \|\widetilde{\xi_0}\| \, \|\eta\| ,$$

so that

$$1 < \|\widetilde{\xi_0}\| .$$

Thus

$$\|\widetilde{\xi_0}\| \leq \|\widetilde{\xi}_0\|,$$

$$\|\widetilde{\xi_0}\| = \|\widetilde{\xi}_0\|,$$

and the map

$$H_\infty \longrightarrow H_1', \quad \xi \longmapsto \widetilde{\xi}$$

is an isometry. By Theorem 4.4.2.21, H_∞ is a W^*-algebra with H_1 as predual. ■

Proposition 7.1.3.8 *Let T_1, T_2 be groups such that T_1 is finite and T_2 is locally finite and commutative and let f_1, f_2 be Schur functions for T_1 and T_2, respectively. We assume f_2 constant and, in the real case,*

$$t_2 = t_2^{-1}$$

for all $t_2 \in T_2$. Put

$$T := T_1 \times T_2$$

and denote by f the Schur function for T

$$f : T \times T \longrightarrow \{\alpha \in \mathbb{K} | \, \|\alpha\| = 1\}, \quad ((s_1, s_2), (t_1, t_2)) \longmapsto f_1(s_1, t_1).$$

We identify $\sigma(\ell^1(T_2))$ with $\sigma(\mathcal{S}_c(f_2))$ and denote it by R, by λ the Haar measure of R with $\lambda(R) = 1$ and by v the isomorphism of Hilbert spaces

$$v : \ell^2(T_2) \longrightarrow L^2(\lambda), \quad x \longmapsto \sum_{t_2 \in T_2} x(t_2)\widehat{e}_{t_2},$$

(Proposition 7.1.2.6 a),b),c)). Put

$$\widetilde{x} : T_1 \times R \longrightarrow \mathbb{K}, \quad (t_1, \cdot) \longmapsto vx(t_1, \cdot)$$

for every $x \in \ell^2(T)$. For every $p \in [1, \infty]$ denote by H_p (by H_0) the set of maps

$$\xi : T_1 \times R \longrightarrow \mathbb{K}$$

such that

$$\xi(t_1, \cdot) \in L^p(\lambda) \quad (\xi(t_1, \cdot) \in \mathcal{C}(R))$$

for every $t_1 \in T_1$, endow H_2 with the structure of a Hilbert $\mathcal{S}(f_1)$-module (Proposition 7.1.3.7 d)), and put

$$\int \xi d\lambda : T_1 \longrightarrow \mathbb{K}, \quad t_1 \longmapsto \int \xi(t_1, \cdot) d\mu$$

for every $\xi \in H_1$.

a) *If* $t_1 \in T_1$, $p \in [1,\infty]$, *and* $k \in L^p(\mu)$ *then* $ke_{t_1} \in H_p$ *and*

$$(ke_{t_1})(s_1,\cdot) = \delta_{s_1,t_1}k$$

for every $s_1 \in T_1$.

b) $x \in \ell^2(T) \Rightarrow \widetilde{x} \in H_2$.

c) *For every* $x, y \in \ell^2(T)$ *and* $t_1 \in T_1$,

$$\langle \widetilde{x}|\widetilde{y}\rangle(t_1) = \sum_{s_1 \in T_1} \langle \widetilde{x}(s_1,\cdot)|f_1(s_1t_1^{-1},t_1)\widetilde{y}(s_1t_1^{-1},\cdot)\rangle .$$

d) $x \in \ell^2(T) \Rightarrow \|\widetilde{x}\| \leq \|x\|_2\sqrt{\operatorname{Card} T_1}$.

e) *The map*

$$\ell^2(T) \longrightarrow H_2, \quad x \longmapsto \widetilde{x}$$

is an isomorphism of Banach spaces. In particular, there is an $\alpha \in \mathbb{R}_+$ *such that*

$$\|x\|_2 \leq \alpha\|\widetilde{x}\|$$

for every $x \in \ell^2(T)$.

f) *An element* x *of* $\ell^2(T)$ *belongs to* $\mathcal{S}_w(T)$ *iff* $\widetilde{x} \in H_\infty$.

g) $x \in \mathcal{S}_w(T)$, $y \in \ell^2(T) \Rightarrow \widetilde{x*y} = \widetilde{x}*\widetilde{y}$ *(Proposition 7.1.3.7 a)).*

h) *For every* $\xi \in H_\infty$,

$$\sup_{t_1 \in T_1} \|\xi(t_1,\cdot)\|_\infty \leq \alpha\|\overset{\sim}{\xi}\|$$

where

$$\overset{\sim}{\xi} : H_2 \longrightarrow H_2, \quad \eta \longmapsto \xi*\eta$$

and $\overset{\sim}{\xi} \in \mathcal{L}_{\mathcal{S}(f_1)}(H_2)$ *(Proposition 7.1.3.7 g)).*

i) *We endow* H_∞ *with the norm*

$$H_\infty \longrightarrow \mathbb{R}_+, \quad \xi \longmapsto \|\overset{\sim}{\xi}\|,$$

put

$$\widetilde{\xi} : H_1 \longrightarrow \mathbb{K}, \quad \eta \longmapsto \sum_{t_1 \in T_1} \int \xi(t_1, \cdot)\eta(t_1, \cdot)d\lambda$$

for every $\xi \in H_\infty$, *and endow* H_1 *with the norm*

$$H_1 \longrightarrow \mathbb{R}_+, \quad \eta \longmapsto \sup_{\xi \in H_\infty^\#} |\widetilde{\xi}(\eta)|.$$

Then H_∞ *is a* W^**-algebra with predual* H_1 *and*

$$\|ke_{t_1}\| = \|k\|_\infty$$

for every $t_1 \in T_1$ *and* $k \in L^\infty(\lambda)$.

j) *Let* $H(f)$, $H(f_1)$, *and* $H(f_2)$ *be the Banach spaces defined in Proposition 7.1.1.4 a) with respect to* f, f_1, *and* f_2, *respectively, and for every* $\eta \in H_1$ *put*

$$\overset{\frown}{\eta} : T \longrightarrow \mathbb{K}, \quad (t_1, t_2) \longmapsto (w\eta(t_1, \cdot))(t_2),$$

where w *is the isometry* $L^1(\lambda) \longmapsto H(f_2)$ *such that*

$$w\widehat{e}_{t_2} = e_{t_2^{-1}}$$

for every $t_2 \in T_2$ *(Proposition 7.1.2.6 e)). Then the map*

$$\mathcal{S}_w(f) \longrightarrow H_\infty, \quad x \longmapsto \widetilde{x}$$

is an isometry of W^**-algebras with pretranspose*

$$H_1 \longrightarrow H(f), \quad \eta \longmapsto \overset{\frown}{\eta}.$$

k) *An element* x *of* $\ell^2(T)$ *belongs to* $\mathcal{S}_c(f)$ *iff* $\widetilde{x} \in H_0$. *In particular,* H_∞ *is the* W^**-subalgebra of* H_∞ *generated by* H_0.

a) and b) are easy to see.

c) and d) follow from Proposition 7.1.3.7 f) (and Parseval's Equation).

e) By b),c), and the Principle of Inverse Operators, it is sufficient to show that the map is bijective.

Take $x \in \ell^2(T)$ with $\widetilde{x} = 0$. For every $t_1 \in T_1$,

$$vx(t_1, \cdot) = \widetilde{x}(t_1, \cdot) = 0$$

so that

$$x(t_1, \cdot) = 0 .$$

Hence $x = 0$ and the map is injective.

Take $\xi \in H_2$ and put

$$x : T \longrightarrow \mathbb{K}, \quad (t_1, t_2) \longmapsto (v^{-1}\xi(t_1, \cdot))(t_2) .$$

For every $t_1 \in T_1$,

$$\widetilde{x}(t_1, \cdot) = vx(t_1, \cdot) = \xi(t_1, \cdot)$$

so that $\widetilde{x} = \xi$ and the map is surjective.

f) Take $t_1 \in T_1$. By Proposition 7.1.2.6 h),

$$x(t_1, \cdot) \in \mathcal{S}_w(f_2) \Longleftrightarrow \widetilde{x}(t_1, \cdot) \in L^\infty(\lambda) .$$

Since T_1 is finite,

$$x \in \mathcal{S}_w(f) \Longleftrightarrow \widetilde{x} \in H_\infty .$$

g) Take $(t_1, t_2) \in T$. Then

$$(x*y)(t_1, t_2) = \sum_{s \in T} f_1(t_1 s_1^{-1}, s_1) x(t_1 s_1^{-1}, t_2 s_2^{-1}) y(s_1, s_2) =$$

$$= \sum_{s_1 \in T_1} f_1(t_1 s_1^{-1}, s_1) \sum_{s_2 \in T_2} x(t_1 s_1^{-1}, t_2 s_2^{-1}) y(s_1, s_2) =$$

$$= \sum_{s_1 \in T_1} f_1(t_1 s_1^{-1}, s_1)(x(t_1 s_1^{-1}, \cdot) * y(s_1, \cdot))(t_2)$$

so that

$$(x*y)(t_1, \cdot) = \sum_{s_1 \in T_1} f_1(t_1 s_1^{-1}, s_1) x(t_1 s_1^{-1}, \cdot) * y(s_1, \cdot) .$$

By Proposition 7.1.2.6 f),

$$\widetilde{x*y}(t_1, \cdot) = v((x*y)(t_1, \cdot)) =$$

$$= \sum_{s_1 \in T_1} f_1(t_1 s_1^{-1}, s_1)(vx(t_1 s_1^{-1}, \cdot))(vy(s_1, \cdot)) =$$

$$= \sum_{s_1 \in T_1} f_1(t_1 s_1^{-1}, s_1)\widetilde{x}(t_1 s_1^{-1}, \cdot)\widetilde{y}(s_1, \cdot) = (\widetilde{x} * \widetilde{y})(t_1, \cdot)\,.$$

Hence

$$\widetilde{x * y} = \widetilde{x} * \widetilde{y}\,.$$

h) By Proposition 7.1.2.6 h), for every $t_1 \in T_1$ there is a $z \in \mathcal{S}_w(f_2)$ such that

$$vz = \xi(t_1, \cdot)\,.$$

Hence there is an $x \in \mathcal{S}_w(f)$ with $\widetilde{x} = \xi$. Take $t_1 \in T_1$ and

$$\beta \in]0, \|\xi(t_1, \cdot)\|_\infty[\,.$$

There is a $k \in L^2(\lambda)^\#$ such that

$$\|\xi(t_1, \cdot)k\|_2 > \beta\,.$$

Put

$$y : T \longrightarrow \mathbb{K}\,, \quad (s_1, s_2) \longmapsto \delta_{s_1, t_1}\langle k | \widehat{e}_{s_2}\rangle\,.$$

Then $y \in \ell^2(T)$ and

$$y(s_1, \cdot) = \delta_{s_1, t_1} v^{-1} k\,,$$

$$\widetilde{y}(s_1, \cdot) = vy(s_1, \cdot) = \delta_{s_1, t_1} k$$

for every $s_1 \in T_1$ so that

$$\widetilde{y} = k e_{t_1}\,.$$

By g),

$$\xi * k e_{t_1} = \widetilde{x} * \widetilde{y} = \widetilde{x * y}\,.$$

For $(s_1, s_2) \in T$,

$$(x * y)(s_1, s_2) = \sum_{(r_1, r_2) \in T} f_1(s_1 r_1^{-1}, r_1)x(s_1 r_1^{-1}, s_2 r_2^{-1})y(r_1, r_2) =$$

$$= \sum_{r_2 \in T_2} \sum_{r_1 \in T_1} f_1(s_1 r_1^{-1}, r_1)x(s_1 r_1^{-1}, s_2 r_2^{-1})\delta_{r_1, t_1}(v^{-1}k)(r_2) =$$

$$= \sum_{r_2 \in T_2} f_1(s_1 t_1^{-1}, r_1) x(s_1 t_1^{-1}, s_2 r_2^{-1})(v^{-1}k)(r_2) =$$

$$= f_1(s_1 t_1^{-1}, r_1)(x(s_1 t_1^{-1}, \cdot) * (v^{-1}k))(s_2) .$$

By Proposition 7.1.2.6 c),f),

$$\|x*y\|_2^2 = \sum_{(s_1,s_2) \in T} |(x*y)(s_1, s_2)|^2 =$$

$$= \sum_{s_1 \in T_1} \sum_{s_2 \in T_2} |x(s_1 t_1^{-1}, \cdot) * (v^{-1}k)(s_2)|^2 =$$

$$= \sum_{s_1 \in T_1} \|v(x(s_1 t_1^{-1}, \cdot) * (v^{-1}k))\|_2^2 = \sum_{s_1 \in T_1} \|vx(s_1 t_1^{-1}, \cdot)k\|_2^2 =$$

$$= \sum_{s_1 \in T_1} \|\xi(s_1 t_1^{-1}, \cdot)k\|_2^2 \geq \|\xi(t_1, \cdot)k\|_2^2 > \beta^2 .$$

By Proposition 7.1.3.7 e),

$$\|k e_{t_1}\| = \|k\|_2 \leq 1$$

so that by e) and g),

$$\|\widetilde{\xi}\| \geq \|\widetilde{\xi}\| \, \|k e_{t_1}\| \geq \|\widetilde{\xi}(k e_{t_1})\| = \|\xi * (k e_{t_1})\| =$$

$$= \|\widetilde{x} * \widetilde{y}\| = \|\widetilde{x*y}\| \geq \frac{1}{\alpha} \|x*y\|_2 > \frac{\beta}{\alpha} .$$

Since β and t_1 are arbitrary, it follows

$$\|\xi(t_1, \cdot)\|_\infty \leq \alpha \|\widetilde{\xi}\| ,$$

$$\sup_{t_1 \in T_1} \|\xi(t_1, \cdot)\|_\infty \leq \alpha \|\widetilde{\xi}\| .$$

i) follows from h) and Proposition 7.1.3.7 i).

j) By Theorem 7.1.2.1 c) and Proposition 7.1.2.3 e), $\mathcal{S}_w(f)$ is a W^*–algebra with predual $H(f)$. By f) and g), the map

$$\mathcal{S}_w(f) \longrightarrow H_\infty , \quad x \longmapsto \widetilde{x}$$

is an isomorphism of algebras. This isomorphism is obviously involutive.

Take $\eta \in H_1$. By Proposition 7.1.2.6 e),

$$\overset{\smile}{\eta}(t_1, \cdot) = w\eta(t_1, \cdot) \in H(f_2)$$

for every $t_1 \in T_1$ such that $\overset{\smile}{\eta} \in H(f)$. For every $x \in \mathcal{S}_w(f)$, by Proposition 7.1.2.6 g),

$$\langle \eta, \widetilde{x} \rangle = \sum_{t_1 \in T_1} \int \widetilde{x}(t_1, \cdot)\eta(t_1, \cdot)d\lambda =$$

$$= \sum_{t_1 \in T_1} \int vx(t_1, \cdot)\eta(t_1, \cdot)d\lambda = \sum_{t_1 \in T_1} \langle \eta(t_1, \cdot), vx(t_1, \cdot) \rangle =$$

$$= \sum_{t_1 \in T_1} \langle w\eta(t_1, \cdot), x(t_1, \cdot) \rangle = \sum_{t_1 \in T_1} \sum_{t_2 \in T_2} \widetilde{\eta}(t_1, t_2)x(t_1, t_2) =$$

$$= \langle \widetilde{\eta}, x \rangle .$$

Hence the map

$$\mathcal{S}_w(f) \longrightarrow H_\infty , \quad x \longmapsto \widetilde{x}$$

is an isometry of W^*-algebras with pretranspose

$$H_1 \longrightarrow H(f) , \quad \eta \longmapsto \widetilde{\eta} .$$

k) $x \in \mathcal{S}_c(f)$ iff $x(t_1, \cdot) \in \mathcal{S}_c(f_2)$ for every $t_1 \in T_1$ so that by Proposition 7.1.2.6 d), $x \in \mathcal{S}_c(f)$ iff $\widetilde{x} \in H_0$. By j) and Proposition 7.1.2.3 i), H_∞ is the W^*-subalgebra of H_∞ generated by H_0. ■

Proposition 7.1.3.9 *Let S, T be groups, f and g Schur functions for S and T, respectively, and $n \in \mathbb{N}$. We assume that $\mathcal{S}(f)$ is isomorphic to $\mathbb{K}_{n,n}$ and T is locally finite. Put*

$$h : (S \times T) \times (S \times T) \longrightarrow \{\alpha \in \mathbb{K} |\ |\alpha| = 1\} ,$$

$$((s_1, t_1), (s_2, t_2)) \longmapsto f(s_1, s_2)g(t_1, t_2) .$$

Then h is a Schur function for $S \times T$ and $\mathcal{S}_c(h)$ is isomorphic to $(\mathcal{S}_c(g))_{n,n}$.

It is easy to see that h is a Schur function for $S \times T$. Let φ be the composition of an isomorphism of C^*–algebras

$$\mathcal{S}(f) \longrightarrow \mathbb{K}_{n,n}$$

with the canonical embedding

$$\mathbb{K}_{n,n} \longrightarrow (\mathcal{S}_c(g))_{n,n}\,.$$

Put

$$ue_{(s,t)} := (\varphi e_s)e_t \in (\mathcal{S}_c(g))_{n,n}$$

for every $(s,t) \in S \times T$ and extend u linearly to a map

$$\mathbb{K}^{(S\times T)} \longrightarrow (\mathcal{S}_c(g))_{n,n}\,.$$

It is easy to see that u is a unital, involutive algebra homomorphism. Since $S \times T$ is locally finite, it follows that u has norm at most 1 with respect to the norm induced on $\mathbb{K}^{(S\times T)}$ by $\mathcal{S}_c(h)$ (Corollary 4.1.1.20). Hence u may be extended to a unital involutive algebra homomorphism

$$\mathcal{S}_c(h) \longrightarrow (\mathcal{S}_c(g))_{n,n}\,.$$

We want to show that u is injective. Let $x \in \mathcal{S}_c(h)$ with

$$ux = 0\,.$$

Put

$$y_s := \lim_{R,\mathfrak{S}_T} \sum_{t\in R} x(s,t)e_t$$

for every $s \in S$ (Proposition 7.1.1.3 $\mathrm{b}_2 \Rightarrow \mathrm{b}_1$). Then

$$x = \sum_{s\in S} e_s y_s\,,$$

$$\sum_{s\in S} (\varphi e_s)y_s = ux = 0\,.$$

For every $x' \in (\mathcal{S}_c(g))'$,

$$\sum_{s\in S} (\varphi e_s)x'(y_s) = 0$$

so that

$$x'(y_s) = 0$$

for every $s \in S$. Since x' is arbitrary, $y_s = 0$ for every $s \in S$, i.e. $x = 0$ and u is injective.

Let $p, q \in \mathbb{N}_n$ and $t \in T$. There is an $x \in \mathcal{S}(f)$ such that

$$\varphi x = [\delta_{ip}\delta_{jq}]_{i,j\in\mathbb{N}_n} \in \mathbb{K}_{n,n}.$$

Put

$$y := \sum_{s\in S} x(s)e_{(s,t)} \in \mathcal{S}_c(h).$$

Then

$$uy = \sum_{s\in S} x(s)(\varphi e_s)e_t = (\varphi x)e_t = [\delta_{ip}\delta_{jq}e_t]_{i,j\in\mathbb{N}_n} \in (\mathcal{S}(g))_{n,n}.$$

Hence

$$(\mathcal{S}(g))_{n,n} \subset \operatorname{Im} u.$$

Since $\operatorname{Im} u$ is closed (Theorem 4.2.6.6),

$$(\mathcal{S}_c(g))_{n,n} \subset \operatorname{Im} u$$

and u is surjective. Hence u is an isomorphism of C^*-algebras. ■

Remark. For every $n \in \mathbb{N}$, there is a Schur function f on $\mathbb{Z}_n^2$ such that $\mathcal{S}^{\mathbb{C}}(f)$ is isomorphic to $\mathbb{C}_{n,n}$ (Proposition 7.1.4.9 d)).

Proposition 7.1.3.10 *Assume* $\mathbb{K} = \mathbb{R}$ *and let* f *be the Schur function for a finite group* T. *Put*

$$p := \operatorname{Card}\{t \in T | t^2 = 1,\ \widetilde{f}(t) = 1\},$$

$$q := \operatorname{Card}\{t \in T | t^2 = 1,\ \widetilde{f}(t) = -1\},$$

$$r := \operatorname{Card} T.$$

a) $\operatorname{Dim}\operatorname{Re}\mathcal{S}(f) = \frac{p-q+r}{2}$.

b) *If* n *is an odd natural number,* $n \neq 1$, *then* $\mathcal{S}(f)$ *is not isomorphic to* $\mathbb{R}_{n,n}$.

a) Take $t \in T$ with $t^2 = 1$. Then

$$e_t^* = \widetilde{f}(t) e_t$$

so that e_t is selfadjoint iff $\widetilde{f}(t) = 1$.

Take $t \in T$ with $t^2 \neq 1$. For $\alpha, \beta \in \mathbb{R}$,

$$(\alpha e_t + \beta e_{t^{-1}})^* = \alpha \widetilde{f}(t) e_{t^{-1}} + \beta \widetilde{f}(t) e_t = \widetilde{f}(t)(\beta e_t + \alpha e_{t^{-1}})$$

such that $\alpha e_t + \beta e_{t^{-1}}$ is selfadjoint iff $\beta = \alpha \widetilde{f}(t)$.

By the above

$$\operatorname{Dim} \operatorname{Re} \mathcal{S}(f) = p + \frac{r - p - q}{2} = \frac{p - q + r}{2}.$$

b) Assume $\mathcal{S}(f)$ is isomorphic to $\mathbb{R}_{n,n}$. Since n^2 is odd,

$$\{t \in T \mid t^2 = 1\} = \{1\}.$$

Hence

$$p = 1, \quad q = 0.$$

By a),

$$\frac{1 + n^2}{2} = \operatorname{Dim} \operatorname{Re} \mathcal{S}(f) = \operatorname{Dim} \operatorname{Re} \mathbb{R}_{n,n} = \frac{n(n+1)}{2},$$

which is a contradiction. ■

Remark. Let $n \in \mathbb{N}$. If there is a $p \in \mathbb{N} \cup \{0\}$ with $n = 2^p$, then by Theorem 7.2.2.7 k), there is a Schur function f on a group such that $\mathcal{S}(f)$ is isomorphic to $\mathbb{R}_{n,n}$. I do not know if the converse of this implication also holds.

7.1.4 Examples

Proposition 7.1.4.1 *Let f be a Schur function for the finite group T and let $\mathfrak{Q}$ be the set of conjugacy classes Q of T such that*

$$\overline{f(r, r^{-1}tr)}\ f(t,r) = \overline{f(s, s^{-1}ts)}\ f(t,s)$$

for every $t \in Q$ and for all $r, s \in T$ with

$$r^{-1}tr = s^{-1}ts\,.$$

a) *There is a unique $\varphi \in (\mathbb{N} \cup \{0\})^{(\mathbb{N})}$ such that $\mathcal{S}^{\mathbb{C}}(f)$ is isomorphic to the C^*-direct product of the family $\left(\mathbb{C}_{n,n}^{\varphi(n)}\right)_{n\in\mathbb{N}}$ of complex C^*-algebras (where $\mathbb{C}_{n,n}^0 = \{0\}$). We have*

$$\mathrm{Dim}(\mathcal{S}^{\mathbb{C}}(f))^c = \sum_{n\in\mathbb{N}} \varphi(n) = \mathrm{Card}\,\mathfrak{Q}\,, \quad \sum_{n\in\mathbb{N}} n^2\varphi(n) = \mathrm{Card}\,T\,.$$

If T is commutative and

$$f(s,t) = f(t,s)$$

for all $s, t \in T$, then $\mathcal{S}^{\mathbb{C}}(f)$ is isomorphic to $\mathbb{C}^{\mathrm{Card}\,T}$.

b) *There are uniquely $\lambda, \mu, \nu \in (\mathbb{N} \cup \{0\})^{(\mathbb{N})}$ such that ν vanishes at the odd natural numbers,*

$$\lambda + 2\mu + \nu = \varphi\,,$$

and $\mathcal{S}^{\mathbb{R}}(f)$ is isomorphic to the C^*-direct product of the family

$$\left(\mathbb{R}_{n,n}^{\lambda(n)} \times \mathbb{C}_{n,n}^{\mu(n)} \times \mathbb{H}_{\frac{n}{2},\frac{n}{2}}^{\nu(n)}\right)_{n\in\mathbb{N}}$$

of real C^*-algebras. We have

$$\mathrm{Dim}\,\mathrm{Re}\,\mathcal{S}^{\mathbb{R}}(f) =$$

$$= \mathrm{Card}\,\{t \in T \mid t = t^{-1}, \widetilde{f}(t) = 1\} + \frac{1}{2}\mathrm{Card}\,\{t \in T \mid t \neq t^{-1}\} =$$

$$= \sum_{n\in\mathbb{N}} \left(\frac{n(n+1)}{2}\lambda(n) + n^2\mu(n) + n(2n-1)\nu(2n)\right)\,,$$

$$\operatorname{Dim}\operatorname{Re}(\mathcal{S}^{\mathbb{R}}(f))^c = \sum_{n\in\mathbb{N}}(\lambda(n)+\mu(n)+\nu(n))\,.$$

If T *is commutative and*

$$f(s,t) = f(t,s)$$

for all $s,t \in T$, *then there are* $p,q \in \mathbb{N}\cup\{0\}$ *such that* $\mathcal{S}^{\mathbb{R}}(f)$ *is isomorphic to* $\mathbb{R}^p \times \mathbb{C}^q$ *and*

$$p+2q = \operatorname{Card} T\,.$$

$q=0$ *iff* $t=t^{-1}$ *and* $\tilde{f}(t)=1$ *for every* $t\in T$.

a) $\mathcal{S}^{\mathbb{C}}(f)$ is finite–dimensional. By Corollary 6.3.6.5, there is a unique $\varphi \in (\mathbb{N}\cup\{0\})^{(\mathbb{N})}$ such that $\mathcal{S}^{\mathbb{C}}(f)$ is isomorphic to the C^*–direct product of the family $\left(\mathbb{C}_{n,n}^{\varphi(n)}\right)$ of complex C^*–algebras. We have

$$\sum_{n\in\mathbb{N}} n^2\varphi(n) = \operatorname{Dim}\mathcal{S}^{\mathbb{C}}(f) = \operatorname{Card} T\,.$$

$(\mathcal{S}^{\mathbb{C}}(f))^c$ is a Gelfand C^*–algebra. By Corollary 4.1.2.5, Proposition 4.2.7.20, and Theorem 7.1.1.5 k),

$$\sum_{n\in\mathbb{N}} \varphi(n) = \operatorname{Dim}(\mathcal{S}(f))^c = \operatorname{Card}\mathfrak{Q}\,.$$

Now assume T commutative and

$$f(s,t) = f(t,s)$$

for all $s,t\in T$. By Theorem 7.1.1.5 k),

$$\operatorname{Card} T = \operatorname{Dim}\mathcal{S}^{\mathbb{C}}(f) = \operatorname{Card}\mathfrak{Q}\,.$$

By the above

$$\varphi(n) = \begin{cases} \operatorname{Card} T & \text{if } n=1 \\ 0 & \text{if } n\neq 1\,. \end{cases}$$

Hence $\mathcal{S}^{\mathbb{C}}(f)$ is isomorphic to $\mathbb{C}^{\operatorname{Card} T}$.

b) By Theorem 7.1.2.1 d), $\mathcal{S}^{\mathbb{C}}(f)$ is isomorphic to the complexification of $\mathcal{S}^{\mathbb{R}}(f)$. By a) and Corollary 5.6.6.10, there are uniquely λ,μ,ν with the given properties.

Let $t\in T$ such that $t=t^{-1}$. Then

$$e_t^* = \widetilde{f}(t)e_t\,,$$

so that e_t is selfadjoint iff $\widetilde{f}(t) = 1$. Hence the intersection of $\mathrm{Re}\,\mathcal{S}^{\mathbb{R}}(f)$ with the vector subspace of $\mathcal{S}^{\mathbb{R}}(f)$ generated by $\{e_t \mid t \in T\,,\ t = t^{-1}\}$ has dimension

$$\mathrm{Card}\,\{t \in T \mid t = t^{-1}\,,\ \widetilde{f}(t) = 1\}\,.$$

Let $t \in T$ such that $t \neq t^{-1}$ and let $\alpha, \beta \in \mathbb{R}$. Then

$$(\alpha e_t + \beta e_{t^{-1}})^* = \alpha e_t^* + \beta e_{t^{-1}}^* = \widetilde{f}(t)(\alpha e_{t^{-1}} + \beta e_t)\,,$$

so that $\alpha e_t + \beta e_{t^{-1}}$ is selfadjoint iff

$$\beta = \alpha \widetilde{f}(t)\,,$$

i.e.

$$\{\alpha e_t + \beta e_{t^{-1}} \mid \alpha, \beta \in \mathbb{R}\} \cap \mathrm{Re}\,\mathcal{S}^{\mathbb{R}}(f)$$

is one-dimensional. Hence the intersection of $\mathrm{Re}\,\mathcal{S}^{\mathbb{R}}(f)$ with the vector subspace of $\mathcal{S}^{\mathbb{R}}(f)$ generated by $\{e_t \mid t \in T\,,\ t \neq t^{-1}\}$ has dimension

$$\frac{1}{2}\mathrm{Card}\,\{t \in T \mid t \neq t^{-1}\}\,.$$

Putting together the above results, one obtains

$$\mathrm{Dim}\,\mathrm{Re}\,\mathcal{S}^{\mathbb{R}}(f,g) = \mathrm{Card}\,\{t \in T \mid t = t^{-1}, \widetilde{f}(t) = 1\} + \frac{1}{2}\mathrm{Card}\,\{t \in T \mid t \neq t^{-1}\}\,.$$

The relation

$$\mathrm{Dim}\,\mathrm{Re}\,(\mathcal{S}^{\mathbb{R}}(f))^c = \sum_{n \in \mathbb{N}} (\lambda(n) + \mu(n) + \nu(n))$$

is easy to see. The other equality follows from Corollary 5.6.6.9.

The final assertion is easy to see. ∎

Remark. It may be a very difficult problem to find φ, λ, μ, and ν in concrete cases. Even if they are known it may be difficult to find a corresponding isomorphism.

Example 7.1.4.2 *Let $n \in \mathbb{N}$ and f a Schur function for $\mathbb{Z}_n$.*

a) *$f(s,t) = f(t,s)$ for all $s,t \in \mathbb{Z}_n$ and $\mathcal{S}^{\mathbb{C}}(f)$ is isomorphic to $\mathbb{C}^n$.*

b) *If n is odd then $\mathcal{S}^{\mathbb{R}}(f)$ is isomorphic to $\mathbb{R} \times \mathbb{C}^{\frac{n-1}{2}}$.*

c) *If* n *is even then* $\mathcal{S}^{\mathbb{R}}(f)$ *is isomorphic to* $\mathbb{C}^{\frac{n}{2}}$ *if* $\widetilde{f}(\frac{n}{2}) = -1$ *and is isomorphic to* $\mathbb{R} \times \mathbb{R} \times \mathbb{C}^{\frac{n}{2}-1}$ *if* $\widetilde{f}(\frac{n}{2}) = 1$.

d) *If* T *is a group and* f *a Schur function for* T *then* $\mathcal{S}(f)$ *is not isomorphic to* $\mathbb{K}_{2,2} \times \mathbb{K}$.

a) By Theorem 7.1.1.5 l), $f(s,t) = f(t,s)$ for all $s, t \in \mathbb{Z}_n$. By Proposition 7.1.4.1 a), $\mathcal{S}^{\mathbb{C}}(f)$ is isomorphic to $\mathbb{C}^n$.

b) and c). By a) and Proposition 7.1.4.1 b), there are $p, q \in \mathbb{N} \cup \{0\}$ such that $\mathcal{S}^{\mathbb{R}}(f)$ is isomorphic to $\mathbb{R}^p \times \mathbb{C}^q$ and such that

$$p + 2q = n,$$

$$p + q = \begin{cases} 1 + \frac{n-1}{2} & \text{if } n \text{ is odd} \\ 1 + \delta_{1,\widetilde{f}(\frac{n}{2})} + \frac{n-2}{2} & \text{if } n \text{ is even.} \end{cases}$$

It follows that

$$q = \begin{cases} n - 1 - \frac{n-1}{2} = \frac{n-1}{2} & \text{if } n \text{ is odd} \\ n - 1 - \delta_{1,\widetilde{f}(\frac{n}{2})} - \frac{n-2}{2} = \frac{n}{2} - \delta_{1,\widetilde{f}(\frac{n}{2})} & \text{if } n \text{ is even,} \end{cases}$$

$$p = \begin{cases} 1 & \text{if } n \text{ is odd} \\ 2\delta_{1,\widetilde{f}(\frac{n}{2})} & \text{if } n \text{ is even.} \end{cases}$$

d) Since

$$\mathrm{Dim}\,(\mathbb{K}_{2,2} \times \mathbb{K}) = 5$$

it follows

$$\mathrm{Card}\,\mathrm{T} = 5$$

$$T = \mathbb{Z}_S.$$

By a) and b), $\mathcal{S}(f)$ is not isomorphic to $\mathbb{K}_{2,2} \times \mathbb{K}$. ■

Remark. Explicit isomorphisms for $\mathcal{S}(f)$ are presented in Example 7.1.4.10.

Example 7.1.4.3 *Let $\mathcal{F}$ be the set of Schur functions for $\mathbb{Z}_2^2$. Put*

$$a := (1,0)\,, \quad b := (0,1)\,, \quad c := (1,1)\,,$$

$$A := \{\alpha \in \mathbb{C} \mid |\alpha| = 1\}$$

and for every $\theta := (\alpha, \beta, \gamma, \varepsilon) \in A^3 \times \{-1, +1\}$ denote by f_θ the function defined by the following table:

f_θ	a	b	c
a	$\alpha\beta$	α	β
b	$\varepsilon\alpha$	$\varepsilon\alpha\gamma$	γ
c	$\varepsilon\beta$	$\varepsilon\gamma$	$\beta\gamma$

,

a) *$f_\theta \in \mathcal{F}$ for every $\theta \in A^3 \times \{-1, +1\}$ and the map*

$$A^3 \times \{-1, +1\} \longrightarrow \mathcal{F}\,, \quad \theta \longmapsto f_\theta\,,$$

is bijective.

b) *For each $\theta := (\alpha, \beta, \gamma, \varepsilon) \in A^3 \times \{-1, +1\}$,*

$$f_\theta(a,b)\ \overline{f_\theta(b,a)} = f_\theta(b,c)\ \overline{f_\theta(c,b)} = f_\theta(c,a)\ \overline{f_\theta(a,c)} = \varepsilon\,.$$

c) *If $f \in \mathcal{F}$ and $f(a,b) = f(b,a)$, then $\mathcal{S}^{\mathbb{C}}(f)$ is isomorphic to $\mathbb{C}^4$ and $\mathcal{S}^{\mathbb{R}}(f)$ is isomorphic to $\mathbb{R}^{2p-4} \times \mathbb{C}^{4-p}$, where*

$$p := \operatorname{Card}\{t \in \mathbb{Z}_2^2 \mid \widetilde{f}(t) = 1\} \in \{2, 4\}\,.$$

Moreover,

$$\|x\| = \sup\{|x(0) + \varphi x(a) + \psi x(b) + \varphi\psi x(c)| \ \mid \varphi, \psi \in \{+1, -1\}\}$$

for every $x \in \mathcal{S}^{\mathbb{C}}(f)$.

d) *If $f \in \mathcal{F}$ and $f(a,b) \neq f(b,a)$ then $\mathcal{S}^{\mathbb{C}}(f)$ is isomorphic to $\mathbb{C}_{2,2}$ and $\mathcal{S}^{\mathbb{R}}(f)$ is isomorphic to $\mathbb{R}_{2,2}$ (to $\mathbb{H}$) iff*

$$\operatorname{Card}\{t \in \mathbb{Z}_2^2 \mid \widetilde{f}(t) = 1\}$$

is equal to 3 (to 1).

a) is the result of a long calculation.

b) follows from a).

c) By b) (and Theorem 7.1.1.5 a)),

$$f(s,t) = f(t,s)$$

for all $s,t \in \mathbb{Z}_2^2$, so that by Proposition 7.1.2.5 d $\Rightarrow$ a, $\mathcal{S}^{\mathbb{C}}(f)$ is isomorphic to $\mathbb{C}^4$.

Since $\mathcal{S}^{\mathbb{C}}(f)$ is a Gelfand C^*–algebra,

$$\|x\| = \|\widehat{x}\|$$

for every $x \in \mathcal{S}^{\mathbb{C}}(f)$ and it is easy to see that

$$\|\widehat{x}\| = \sup\{|x(0) + \varphi x(a) + \psi x(b) + \varphi\psi x(c)| \mid \varphi, \psi \in \{+1, -1\}\}.$$

Now suppose $\mathbb{K} = \mathbb{R}$. By a) and b),

$$\widetilde{f}(a)\widetilde{f}(b)\widetilde{f}(c) = 1,$$

so that

$$p \in \{2, 4\}.$$

The system of equations

$$\begin{cases} \lambda + 2\mu = 4 \\ \lambda + \mu = p \end{cases}$$

has the unique solution

$$\lambda = 2p - 4, \quad \mu = 4 - p.$$

By Proposition 7.1.4.1 b), $\mathcal{S}^{\mathbb{R}}(f)$ is isomorphic to $\mathbb{R}^{2p-4} \times \mathbb{C}^{4-p}$.

d) By Proposition 7.1.2.5 a $\Rightarrow$ d, $\mathcal{S}^{\mathbb{C}}(f)$ is not isomorphic to $\mathbb{C}^4$, so it is isomorphic to $\mathbb{C}_{2,2}$ (Proposition 7.1.4.1 a)).

Now suppose $\mathbb{K} = \mathbb{R}$. By a) and b),

$$\widetilde{f}(a)\widetilde{f}(b)\widetilde{f}(c) = -1,$$

so that

$$p := \operatorname{Card}\{t \in \mathbb{Z}_2^2 \mid \widetilde{f}(t) = 1\} \in \{1, 3\}.$$

The system of equations

$$\begin{cases} \lambda + \nu = 1 \\ 3\lambda + \nu = p \end{cases}$$

has the unique solution

$$\lambda = \frac{p-1}{2}, \quad \nu = \frac{3-p}{2}.$$

By Proposition 7.1.4.1 b), $\mathcal{S}^{\mathbb{R}}(f)$ is isomorphic to $\mathbb{R}_{2,2}$ (to $\mathbb{H}$) iff $p = 3$ (iff $p = 1$). ■

Proposition 7.1.4.4 *Let X be a set, T a subgroup of the permutation groups of X such that*

$$x, y \in X \Longrightarrow (xy) \in T,$$

and f a real Schur function for T (we use here and in the following the usual notation of permutation groups: e.g. (xy) denotes the permutation of X, which interchanges x and y and keeps all other elements fixed).

a) *Let $(x_j)_{j \in \mathbb{N}_{2n+1}}$ be a family of distinct elements of X for some $n \in \mathbb{N}$. Put*

$$a := (x_{2n-1}x_{2n}), \quad b := (x_{2n-1}x_{2n+1}), \quad d := (x_{2n-1}x_{2n}x_{2n+1}),$$

$$t := \begin{cases} \prod\limits_{j \in \mathbb{N}_{n-1}} (x_{2j-1}x_{2j}) & \text{if } n \neq 1 \\ 1 & \text{if } n = 1. \end{cases}$$

Then

$$\widetilde{f}(ta) = \widetilde{f}(tb) = f(ta, ta),$$

$$\widetilde{f}(d) = f(ta, tb)f(tb, ta).$$

b) *Let $(x_j)_{j \in \mathbb{N}_{2n}}$ be a family of distinct elements of X with $n \in \mathbb{N}\backslash\{1\}$. Put*

$$a := (x_1x_2)(x_3x_4), \quad b := (x_1x_3)(x_2x_4),$$

$$t := \begin{cases} \prod\limits_{j=3}^{n} (x_{2j-1}x_{2j}) & \text{if } n \neq 2 \\ 1 & \text{if } n = 2. \end{cases}$$

Then

$$\widetilde{f}(at) = \widetilde{f}(bt),$$

$$\widetilde{f}(a) = f(at, bt)f(bt, at).$$

c) *If* $(x_j)_{j\in\mathbb{N}_{2n}}$, $(y_j)_{j\in\mathbb{N}_{2n}}$ *are two families of distinct elements of* X *for some* $n \in \mathbb{N}$, *then*

$$\widetilde{f}\left(\prod_{j\in\mathbb{N}_n}(x_{2j-1}x_{2j})\right) = \widetilde{f}\left(\prod_{j\in\mathbb{N}_n}(y_{2j-1}y_{2j})\right).$$

d) *Let* $(x_j)_{j\in\mathbb{N}_{4n}}$ *be a family of distinct elements of* X *for some* $n \in \mathbb{N}$. *Put*

$$s := \prod_{j=1}^{n}(x_{2j-1}x_{2j}), \quad t := \prod_{j=n+1}^{2n}(x_{2j-1}x_{2j}).$$

Then

$$\widetilde{f}(st) = f(s,t)f(t,s).$$

e) *Assume* X *finite and* $\widetilde{f}((xy)) = 1$ *for all* $x, y \in X$. *If* E *is a real* C^**-algebra with*

$$\operatorname{Dim}\operatorname{Re}E = 1$$

and $u : \mathcal{S}^{\mathbb{R}}(f) \to E$ *is an involutive algebra homomorphism then*

$$\operatorname{Im}u \subset \operatorname{Re}E.$$

f) *If* X *is infinite and*

$$\{x \in X | t(x) \neq x\}$$

is finite for every $t \in T$ *then*

$$(\mathcal{S}_w(f))^c = \mathbb{K}1.$$

a) Put

$$c := (x_{2n}x_{2n+1}), \quad e := (x_{2n-1}x_{2n+1}x_{2n}).$$

By Proposition 7.1.1.9 a),d),

$$\begin{cases} f(c,ta)f(te,c) = f(c,td)f(ta,c) \\ f(c,ta)\widetilde{f}(c)\widetilde{f}(ta) = f(ta,c)\widetilde{f}(te) \\ f(c,td)\widetilde{f}(c)\widetilde{f}(td) = f(te,c)\widetilde{f}(tb) \\ \widetilde{f}(td) = \widetilde{f}(te), \end{cases}$$

since

$$te = (td)^{-1}.$$

It follows that

$$\tilde{f}(c)\tilde{f}(ta)\tilde{f}(te) = f(c,ta)f(ta,c) =$$

$$= f(te,c)f(c,td) = \tilde{f}(c)\tilde{f}(td)\tilde{f}(tb),$$

so that

$$f(ta,ta) = f(ta,(ta)^{-1}) = \tilde{f}(ta) = \tilde{f}(tb),$$

$$f(ta,tb) = f(ta,tb)\tilde{f}(ta)\tilde{f}(tb) = f(tb,ta)\tilde{f}(tbta) =$$

$$= f(tb,ta)\tilde{f}(d)$$

(Proposition 7.1.1.9 d)), so that

$$\tilde{f}(d) = \tilde{f}(e) = f(ta,tb)f(tb,ta).$$

b) Put

$$c := (x_2x_3), \quad d := (x_1x_3x_4x_2), \quad e := (x_1x_2x_4x_3).$$

By Proposition 7.1.1.9 a),d),

$$\begin{cases} f(c,at)f(dt,c) = f(c,et)f(at,c) \\ f(c,at)\tilde{f}(c)\tilde{f}(at) = f(at,c)\tilde{f}(dt) \\ f(dt,c)\tilde{f}(dt)\tilde{f}(c) = f(c,et)\tilde{f}(bt) \\ \tilde{f}(dt) = \tilde{f}(et), \end{cases}$$

since

$$et = (dt)^{-1}.$$

It follows that

$$\tilde{f}(c)\tilde{f}(at)\tilde{f}(dt) = f(c,at)f(at,c) =$$

$$= f(dt,c)f(c,et) = \tilde{f}(dt)\tilde{f}(c)\tilde{f}(bt),$$

so that

$$\widetilde{f}(at) = \widetilde{f}(bt)\,.$$

We deduce

$$f(at,bt) = f(at,bt)\widetilde{f}(at)\widetilde{f}(bt) = f(bt,at)\widetilde{f}(ab)$$

(Proposition 7.1.1.9 d)),

$$\widetilde{f}(a) = \widetilde{f}((x_1x_4)(x_2x_3)) = \widetilde{f}(ab) = f(at,bt)f(bt,at)\,.$$

c) follows from a) and b).

d) By c) and Proposition 7.1.1.9 d),

$$f(s,t) = f(s,t)\widetilde{f}(s)\widetilde{f}(t) = f(t,s)\widetilde{f}(st)\,,$$

so that

$$\widetilde{f}(st) = f(s,t)f(t,s)\,.$$

e) follows from Theorem 7.1.2.1 n).

f) Take $t \in T\backslash\{1\}$ and put

$$A := \{x \in X | t(x) \neq x\}\,.$$

Let $(A_n)_{n\in\mathbb{N}}$ be a sequence of pairwise disjoint subsets of $X\backslash A$ and for each $n \in \mathbb{N}$ let $\varphi_n : A \to A_n$ be a bijection. For every $n \in \mathbb{N}$, put

$$s_n : X \longrightarrow X\,, \quad x \longmapsto \begin{cases} \varphi_n(x) & \text{if } x \in A \\ \varphi_n^{-1}(x) & \text{if } x \in A_n \\ x & \text{if } x \in X\backslash(A \cup A_n) \end{cases}$$

Then $(s_n t s_n^{-1})_{n\in\mathbb{N}}$ is an injective sequence in T, so that the conjugacy class of t is infinite. By Theorem 7.1.1.5 k),

$$(\mathcal{S}_w(f))^c = \mathbb{K}1\,. \qquad \blacksquare$$

Example 7.1.4.5 *Let S_3 be the permutation group of $\mathbb{N}_3$, $\mathcal{F}$ the set of Schur functions for S_3, and $f \in \mathcal{F}$. Put*

$$A := \{\alpha \in \mathbb{K} \mid |\alpha| = 1\}\,,$$

$$a := (12)\,, \quad b := (13)\,, \quad c := (23)\,, \quad d := (123)\,, \quad e := (132)$$

and denote for every

$$\theta := (\alpha,\beta,\gamma,\delta,\varepsilon) \in A^5$$

by f_θ the function defined by the following table:

f_θ	a	b	c	d	e
a	$\alpha^2\delta$	$\overline{\gamma}$	$\overline{\beta}\overline{\varepsilon}$	$\alpha^2\beta\delta\varepsilon$	$\alpha^2\gamma\delta$
b	$\overline{\gamma\varepsilon}$	$\overline{\beta}\overline{\varepsilon}$	$\overline{\alpha}$	$\beta^2\gamma\delta\varepsilon$	$\alpha\beta^2\delta$
c	β	$\overline{\alpha\varepsilon}$	$\gamma^2\delta$	$\alpha\gamma^2\delta\varepsilon$	$\beta\gamma^2\delta$
d	$\alpha^2\gamma\delta\varepsilon$	$\alpha\beta^2\delta\varepsilon$	$\beta\gamma^2\delta\varepsilon$	$\alpha\beta\gamma\overline{\delta}\varepsilon^2$	$\alpha^2\beta^2\gamma^2\delta^2\varepsilon$
e	$\alpha^2\beta\delta$	$\beta^2\gamma\delta$	$\alpha\gamma^2\delta$	$\alpha^2\beta^2\gamma^2\delta^2\varepsilon$	$\overline{\alpha}\overline{\beta}\overline{\gamma}\overline{\delta}\overline{\varepsilon}$

a) *$f_\theta \in \mathcal{F}$ for every $\theta \in A^5$ and the map*

$$A^5 \longrightarrow \mathcal{F}, \quad \theta \longmapsto f_\theta$$

is bijective.

b) *$\mathcal{S}^{\mathbb{C}}(f)$ is isomorphic to $\mathbb{C} \times \mathbb{C} \times \mathbb{C}_{2,2}$.*

c) *If $\widetilde{f}((12)) = 1$, then $\mathcal{S}^{\mathbb{R}}(f)$ is isomorphic to $\mathbb{R} \times \mathbb{R} \times \mathbb{R}_{2,2}$.*

d) *If $\widetilde{f}((1,2)) = -1$, then $\mathcal{S}^{\mathbb{R}}(f)$ is isomorphic to $\mathbb{C} \times \mathbb{H}$.*

a) is the result of a long calculation.

b) The system

$$\begin{cases} \sum\limits_{n\in\mathbb{N}} \varphi(n) \leq 3 \\ \sum\limits_{n\in\mathbb{N}} n^2\varphi(n) = \operatorname{Card} S_3 \end{cases}$$

has a unique solution in $\varphi \in (\mathbb{N} \cup \{0\})^{(\mathbb{N})}$, namely

$$\varphi(1) = 2, \quad \varphi(2) = 1, \quad n > 2 \Longrightarrow \varphi(n) = 0.$$

By Proposition 7.1.4.1 a) and Theorem 2.2.2.7 d_4), $\mathcal{S}^{\mathbb{C}}(f)$ is isomorphic to $\mathbb{C} \times \mathbb{C} \times \mathbb{C}_{2,2}$.

c) and d). By Proposition 7.1.4.4 a),

$$\widetilde{f}((12)) = \widetilde{f}((13)) = \widetilde{f}((23)).$$

Hence, by Proposition 7.1.4.1 b),

$$d := \operatorname{Dim} \operatorname{Re} \mathcal{S}^{\mathbb{R}}(f) = \begin{cases} 5 & \text{if } \widetilde{f}((12)) = 1 \\ 2 & \text{if } \widetilde{f}((12)) = -1 . \end{cases}$$

Let $\lambda, \mu, \nu \in (\mathbb{N} \cup \{0\})^2$ such that $\nu(1) = 0$ and

$$\begin{cases} \lambda(1) + 2\mu(1) = 2 \\ \lambda(2) + 2\mu(2) + \nu(2) = 1 \\ \lambda(1) + \mu(1) + \nu(2) + 3\lambda(2) + 4\mu(2) = d . \end{cases}$$

Then $\mu(2) = 0$ and

$$-\mu(1) + 2\lambda(2) = d - 3 .$$

If $\mu(1) = 0$, then

$$\lambda(1) = 2 , \quad \lambda(2) = \frac{d-3}{2} .$$

This implies

$$d = 5 , \quad \lambda(2) = 1 , \quad \nu(2) = 0 .$$

If $\mu(1) \neq 0$ then

$$\mu(1) = 1 , \quad \lambda(1) = 0 , \quad \lambda(2) = \frac{d-2}{2} .$$

This implies

$$d = 2 , \quad \lambda(2) = 0 , \quad \nu(2) = 1 .$$

By Proposition 7.1.4.1 b), if $\widetilde{f}((12)) = 1$, then $\mathcal{S}^{\mathbb{R}}(f)$ is isomorphic to $\mathbb{R} \times \mathbb{R} \times \mathbb{R}_{2,2}$ and if $\widetilde{f}((1,2)) = -1$, then $\mathcal{S}^{\mathbb{R}}(f)$ is isomorphic to $\mathbb{C} \times \mathbb{H}$. ■

Example 7.1.4.6 *Let A_4 be the group of even permutations of $\mathbb{N}_4$ and f a real Schur function for A_4. Put*

$$a := (12)(34) , \quad b := (13)(24) , \quad c := (14)(23) ,$$

$$p := \operatorname{Card} \{t \in A_4 | t^{-1} = t , \ \widetilde{f}(t) = 1\} .$$

a) *$\{1, a, b, c\}$ is a subgroup of A_4 isomorphic to $\mathbb{Z}_2^2$.*

b) $f(a,b)f(b,a) = f(b,c)f(c,b) = f(c,a)f(a,c)$.

c) *If* $f(a,b) = f(b,a)$, *then*

$$\tilde{f}(a) = \tilde{f}(b) = \tilde{f}(c) = 1, \quad p = 4,$$

$\mathcal{S}^{\mathbb{C}}(f)$ *is isomorphic to* $\mathbb{C}^3 \times \mathbb{C}_{3,3}$, *and* $\mathcal{S}^{\mathbb{R}}(f)$ *is isomorphic to* $\mathbb{R} \times \mathbb{C} \times \mathbb{R}_{3,3}$.

d) *If* $f(a,b) \neq f(b,a)$ *then*

$$\tilde{f}(a)\tilde{f}(b)\tilde{f}(c) = -1, \quad p \in \{1,3\},$$

$\mathcal{S}^{\mathbb{C}}(f)$ *is isomorphic to* $\mathbb{C}^3_{2,2}$, *and* $\mathcal{S}^{\mathbb{R}}(f)$ *is isomorphic to* $\mathbb{R}_{2,2} \times \mathbb{C}_{2,2}$ *if* $p = 3$ *and isomorphic to* $\mathbb{C}_{2,2} \times \mathbb{H}$ *if* $p = 1$.

a) is easy to verify.

b) follows from a) and Example 7.1.4.3 b).

c) and d). A_4 has 4 conjugacy classes. The equation

$$\sum_{n \in \mathbb{N}} n^2 \varphi(n) = \operatorname{Card} A_4$$

has two solutions in $\varphi \in (\mathbb{N} \cup \{0\})^{(\mathbb{N})}$ with

$$\sum_{n \in \mathbb{N}} \varphi(n) \leq 4,$$

namely

$$\varphi(1) = 3, \quad \varphi(2) = 0, \quad \varphi(3) = 1, \quad n > 3 \Longrightarrow \varphi(n) = 0$$

and

$$\varphi(1) = 0, \quad \varphi(2) = 3, \quad \varphi(3) = 0, \quad n > 3 \Longrightarrow \varphi(n) = 0.$$

By Proposition 7.1.4.1 a), $\mathcal{S}^{\mathbb{C}}(f)$ is isomorphic to $\mathbb{C}^3 \times \mathbb{C}_{3,3}$ in the first case and it is isomorphic to $\mathbb{C}^3_{2,2}$ in the second case.

Assume φ is the first solution. The system of equations

$$\begin{cases} \lambda_1 + 2\mu_1 = 3 \\ \lambda_3 = 1 \\ \lambda_1 + \mu_1 + 6\lambda_3 = 4 + p \end{cases}$$

has a solution in $\mathbb{N} \cup \{0\}$ only if $p = 4$ and in this case, there is a unqiue solution

$$\lambda_1 = \mu_1 = \lambda_3 = 1 .$$

Since $p = 4$,

$$\widetilde{f}(a) = \widetilde{f}(b) = \widetilde{f}(c) = 1$$

and by Example 7.1.4.3 c),d),

$$f(a,b) = f(b,a) .$$

By Proposition 7.1.4.1 b), $\mathcal{S}^{\mathbb{R}}(f)$ is isomorphic to $\mathbb{R} \times \mathbb{C} \times \mathbb{R}_{3,3}$.

Now suppose φ is the second solution. Consider the system of equations

$$\begin{cases} \lambda + 2\mu + \nu = 3 \\ 3\lambda + 4\mu + \nu = 4 + p . \end{cases}$$

It follows that

$$2\mu + 2\nu = 5 - p .$$

Hence p is odd and therefore

$$\widetilde{f}(a)\widetilde{f}(b)\widetilde{f}(c) = -1 .$$

By Example 7.1.4.3 c),d),

$$f(a,b) \neq f(b,a) .$$

There is a $t \in \{a,b,c\}$ such that

$$\widetilde{f}(t) = -1 .$$

For $s := 1$,

$$s^{-1}ts = t^{-1} ,$$

$$f(s, s^{-1}t^{-1}s)f(t^{-1}, s)\widetilde{f}(t) = -1 .$$

By Proposition 7.1.2.2 b),

$$\operatorname{Dim} \operatorname{Re}(\mathcal{S}^{\mathbb{R}}(f))^c \leq 2$$

so that, by Proposition 7.1.4.1 b),

$$\lambda + \mu + \nu \leq 2 .$$

If $p = 1$, then

$$\mu + \nu = 2,$$

and $\lambda = 0, \mu = 1, \nu = 1$ is the unique solution. Hence $\mathcal{S}^{\mathbb{R}}(f)$ is isomorphic to $\mathbb{C}_{2,2} \times \mathbb{H}$.

If $p = 3$, then

$$\mu + \nu = 1$$

and $\lambda = 1, \mu = 1, \nu = 0$ is the unique solution. Hence $\mathcal{S}^{\mathbb{R}}(f)$ is isomorphic to $\mathbb{R}_{2,2} \times \mathbb{C}_{2,2}$. ■

Example 7.1.4.7 *Let S_4 be the permutation group of $\mathbb{N}_4$ and f a real Schur function for S_4. Put*

$$a := (12)(34), \quad b := (13)(24), \quad c := (14)(23).$$

a) $f(a,b)f(b,a) = f(b,c)f(c,b) = f(c,a)f(a,c)$,

$$\widetilde{f}(a) = \widetilde{f}(b) = \widetilde{f}(c),$$

$$\widetilde{f}((12)) = \widetilde{f}((13)) = \widetilde{f}((14)) = \widetilde{f}((23)) = \widetilde{f}((24)) = \widetilde{f}((34)).$$

b) *If $f(a,b) = f(b,a)$, then*

$$\widetilde{f}(a) = \widetilde{f}(b) = \widetilde{f}(c) = 1,$$

$\mathcal{S}^{\mathbb{C}}(f)$ is isomorphic to $\mathbb{C}^2 \times \mathbb{C}_{2,2} \times \mathbb{C}_{3,3}^2$, and $\mathcal{S}^{\mathbb{R}}(f)$ is isomorphic to

$$\begin{cases} \mathbb{R}^2 \times \mathbb{R}_{2,2} \times \mathbb{R}_{3,3}^2 & \text{if } \widetilde{f}((12)) = 1 \\ \mathbb{C} \times \mathbb{H} \times \mathbb{C}_{3,3} & \text{if } \widetilde{f}((12)) = -1. \end{cases}$$

c) *If $f(a,b) \neq f(b,a)$, then*

$$\widetilde{f}(a) = \widetilde{f}(b) = \widetilde{f}(c) = -1,$$

$\mathcal{S}^{\mathbb{C}}(f)$ is isomorphic to $\mathbb{C}_{2,2}^2 \times \mathbb{C}_{4,4}$, and $\mathcal{S}^{\mathbb{R}}(f)$ is isomorphic to

$$\begin{cases} \mathbb{C}_{2,2} \times \mathbb{R}_{4,4} & \text{if } \widetilde{f}((12)) = 1 \\ \mathbb{H}^2 \times \mathbb{H}_{2,2} & \text{if } \widetilde{f}((12)) = -1. \end{cases}$$

a) follows from Example 7.1.4.6 b) and Proposition 7.1.4.4 c).

b & c. S_4 has five conjugacy classes (Theorem 2.2.2.7 d_4)). The equation

$$\sum_{n\in\mathbb{N}} n^2\varphi(n) = \operatorname{Card} S_4$$

has two solutions in $\varphi \in (\mathbb{N}\cup\{0\})^{(\mathbb{N})}$ with

$$\sum_{n\in\mathbb{N}} \varphi(n) \leq 5\,,$$

namely

$$\varphi(1)=2\,,\quad \varphi(2)=1\,,\quad \varphi(3)=2\,,\quad n>3 \Longrightarrow \varphi(n)=0$$

and

$$\varphi(1)=\varphi(3)=0\,,\quad \varphi(2)=2\,,\quad \varphi(4)=1\,,\quad n>4 \Longrightarrow \varphi(n)=0\,.$$

By Proposition 7.1.4.1 a), $\mathcal{S}^{\mathbb{C}}(f)$ is isomorphic to $\mathbb{C}^2\times\mathbb{C}_{2,2}\times\mathbb{C}_{3,3}^2$ in the first case and to $\mathbb{C}_{2,2}^2\times\mathbb{C}_{4,4}$ in the second case. Put

$$p := \operatorname{Card}\{t\in S_4 \mid t^{-1}=t\,,\quad \widetilde{f}(t)=1\}\,.$$

Assume φ is the first solution. By Corollary 2.1.4.13 c), $\mathbb{C}_{2,2}^3$ is not isomorphic to a subalgebra of $\mathbb{C}^2\times\mathbb{C}_{2,2}\times\mathbb{C}_{3,3}^2$. By Example 7.1.4.6 c),d),

$$f(a,b)=f(b,a)\,,$$

$$\widetilde{f}(a)=\widetilde{f}(b)=\widetilde{f}(a)=1\,,$$

since A_4 is a subgroup of S_4. By a), $p\in\{4,10\}$. Consider the system of equations

$$\begin{cases} \lambda_1+2\mu_1=2 \\ \lambda_2+\nu_2=1 \\ \lambda_3+2\mu_3=2 \\ \lambda_1+\mu_1+\nu_2+3\lambda_2+6\lambda_3+9\mu_3=7+p\,. \end{cases}$$

We have

$$\mu_1+2\nu_2+3\mu_3=10-p\,.$$

If $p=10$, then $\widetilde{f}((12))=1$ and

$$\mu_1 = \nu_2 = \mu_3 = 0\,, \quad \lambda_1 = \lambda_3 = 2\,, \quad \lambda_2 = 1$$

and if $p = 4$, then $\widetilde{f}((12)) = -1$ and

$$\mu_1 = \nu_2 = \mu_3 = 1\,, \quad \lambda_1 = \lambda_2 = \lambda_3 = 0\,.$$

By Proposition 7.1.4.1 b), $\mathcal{S}^{\mathbb{R}}(f)$ is isomorphic to $\mathbb{R}^2 \times \mathbb{R}_{2,2} \times \mathbb{R}^2_{3,3}$ in the first case and to $\mathbb{C} \times \mathbb{H} \times \mathbb{C}_{3,3}$ in the second case.

Now suppose φ is the second solution. By Corollary 2.1.4.13 c), $\mathbb{C}^3 \times \mathbb{C}_{3,3}$ is not isomorphic to a subalgebra of $\mathbb{C}^2_{2,2} \times \mathbb{C}_{4,4}$. By Example 7.1.4.6 c),d),

$$f(a,b) \neq f(b,a)\,,$$

$$\widetilde{f}(a)\widetilde{f}(b)\widetilde{f}(c) = -1\,,$$

since A_4 is a subgroup of S_4. By a),

$$\widetilde{f}(a) = \widetilde{f}(b) = \widetilde{f}(c) = -1$$

and $p \in \{1, 7\}$. Consider the system of equations

$$\begin{cases} \lambda_2 + 2\mu_2 + \nu_2 = 2 \\ \lambda_4 + \nu_4 = 1 \\ 3\lambda_2 + 10\lambda_4 + 4\mu_2 + \nu_2 + 6\nu_4 = 7 + p\,. \end{cases}$$

We have

$$2\mu_2 + 2\nu_2 + 4\nu_4 = 9 - p\,.$$

If $p = 1$, then $\widetilde{f}((12)) = -1$ and

$$\nu_2 = 2\,, \quad \nu_4 = 1\,, \quad \lambda_2 = \lambda_4 = \mu_2 = 0$$

and if $p = 7$, then $\widetilde{f}((12)) = 1$ and either

$$\lambda_4 = \mu_2 = 1\,, \quad \lambda_2 = \nu_2 = \nu_4 = 0$$

or

$$\lambda_2 = \lambda_4 = \nu_2 = 1\,, \quad \mu_2 = \nu_4 = 0\,.$$

By Proposition 7.1.4.1 b), $\mathcal{S}^{\mathbb{R}}(f)$ is isomorphic to $\mathbb{H}^2 \times \mathbb{H}_{2,2}$ in the first case while in the second case it can be isomorphic either to $\mathbb{C}_{2,2} \times \mathbb{R}_{4,4}$ or to $\mathbb{R}_{2,2} \times \mathbb{H} \times \mathbb{R}_{4,4}$. But the last case cannot occur by a) and Proposition 7.1.4.4 e). ■

Lemma 7.1.4.8 *Take* $n \in \mathbb{N}$ *and put*

$$\omega := e^{\frac{2\pi i}{n}}, \quad x := \left[\delta_{jk}\omega^{k-1}\right]_{j,k\in\mathbb{N}_n} \in \mathbb{C}_{n,n}, \quad y := \left[\delta_{1+j,k}\right]_{j,k\in\mathbb{N}_n} \in \mathbb{C}_{n,n},$$

where, in the definition of y, $\mathbb{N}_n$ *is canonically identified with* $\mathbb{Z}_n$.

a) $x, y \in \mathrm{Un}(\mathbb{C}_{n,n})$.

b) *For* $p, q \in \mathbb{Z}$,

$$x^p y^q = \left[\omega^{p(j-1)}\delta_{q+j,k}\right]_{j,k\in\mathbb{N}_n},$$

$$y^q x^p = \omega^{pq} x^p y^q.$$

c) $x^n = y^n = 1$.

d) $(x^p y^q)_{(p,q)\in\mathbb{N}_n^2}$ *is an algebraic basis of the vector space* $\mathbb{C}_{n,n}$.

a) is easy to see.

b) Obviously

$$x^p = \left[\delta_{jk}\omega^{p(k-1)}\right]_{j,k\in\mathbb{N}_n}.$$

By induction

$$y^q = \left[\delta_{q+j,k}\right]_{j,k\in\mathbb{N}_n}.$$

Hence

$$x^p y^q = \left[\sum_{\ell\in\mathbb{Z}_n} \delta_{j\ell}\omega^{p(\ell-1)}\delta_{q+\ell,k}\right]_{j,k\in\mathbb{N}_n} =$$

$$= \left[\omega^{p(j-1)}\delta_{q+j,k}\right]_{j,k\in\mathbb{N}_n},$$

$$y^q x^p = \left[\sum_{\ell\in\mathbb{Z}_n} \delta_{q+j,\ell}\delta_{\ell,k}\omega^{p(k-1)}\right]_{j,k\in\mathbb{N}_n} =$$

$$= \left[\delta_{q+j,k}\omega^{p(k-1)}\right]_{j,k\in\mathbb{N}_n} = \left[\delta_{q+j,k}\omega^{p(q+j-1)}\right]_{j,k\in\mathbb{N}_n} =$$

$$= \omega^{pq}\left[\delta_{q+j,k}\omega^{p(j-1)}\right]_{j,k\in\mathbb{N}_n} = \omega^{pq} x^p y^q.$$

c) follows from b).

d) Let $(\alpha_{pq})_{p,q\in\mathbb{N}_n}$ be a family in $\mathbb{C}$ such that

$$\sum_{p,q\in\mathbb{N}_n} \alpha_{pq} x^p y^q = 0 .$$

By b), for $j,k \in \mathbb{N}_n$,

$$0 = \sum_{p,q\in\mathbb{N}_n} \alpha_{pq}\omega^{p(j-1)}\delta_{q+j,k} = \sum_{p\in\mathbb{N}_n} \alpha_{p,k-j}\omega^{p(j-1)} ,$$

$$\sum_{p\in\mathbb{N}_n} \alpha_{p,q}\omega^{p(j-1)} = 0 .$$

Since

$$\mathrm{Det}\left[\omega^{p(j-1)}\right]_{p,j\in\mathbb{N}_n} \neq 0$$

it follows

$$\alpha_{p,q} = 0$$

for all $p,q \in \mathbb{N}_n$. Hence $(x^p y^q)_{p,q\in\mathbb{N}_n^2}$ is linearly independent, i.e. it is an algebraic basis for the vector space $\mathbb{C}_{n,n}$. ■

Proposition 7.1.4.9 *Take $n \in \mathbb{N}$ and $j_1, j_2, k_1, k_2, \ell \in \mathbb{Z}$ such that nj_1, nj_2, $j_1 + k_1$, and $j_2 + k_2$ are even and put*

$$\omega := e^{\frac{\pi i}{n}} ,$$

$$f_{j_m} : \mathbb{Z}_n \times \mathbb{Z}_n \longrightarrow \{\alpha \in \mathbb{C} \mid |\alpha| = 1\} , \quad (p,q) \longmapsto \omega^{2j_m pq}$$

for $m \in \{1,2\}$,

$$x := \left[\delta_{jk}\omega^{2(k-1)}\right]_{j,k\in\mathbb{N}_n} \in \mathbb{C}_{n,n} ,$$

$$y := \left[\delta_{1+j,k}\right]_{j,k\in\mathbb{N}_n} \in \mathbb{C}_{n,n} ,$$

where, in the definition of y, $\mathbb{N}_n$ is canonically identified with $\mathbb{Z}_n$.

a) *f_{j_m} is a Schur function for $\mathbb{Z}_n$ for every $m \in \{1,2\}$.*

b) *There is a unique Schur function* f *for* $\mathbb{Z}_n \times \mathbb{Z}_n$ *such that*

$$f((p,0),(q,0)) = \omega^{2j_1pq}, \quad f((0,p),(0,q)) = \omega^{2j_2pq},$$

$$f((p,0),(0,q)) = \omega^{2\ell pq}, \quad f((0,q),(p,0)) = \omega^{2(\ell+1)pq}$$

for all $p,q \in \mathbb{Z}_n$. *For* $(p,q),(r,s) \in \mathbb{Z}_n \times \mathbb{Z}_n$,

$$f((p,q),(r,s)) = \omega^{2j_1pr+2j_2qs+2\ell ps+2(\ell+1)qr}.$$

c) *For* $(p,q) \in \mathbb{Z}_n \times \mathbb{Z}_n$,

$$e^*_{(p,q)} = \omega^{2j_1p^2+2j_2q^2+2(2\ell+1)pq} e_{(-p,-q)}$$

in $\mathcal{S}(f)$.

d) *There is a unique isomorphism of* C^**-algebras*

$$\varphi : \mathcal{S}(f) \longrightarrow \mathbb{C}_{n,n}$$

such that

$$\varphi(e_{(1,0)}) = \omega^{k_1}x, \quad \varphi(e_{(0,1)}) = \omega^{k_2}y.$$

We have

$$\varphi(e_{(p,q)}) = \omega^{(j_1+k_1)p+(j_2+k_2)q-j_1p^2-j_2q^2-2\ell pq}x^py^q$$

for all $p,q \in \mathbb{Z}_n$.

a) For $r,s,t \in \mathbb{Z}_n$,

$$f_{j_m}(r,s)f_{j_m}(r+s,t) = \omega^{2j_mrs}\omega^{2j_m(r+s)t} =$$

$$\omega^{2j_mr(s+t)}\omega^{2j_mst} = f_{j_m}(r,s+t)f_{j_m}(s,t).$$

Moreover

$$f_{j_m}(0,0) = \omega^0 = 1.$$

b) Define

$$g : \mathbb{Z}_n \times \mathbb{Z}_n \longrightarrow \{\alpha \in \mathbb{C} \mid |\alpha| = 1\}, \quad (p,q) \longmapsto \omega^{2\ell pq},$$

$$h : \mathbb{Z}_n \times \mathbb{Z}_n \longrightarrow \{\alpha \in \mathbb{C} \mid |\alpha| = 1\}, \quad (p,q) \longmapsto \omega^{2(\ell+1)pq}.$$

The claim follows from a) and Proposition 7.1.3.1 a $\Rightarrow$ c.

c) By b), for $(r,s) \in \mathbb{Z}_n \times \mathbb{Z}_n$,

$$\widetilde{f}((r,s)) = \overline{f((r,s),(-r,-s))} = \omega^{2j_1r^2+2j_2s^2+2\ell rs+2(\ell+1)rs},$$

$$e^*_{(p,q)}((r,s)) = \widetilde{f}((r,s))\overline{e_{(p,q)}((-r,-s))} =$$

$$= \omega^{2j_1r^2+2j_2s^2+2(2\ell+1)rs}\delta_{(p,q),(-r,-s)} =$$

$$= \omega^{2j_1p^2+2j_2q^2+2(2\ell+1)pq}e_{(-p,-q)}((r,s)),$$

so that

$$e^*_{(p,q)} = \omega^{2j_1p^2+2j_2q^2+2(2\ell+1)pq}e_{(-p,-q)}.$$

d) The uniqueness is obvious. Define φ as a linear map

$$\varphi : \mathcal{S}(f) \longrightarrow \mathbb{C}_{n,n}$$

such that

$$\varphi(e_{(p,q)}) = \omega^{(j_1+k_1)p+(j_2+k_2)q-j_1p^2-j_2q^2-2\ell pq}x^py^q$$

for all $p,q \in \mathbb{Z}_n$. By Lemma 7.1.4.8 c,d), φ is well-defined and bijective. Take $(p,q),(r,s) \in \mathbb{Z}_n \times \mathbb{Z}_n$. By b),c), and Lemma 7.1.4.8 a),b),

$$\varphi(e_{(p,q)} * e_{(r,s)}) = f((p,q),(r,s))\varphi(e_{(p+r,q+s)}) =$$

$$= \omega^{2j_1pr+2j_2qs+2\ell ps+2(\ell+1)qr} \times$$

$$\times\omega^{(j_1+k_1)(p+r)+(j_2+k_2)(q+s)-j_1(p+r)^2-j_2(q+s)^2-2\ell(p+r)(q+s)}x^{p+r}y^{q+s} =$$

$$= \omega^{(j_1+k_1)p+(j_2+k_2)q-j_1p^2-j_2q^2-2\ell pq} \times$$

$$\times\omega^{(j_1+k_1)r+(j_2+k_2)s-j_1r^2-j_2s^2-2\ell rs}x^py^qx^ry^s = \varphi(e_{(p,q)})\varphi(e_{r,s})),$$

$$\varphi(e^*_{(p,q)}) = \omega^{2j_1p^2+2j_2q^2+2(2\ell+1)pq}\varphi(e_{(-p,-q)}) =$$

$$= \omega^{2j_1p^2+2j_2q^2+2(2\ell+1)pq} \times$$

$$\times \omega^{-(j_1+k_1)p-(j_2+k_2)q-j_1p^2-j_2q^2-2\ell pq} x^{-p} y^{-q} =$$

$$= \omega^{-(j_1+k_1)p-(j_2+k_2)q+j_1p^2+j_2q^2+2\ell pq} y^{-q} x^{-p} =$$

$$= (\omega^{(j_1+k_1)p+(j_2+k_2)q-j_1p^2-j_2q^2-2\ell pq} x^p y^q)^* = \varphi(e_{(p,q)})^* .$$

Hence φ is an involutive algebra homomorphism and therefore an isomorphism of C^*-algebras. ■

Example 7.1.4.10 *Let $n \in \mathbb{N}$ and let $\mathcal{F}$ be the set of Schur functions for $\mathbb{Z}_n$. Denote*

$$A := \{\alpha \in \mathbb{K} | \, |\alpha| = 1\}, \quad \omega := e^{\frac{2\pi i}{n}},$$

and for every $\alpha \in A^{n-1}$ put

$$f_\alpha : \mathbb{Z}_n \times \mathbb{Z}_n \longrightarrow A, \quad (p,q) \longmapsto \left(\prod_{j=1}^{p-1} \overline{\alpha}_j\right) \left(\prod_{k=q}^{p+q-1} \alpha_k\right),$$

where $\mathbb{Z}_n$ and $\mathbb{N}_n$ are canonically identified and $\alpha_n = 1$.

a) *$f_\alpha \in \mathcal{F}$ for every $\alpha \in A^{n-1}$ and the map*

$$A^{n-1} \longrightarrow \mathcal{F}, \quad \alpha \longmapsto f_\alpha$$

is bijective. Fix $\alpha \in A^{n-1}$ and take $\beta \in \mathbb{C}$ with

$$\beta^n = \prod_{j=1}^{n-1} \alpha_j .$$

b) *For every $p \in \mathbb{N}_n$,*

$$\underbrace{e_1 * e_1 * \ldots * e_1}_{p-times} = \left(\prod_{j=1}^{p-1} \alpha_j\right) e_p .$$

c) *For every $k \in \mathbb{Z}_n$ there is exactly one homomorphism of C^*-algebras*

$$u_k : \mathcal{S}(f) \longrightarrow \mathbb{C}$$

with

$$u_k(e_1) = \beta\omega^k .$$

For all $k, p, q \in \mathbb{Z}_n$,

$$u_k(e_p) = \beta^p \omega^{kp} \prod_{j=1}^{p-1} \overline{\alpha}_j, \quad f_\alpha(p,q) = \frac{u_k(e_p) u_k(e_q)}{u_k(e_{p+q})} .$$

d) $\sigma(\mathcal{S}^{\mathbb{C}}(f_\alpha)) = \{u_k | k \in \mathbb{N}_n\}$.

e) *If* $\mathbb{K} = \mathbb{R}$ *and* n *is odd then we may take* $\beta \in \mathbb{R}$ *and*

$$\mathcal{S}^{\mathbb{R}}(f_\alpha) \longrightarrow \mathbb{R} \times \mathbb{C}^{\frac{n-1}{2}}, \quad x \longmapsto \left(u_n(x), (u_k(x))_{k \in \mathbb{N}_{\frac{n-1}{2}}}\right)$$

is an isomorphism of real C^**-algebras.*

f) *If* $\mathbb{K} = \mathbb{R}$, n *is even, and* $\prod_{j=1}^{n-1} \alpha_j = -1$ *then*

$$\mathcal{S}^{\mathbb{R}}(f_\alpha) \longrightarrow \mathbb{C}^{\frac{n}{2}}, \quad x \longmapsto (u_{k-1}(x))_{k \in \mathbb{N}_{\frac{n}{2}}}$$

is an isomorphism of real C^**-algebras.*

g) *If* $\mathbb{K} = \mathbb{R}$, n *is even, and* $\prod_{j=1}^{n-1} \alpha_j = 1$ *the we may take* $\beta = 1$ *and*

$$\mathcal{S}^{\mathbb{R}}(f_\alpha) \longrightarrow \mathbb{R} \times \mathbb{R} \times \mathbb{C}^{\frac{n}{2}-1}, \quad x \longmapsto \left(u_n(x), u_{\frac{n}{2}}(x), \ (u_k(x))_{k \in \mathbb{N}_{\frac{n}{2}-1}}\right)$$

is an isomorphism of real C^**-algebras.*

h) *For every* $x \in \mathbb{C}^n$,

$$\left\| [x_{j-k}]_{j,k \in \mathbb{Z}_n} \right\| = \sup_{k \in \mathbb{N}_n} \left| \sum_{j=1}^{n} e^{\frac{2\pi i j k}{n}} x_j \right| .$$

a) For $p, q, r \in \mathbb{Z}_n$,

$$f_\alpha(p,q) f_\alpha(p+q,r) \overline{f_\alpha(p,q+r)}\, \overline{f_\alpha(q,r)} =$$

$$= \left(\prod_{j=1}^{p-1} \overline{\alpha}_j\right) \left(\prod_{k=q}^{p+q-1} \alpha_k\right) \left(\prod_{j=1}^{p+q-1} \overline{\alpha}_j\right) \left(\prod_{k=r}^{p+q+r-1} \alpha_k\right) \times$$

$$\times \left(\prod_{j=1}^{p-1} \alpha_j\right) \left(\prod_{k=q+r}^{p+q+r-1} \overline{\alpha}_k\right) \left(\prod_{j=1}^{q-1} \alpha_j\right) \left(\prod_{k=r}^{q+r-1} \overline{\alpha}_k\right) = 1 .$$

Hence $f_\alpha \in \mathcal{F}$.

Take $f \in \mathcal{F}$ and put

$$\alpha_j := f(1,j) \in A$$

for every $j \in \mathbb{Z}_n \backslash \{0\}$. By Example 7.1.4.2 a),

$$f(j,1) = \alpha_j$$

for all $j \in \mathbb{Z}_n \backslash \{0\}$. Let $p \in \mathbb{Z}_n \backslash \{0\}$ and assume

$$f(p,q) = \left(\prod_{j=1}^{p-1} \overline{\alpha}_j\right)\left(\prod_{k=q}^{p+q-1} \alpha_k\right)$$

for all $q \in \mathbb{Z}_n \backslash \{0\}$. Then

$$f(p+1,q) = \overline{f(p,1)} f(p,q+1) f(1,q) =$$

$$= \overline{\alpha}_p \left(\prod_{j=1}^{p-1} \overline{\alpha}_j\right)\left(\prod_{k=q+1}^{p+q} \alpha_k\right)\alpha_q = \left(\prod_{j=1}^{p} \overline{\alpha}_j\right)\left(\prod_{k=q}^{p+q} \alpha_k\right)$$

for all $q \in \mathbb{Z}_n \backslash \{0\}$. Hence $f = f_\alpha$ and the map

$$A^{n-1} \longrightarrow \mathcal{F}, \quad \alpha \longmapsto f_\alpha$$

is bijective.

b) follows by complete induction.

c) Uniqueness is obvious. Put

$$u_k(e_p) := \beta^p \omega^{kp} \prod_{j=1}^{p-1} \overline{\alpha}_j$$

for every $p \in \mathbb{Z}_n$ and extend u_k linearly to $\mathcal{S}(f)$. For $p, q \in \mathbb{Z}_n$,

$$u_k(e_p)u_k(e_q) = \beta^p \omega^{kp}\left(\prod_{j=1}^{p-1} \overline{\alpha}_j\right)\beta^q \omega^{kq}\left(\prod_{k=1}^{q-1} \overline{\alpha}_k\right) =$$

$$= \beta^{p+q} \omega^{k(p+q)}\left(\prod_{j=1}^{p-1} \overline{\alpha}_j\right)\left(\prod_{k=q}^{p+q-1} \alpha_k\right)\left(\prod_{\ell=1}^{p+q-1} \overline{\alpha}_\ell\right) = f_\alpha(p,q) u_k(e_{p+q}) =$$

$$= u_k(e_p * e_q),$$

$$u_k(e_p^*) = u_k(\widetilde{f_\alpha}(p) e_{-p}) =$$

$$\overline{f_\alpha(p, n-p)} u_k(e_{n-p}) = \left(\prod_{j=1}^{p-1} \alpha_j\right)\left(\prod_{k=n-p}^{n-1} \overline{\alpha}_k\right)\beta^{n-p} \omega^{k(n-p)} \prod_{\ell=1}^{n-p-1} \overline{\alpha}_\ell =$$

$$= \left(\prod_{j=1}^{p-1} \alpha_j\right) \beta^{-p} \omega^{-kp} = \overline{u_k(e_p)}\,.$$

Hence u_k is an involutive algebra homomorphism.

We have

$$\frac{u_k(e_p)u_k(e_q)}{u_k(e_{p+q})} = \left(\prod_{j=1}^{p-1} \overline{\alpha_j}\right)\left(\prod_{j=1}^{q-1} \overline{\alpha_k}\right)\left(\prod_{\ell=1}^{p+q-1} \alpha_\ell\right) =$$

$$= \left(\prod_{j=1}^{p-1} \overline{\alpha_j}\right)\left(\prod_{k=q}^{p+q-1} \alpha_k\right) = f(p,q)\,.$$

d) follows from c) (and Example 7.1.4.2 a)).

e) Take $x \in \mathcal{S}^{\mathbb{R}}(f_\alpha)$ with

$$\left(u_n(x)\,,\,(u_k(x))_{k\in\mathbb{N}_{\frac{n-1}{2}}}\right) = 0\,.$$

Then for every $k \in \mathbb{N}_{\frac{n-1}{2}}$,

$$\sum_{p=0}^{n-1} x_p \beta^p \prod_{j=1}^{p-1} \alpha_j = 0\,,$$

$$\sum_{p=0}^{n-1} x_p \beta^p \omega^{kp} \prod_{j=1}^{p-1} \alpha_j = 0\,,$$

$$\sum_{p=0}^{n-1} x_p \beta^p \omega^{-kp} \prod_{j=1}^{p-1} \alpha_j = 0\,.$$

It follows $x = 0$ and the map is bijective. By c), it is an isomorphism of real C^*-algebras.

f) Take $x \in \mathcal{S}^{\mathbb{R}}(f_\alpha)$ with

$$(u_{k-1}(x))_{k\in\mathbb{N}_{\frac{n}{2}}} = 0\,.$$

Then for every $k \in \mathbb{N}_{\frac{n}{2}}$,

$$\sum_{p=0}^{n-1} x_p \beta^p \omega^{(k-1)p} \prod_{j=1}^{p-1} \alpha_j = 0\,,$$

$$\sum_{p=0}^{n-1} x_p \beta^{-p} \omega^{-(k-1)p} \prod_{j=1}^{p-1} \alpha_j = 0 .$$

It follows $x = 0$ and the map is bijective. By c), it is an isomorphism of real C^*–algebras.

g) Take $x \in \mathcal{S}^{\mathbb{R}}(f_k)$ with

$$\left(u_n(x)\,,\, u_{\frac{n}{2}}(x)\,,\, (u_k(x))_{k\in\mathbb{N}_{\frac{n}{2}-1}}\right) = 0 .$$

Then for every $k \in \mathbb{N}_{\frac{n}{2}-1}$,

$$\sum_{p=0}^{n-1} x_p \prod_{j=1}^{p-1} \alpha_j = 0 ,$$

$$\sum_{p=0}^{n-1} x_p (-1)^p \prod_{j=1}^{p-1} \alpha_j = 0 ,$$

$$\sum_{p=0}^{n-1} x_p \omega^{kp} \prod_{j=1}^{p-1} \alpha_j = 0 ,$$

$$\sum_{p=0}^{n-1} x_p \omega^{-kp} \prod_{j=1}^{p-1} \alpha_j = 0 .$$

It follows $x = 0$ and the map is bijective. By c), it is an isomorphism of real C^*–algebras.

h) Take $\alpha_j = 1$ for every $j \in \mathbb{Z}_n$ and $\beta = 1$. By Proposition 7.1.2.5 d $\Rightarrow$ a, $\mathcal{S}^{\mathbb{C}}(f_\alpha)$ is a Gelfand C^*–algebra so that the Gelfand transform is an isometry of C^*–algebra. By d),

$$\|x\| = \sup_{k\in\mathbb{N}_n} |u_k(x)| = \sup_{k\in\mathbb{N}_n} \left| \sum_{j=1}^{n} \omega^{kj} x_j \right| .$$

For every $y \in \mathbb{C}^n$ and $j \in \mathbb{N}_n$,

$$(x * y)(j) = \sum_{k=1}^{n} x(j-k) y(k) .$$

Hence if we identify $\mathcal{S}(f)$ with the corresponding subalgebra of $\mathbb{C}_{n,n}$ then x is identified with the matrix

$$[x_{j-k}]_{j,k\in\mathbb{Z}_n} .$$

We get

$$\|[x_{j-k}]_{j,k\in\mathbb{Z}_n}\| = \sup_{k\in\mathbb{N}_n} \left| \sum_{j=1}^{n} e^{\frac{2\pi i j k}{n}} x_j \right| . \qquad \blacksquare$$

7.2 Clifford Algebras

Clifford algebras appear in many domains of mathematics and physics. We present them in this section only as examples of C^*–algebras.

7.2.1 General Clifford Algebras

Throughout this subsection, T denotes a totally ordered set and ρ a map

$$\rho : T \times T \longrightarrow \{-1, +1\},$$

such that

$$\rho(s,t) = \rho(t,s)$$

for all $s, t \in T$. We define

$$\sigma : T \longrightarrow \{-1,+1\}, \quad t \longmapsto \rho(t,t),$$

$$\mathfrak{T} := \mathfrak{P}_f(T),$$

$$\omega : \mathfrak{T} \longrightarrow \mathbb{N} \cup \{0\}, \quad A \longmapsto \frac{(1 + \operatorname{Card} A)\operatorname{Card} A}{2},$$

and endow $\mathfrak{T}$ with the composition

$$\mathfrak{T} \times \mathfrak{T} \longrightarrow \mathfrak{T}, \quad (A,B) \longmapsto A \triangle B.$$

$\mathfrak{T}$ is a locally finite commutative group with $\emptyset$ as neutral element (Proposition 2.4.4.11 a)).

Definition 7.2.1.1 (7) *A* ***word*** *(**on the alphabet** T) is a family of the form $(t_j)_{j \in \mathbb{N}_n}$ in T. If $a := (s_j)_{j \in \mathbb{N}_m}$ and $b := (t_j)_{j \in \mathbb{N}_n}$ are two words, then we denote by ab (**the product of** a **and** b) the word $(r_j)_{j \in \mathbb{N}_{m+n}}$ defined by*

$$r_j := \begin{cases} s_j & \text{if } j \leq m \\ t_{j-m} & \text{if } j > m. \end{cases}$$

We add the ***empty word*** 1 *to the set of words and define its products by*

$$1a = a1 = a$$

for any word a .

If we represent the above words a and b written as

$$a := s_1 s_2 \dots s_m\,, \quad b := t_1 t_2 \dots t_n\,,$$

then ab is represented by

$$s_1 s_2 \dots s_m\, t_1 t_2 \dots t_n$$

which is obtained by concantenation, i.e. by writing the letters of a in order and following them by those of b in order. The empty word has no letters at all. If a, b, c are words, then

$$(ab)c = a(bc)\,.$$

Proposition 7.2.1.2 (7) *Let* W *be the set of words on the alphabet* T . *We consider the vector space* $\mathbb{K}^{(W)}$, *set*

$$1^* = 1$$

and

$$(t_1 t_2 \dots t_n)^* := \left(\prod_{j=1}^{n} \sigma(t_j)\right) t_n \dots t_2 t_1 \in \mathbb{K}^{(W)}$$

for every $t_1 t_2 \dots t_n \in W$, *and endow* $\mathbb{K}^{(W)}$ *with the multiplication*

$$\mathbb{K}^{(W)} \times \mathbb{K}^{(W)} \longrightarrow \mathbb{K}^{(W)}\,, \quad \left(\sum_{a\in W} \alpha_a a\,,\ \sum_{b\in W} \beta_b b\right) \longmapsto \sum_{a,b\in W} \alpha_a \beta_b ab$$

and with the involution

$$\mathbb{K}^{(W)} \longrightarrow \mathbb{K}^{(W)}\,, \quad \sum_{a\in W} \alpha_a a \longmapsto \sum_{a\in W} \overline{\alpha_a} a^*\,,$$

where we identified e_a *with* a *for every* $a \in W$.

a) $\mathbb{K}^{(W)}$ *is an involutive unital algebra with* 1 *as unit.*

b) *The ideal* R *of* $\mathbb{K}^{(W)}$ *generated by*

$$\{st - \rho(s,t)ts \mid s,t \in T, s \neq t\} \cup \{t^2 - \sigma(t)1 \mid t \in T\}$$

is proper and involutive. We denote by $\mathcal{C}\ell(\rho)$ *(or more precisely,* $\mathcal{C}\ell^{\mathbb{K}}(\rho)$ *) the involutive unital algebra* $\mathbb{K}^{(W)}/R$ *and call it* ***the Clifford algebra associated to*** ρ . *If* ρ *is the constant* -1 , *then we write* $\mathcal{C}\ell(T)$ *(or* $\mathcal{C}\ell^{\mathbb{K}}(T)$ *) instead of* $\mathcal{C}\ell(\rho)$ *and call* $\mathcal{C}\ell(T)$ ***the Clifford algebra generated by*** T . *In this case the elements of* T *are called* ***the generators of*** $\mathcal{C}\ell(T)$.

c) *We identify* $\mathfrak{T}$ *with a subset of* $\mathcal{C}\ell(\rho)$ *via the map*

$$\mathfrak{T} \longrightarrow \mathcal{C}\ell(\rho)\,, \quad \begin{cases} \emptyset \longmapsto 1 \\ \{t_1, t_2, \ldots, t_n\} \longmapsto q(t_1 t_2 \ldots t_n)\,, \end{cases}$$

where $t_1 < t_2 < \cdots < t_n$ *and where* q *denotes the quotient map* $\mathbb{K}^{(W)} \to \mathcal{C}\ell(\rho)$. *Then* $\mathfrak{T}$ *is an algebraic basis of the underlying vector space of* $\mathcal{C}\ell(\rho)$. *In particular,*

$$\operatorname{Dim} \mathcal{C}\ell(\rho) = 2^{\operatorname{Card} T}$$

if T *is finite.*

d) *The map*

$$\overset{\circ}{\widetilde{\mathcal{C}\ell^{\mathbb{R}}(\rho)}} \longrightarrow \mathcal{C}\ell^{\mathbb{C}}(\rho)\,, \quad (x, y) \longmapsto x + iy$$

is an isomorphism of involutive unital complex algebras.

e) $A \in \mathfrak{T} \Rightarrow A^* A = 1$.

f) *There are uniquely determined maps*

$$f : \mathfrak{T} \times \mathfrak{T} \longrightarrow \{-1, +1\}\,,$$

$$g : \mathfrak{T} \longrightarrow \{-1, +1\}$$

such that

$$AB = f(A, B) A \triangle B\,, \quad A^* = g(A) A$$

for all $A, B \in \mathfrak{T}$.

g) f *is a Schur function for* $\mathfrak{T}$ *(Definition 7.1.1.7),* $g = \widetilde{f}$, *and* $\mathcal{C}\ell(\rho) = \mathcal{S}(f)$. f *is called* ***the Schur function associated to*** ρ.

h) *Given* $x, y \in \mathcal{C}\ell(\rho)$ *and* $A \in \mathfrak{T}$,

$$(xy)(A) = \sum_{B \in \mathfrak{T}} f(A \triangle B, B) x(A \triangle B) y(B)\,,$$

$$x^*(A) = g(A) \overline{x(A)}\,.$$

i) *If*

$$\rho(s,t) = -1$$

for all $s,t \in T$ *with* $s \neq t$, *then:*

i_1) $A, B \in \mathfrak{T} \Rightarrow AB = (-1)^{\mathrm{Card}\,(A\times B)-\mathrm{Card}\,(A\cap B)}BA$.

i_2) *Take* $A \in \mathfrak{T}$. *Put*

$$n := \mathrm{Card}\,A\,, \quad \alpha := \prod_{t\in A}\sigma(t)\,.$$

Then

$$A^2 = (-1)^{\frac{n(n-1)}{2}}\alpha 1\,, \quad A^* = (-1)^{\frac{n(n-1)}{2}}\alpha A\,.$$

In particular, if ρ *is constant equal to* -1,

$$A^2 = (-1)^{\omega(A)}1\,, \quad A^* = (-1)^{\omega(A)}A\,.$$

i_3) *Take* $A \in \mathfrak{T}$, $t \in A$, *and* $\beta \in \mathbb{K}$ *with* $\beta^4 = 1$. *Put*

$$\alpha := \prod_{s\in A}\sigma(s)\,, \quad x = \beta tA\,.$$

Then

$$x^2 = -(-1)^{\omega(A)}\alpha\beta^2\sigma(t)1\,,$$

$$x^* = (-1)^{\omega(A)}\alpha\beta^2\sigma(t)x\,.$$

a),b),c),d),e), and f) are easy to see.

g) The relation

$$A \in \mathfrak{T} \Longrightarrow f(A,\emptyset) = f(\emptyset,A) = 1$$

is obvious. Take $A, B, C \in \mathfrak{T}$. The relations

$$f(A,B)f(A\triangle B,C) = f(A,B\triangle C)f(B,C)\,,$$

$$f(A,B)g(A)g(B) = f(B,A)g(A\triangle B)$$

follow from Proposition 7.1.1.1. $c_1 \Rightarrow c_2$ and Proposition 7.1.1.6 $a_2 \Rightarrow a_3$, respectively. By e) and f),

$$f(A,B)g(A)A\triangle B = g(A)AB = A^*B =$$

$$= A^*f(A, A\triangle B)A(A\triangle B) = f(A, A\triangle B)A\triangle B\,,$$

so that

$$f(A,B)g(A) = f(A, A\triangle B)\,.$$

We get

$$g(A)f(A,A) = f(A,\emptyset) = 1\,,$$

so that

$$g(A) = f(A,A) = \tilde{f}(A)\,.$$

h) By c) and f),

$$xy = \Big(\sum_{C\in\mathfrak{T}} x(C)C\Big)\Big(\sum_{B\in\mathfrak{T}} y(B)B\Big) =$$

$$= \sum_{B\in\mathfrak{T}}\Big(\sum_{C\in\mathfrak{T}} x(C)y(B)CB\Big) =$$

$$= \sum_{B\in\mathfrak{T}}\sum_{A\in\mathfrak{T}} x(A\triangle B)y(B)(A\triangle B)B =$$

$$= \sum_{B\in\mathfrak{T}}\sum_{A\in\mathfrak{T}} x(A\triangle B)y(B)f(A\triangle B,B)A =$$

$$= \sum_{A\in\mathfrak{T}}\Big(\sum_{B\in\mathfrak{T}} f(A\triangle B,B)x(A\triangle B)y(B)\Big)A\,,$$

$$x^* = \Big(\sum_{A\in\mathfrak{T}} x(A)A\Big)^* = \sum_{A\in\mathfrak{T}}\overline{x(A)}\;A^* = \sum_{A\in\mathfrak{T}}\overline{x(A)}\;g(A)A\,.$$

Thus

$$(xy)(A) = \sum_{B\in\mathfrak{T}} f(A\triangle B,B)x(A\triangle B)y(B)\,,$$

$$x^*(A) = g(A)\;\overline{x(A)}\,.$$

i_1) For every $t \in A$,

$$tB = \begin{cases} (-1)^{\text{Card } B} Bt & \text{if } t \notin B \\ (-1)^{\text{Card } B - 1} Bt & \text{if } t \in B. \end{cases}$$

Hence

$$AB = (-1)^{\text{Card}(A \setminus B) \times \text{Card } B + \text{Card}(A \cap B) \times (\text{Card } B - 1)} BA =$$

$$= (-1)^{\text{Card}(A \times B) - \text{Card}(A \cap B)} BA.$$

i_2) Take

$$A = \{t_1, t_2, \ldots, t_n\}$$

with $t_1 < t_2 < \ldots t_n$. Then

$$A^2 = t_1 t_2 \ldots t_n t_1 t_2 \ldots t_n =$$

$$= (-1)^{(n-1)+\cdots+2+1} t_1^2 t_2^2 \ldots t_n^2 = (-1)^{\frac{n(n-1)}{2}} \alpha 1,$$

$$A^* = (t_1 t_2 \ldots t_n)^* = t_n^* \ldots t_2^* t_1^* = \alpha t_n \ldots t_2 t_1 =$$

$$= \alpha (-1)^{(n-1)+\cdots+2+1} t_1 t_2 \ldots t_n = (-1)^{\frac{n(n-1)}{2}} \alpha A.$$

i_3) Put

$$n := \text{Card } A.$$

By i_2),

$$x^2 = \beta^2 tAtA = (-1)^{n-1} \beta^2 t^2 A^2 = (-1)^{n-1} \beta^2 \sigma(t) (-1)^{\frac{n(n-1)}{2}} \alpha 1 =$$

$$= -(-1)^{\omega(A)} \alpha \beta^2 \sigma(t) 1,$$

$$x^* = \overline{\beta} A^* t^* = \overline{\beta} (-1)^{\frac{n(n-1)}{2}} \alpha \sigma(t) At = \frac{\overline{\beta}}{\beta} (-1)^{\frac{n(n-1)}{2}} \alpha \beta \sigma(t) (-1)^{n-1} tA =$$

$$= -(-1)^{\omega(A)} \alpha \beta^2 \sigma(t) x. \qquad \blacksquare$$

Definition 7.2.1.3 (7) *Let f be the Schur function associated to ρ (Proposition 7.2.1.2 g)). We set*

$$\mathcal{C}\ell_w^{\mathbb{K}}(\rho) := \mathcal{S}_w^{\mathbb{K}}(f)\,, \quad \mathcal{C}\ell_c^{\mathbb{K}}(\rho) := \mathcal{S}_c^{\mathbb{K}}(\rho)\,,$$

and call them ***the Clifford W^*-algebra*** *and* ***the Clifford C^*-algebra associated to*** *ρ, respectively. In this context we denote the product by xy rather than by $x * y$ and set*

$$x_S : \mathfrak{T} \longrightarrow \mathbb{K}\,, \quad A \longmapsto \begin{cases} x(A) & \text{if } A \subset S \\ 0 & \text{if } A \not\subset S \end{cases}$$

for all $x, y \in \ell^2(\mathfrak{T})$ and $S \subset T$. If ρ is the constant -1 then we write $\mathcal{C}\ell_w^{\mathbb{K}}(T)$ and $\mathcal{C}\ell_c^{\mathbb{K}}(T)$ instead of $\mathcal{C}\ell_w^{\mathbb{K}}(\rho)$, and $\mathcal{C}\ell_c^{\mathbb{K}}(\rho)$, respectively, and call them ***the Clifford W^*-algebra*** *and* ***the Clifford C^*-algebra of*** *T. The exponent $\mathbb{K}$ will be omitted in general.*

If T is finite, then

$$\mathcal{C}\ell(\rho) = \mathcal{C}\ell_c(\rho) = \mathcal{C}\ell_w(\rho)\,.$$

By Proposition 7.2.1.2 i_1), i_2), we may drop the condition that T be totally ordered for $\mathcal{C}\ell(T)$, $\mathcal{C}\ell_c(T)$, and $\mathcal{C}\ell_w(T)$.

Proposition 7.2.1.4 (7) *We identify $\ell^2(T)$ in a natural way with a vector subspace of $\ell^2(\mathfrak{T})$. If*

$$\rho(s,t) = -1$$

for all $s, t \in T$, $s \neq t$, then:

a) $\ell^2(T) \subset \mathcal{C}\ell_c(\rho)$.

b) $x, y \in \ell^2(T) \Rightarrow \frac{1}{2}(xy^* + y^*x) = \langle x|y\rangle 1$.

c) $x \in \ell^2(T) \Rightarrow \|\vec{x}\| \leq \sqrt{2}\|x\|_2\,,\ \vec{x} = \sum\limits_{t\in T} x(t)\vec{t}$.

d) *If $\mathbb{K} = \mathbb{R}$ and if σ is constant, then*

$$x \in \ell^2(T) \Longrightarrow x^*x = \|x\|_2^2 1\,,\ \|\vec{x}\| = \|x\|_2\,.$$

a),b), and c). Let $x, y \in \mathbb{K}^{(T)}$. Then

$$xy^* + y^*x = \Big(\sum_{s\in T} x(s)s\Big)\ \Big(\sum_{t\in T} \sigma(t)\overline{y(t)}\, t\Big) + \Big(\sum_{s\in T} \sigma(s)\overline{y(s)}\, s\Big)\ \Big(\sum_{t\in T} x(t)t\Big) =$$

$$= 2\sum_{t\in T} x(t)\sigma(t)\overline{y(t)}\, t^2 +$$

$$+ \sum_{\substack{s,t\in T\\ s<t}} (x(s)\sigma(t)\overline{y(t)}\, st + x(t)\sigma(s)\overline{y(s)}\, ts + \sigma(s)\overline{y(s)}\, x(t)st + \sigma(t)\overline{y(t)}\, x(s)ts) =$$

$$= 2\langle x|y\rangle 1\,.$$

In particular,

$$\frac{1}{2}(xx^* + x^*x) = \|x\|_2^2 1\,,$$

$$\frac{1}{2}\|\vec{x}\|^2 = \frac{1}{2}\|x^*x\| = \|\frac{1}{2}x^*x\| \le \|\frac{1}{2}(xx^* + x^*x)\| = \|x\|_2^2$$

(Theorem 7.1.2.1 c), Corollary 4.2.1.18), so that

$$\|\vec{x}\| \le \sqrt{2}\|x\|_2\,.$$

Let now $x \in \ell^2(T)$. Then

$$\|\vec{x_R} - \vec{x_S}\| \le \sqrt{2}\|x_R - x_S\|_2$$

for all $R, S \in \mathfrak{T}$, $R \subset S$. Hence if $\mathfrak{F}$ denotes the upper section filter of $\mathfrak{T}$, then

$$u := \lim_{S,\mathfrak{F}} \vec{x_S}$$

exists in $\mathcal{L}(\ell^2(T))$ and

$$\|u\| \le \sqrt{2}\|x\|_2\,.$$

We have for every $y \in \ell^2(T)$ and $A \in \mathfrak{T}$,

$$uy = \lim_{S,\mathfrak{F}} \vec{x_S}y = \lim_{S,\mathfrak{F}} x_S y\,,$$

$$(uy)(A) = \lim_{S,\mathfrak{F}} (x_S y)(A) = (xy)(A)\,,$$

$$xy = uy\,.$$

Hence $x \in \mathcal{C}\ell_w(\rho)$ and $\vec{x} = u$, so

$$\|\vec{x}\| \leq \sqrt{2}\|x\|_2\,.$$

By Proposition 7.1.1.3 $b_4 \Rightarrow b_2$, $x \in \mathcal{C}\ell_c(\rho)$. b) and

$$\vec{x} = \sum_{t\in T} x(t)\vec{t}$$

follow by continuity.

d) For $x \in \mathbb{R}^{(T)}$,

$$x^*x = \Big(\sum_{s\in T} x(s)\sigma(s)s\Big)\Big(\sum_{t\in T} x(t)t\Big) =$$

$$= \sum_{t\in T} x(t)^2\sigma(t)t^2 + \sum_{\substack{s,t\in T\\ s<t}} (x(s)\sigma(s)x(t)st + x(t)\sigma(t)x(s)ts) =$$

$$= \|x\|_2^2 1\,.$$

By continuity, the above result holds for an arbitrary $x \in \ell^2(T)$. We get

$$\|\vec{x}\|^2 = \|x^*x\| = \|x\|_2^2\,. \qquad \blacksquare$$

Proposition 7.2.1.5 (7)

a) *There are unital C^*-algebras containing $\mathcal{C}\ell(\rho)$ as an involutive unital subalgebra and all such C^*-algebras induce the same norm on $\mathcal{C}\ell(\rho)$. We take $\mathcal{C}\ell(\rho)$ with this norm.*

b) *Let E be a unital C^*-algebra and $(x_t)_{t\in T}$ a family in E such that*

$$x_s x_t = \rho(s,t)x_t x_s\,,$$

$$x_t^2 = \sigma(t)1\,, \quad x_t^* = \sigma(t)x_t$$

for all $s,t \in T$, $s \neq t$. Then there are uniquely determined continuous homomorphisms of involutive unital algebras

$$u : \mathcal{C}\ell(\rho) \longrightarrow E\,,$$

$$v : \mathcal{C}\ell_c(\rho) \longrightarrow E\,,$$

such that $\|u\| \leq 1$ *and*

$$ut = vt = x_t$$

for every $t \in T$. *If, in addition, the set*

$$\{x_{t_1} x_{t_2} \dots x_{t_n} \mid t_1, t_2, \dots t_n \in T\,,\ t_1 < t_2 < \dots < t_n\}$$

is linearly independent, then u *and* v *are injective.*

c) *If* E *is a unital* C^**-algebra containing* $\mathcal{C}\ell(\rho)$ *as a dense involutive unital subalgebra, then there is a unique isomorphism of* C^**-algebras* $\mathcal{C}\ell_c(\rho) \to E$ *which fixes every element of* $\mathcal{C}\ell(\rho)$.

d) *If*

$$\rho' : T \times T \longrightarrow \{-1, +1\}$$

is a map such that

$$\rho'(s,t) = \rho(s,t)$$

for all distinct $s, t \in T$, *then there is a unique isomorphism*

$$u : \mathcal{C}\ell^{\mathbb{C}}(\rho) \longrightarrow \mathcal{C}\ell^{\mathbb{C}}(\rho') \quad (\textit{resp. } u : \mathcal{C}\ell_c^{\mathbb{C}}(\rho) \longrightarrow \mathcal{C}\ell_c^{\mathbb{C}}(\rho'))$$

such that

$$ut = \begin{cases} t & \text{if } \rho(t,t) = \rho'(t,t) \\ it & \text{if } \rho(t,t) \neq \rho'(t,t) \end{cases}$$

for every $t \in T$.

a) By Theorem 7.1.2.1 c), there are unital C^*-algebras containing $\mathcal{C}\ell(\rho)$ as an involutive unital subalgebra. The uniqueness of the norm is obvious if T is finite (Corollary 4.1.1.21). But every element of $\mathcal{C}\ell(\rho)$ is contained in an involutive unital subalgebra of $\mathcal{C}\ell(\rho)$ generated by a finite subset of T.

b) Let W be the set of words on the alphabet T and $w : \mathbb{K}^{(W)} \to E$ the homomorphism of unital algebras such that

$$wt = x_t$$

for every $t \in T$. By the relations fulfilled by the family $(x_t)_{t\in T}$, w factorizes to a homomorphism of involutive unital algebras $u : \mathcal{C}\ell(\rho) \to E$. Since every element of $\mathcal{C}\ell(\rho)$ is contained in an involutive unital subalgebra of $\mathcal{C}\ell(\rho)$ generated by a finite subset of T, it follows from Corollary 4.1.1.20 that u is continuous and $\|u\| \leq 1$. Hence u may be extended to a continuous homomorphism of involutive unital algebras $v : \mathcal{C}\ell_c(\rho) \to E$. The uniqueness is obvious.

If the supplementary condition is fulfilled, then u is injective, so it preserves the norms (Theorem 4.2.6.6). It follows that v is injective.

c) follows from b).

d) Put

$$x_t := \begin{cases} t & \text{if } \rho(t,t) = \rho'(t,t) \\ it & \text{if } \rho(t,t) \neq \rho'(t,t) \end{cases}$$

for all $t \in T$ and apply b) and c). ■

Corollary 7.2.1.6 (7) *Let S be a totally ordered set. Then the following are equivalent.*

a) *$\mathcal{C}\ell(S)$ and $\mathcal{C}\ell(T)$ are isomorphic.*

b) *$\mathcal{C}\ell_c(S)$ and $\mathcal{C}\ell_c(T)$ are isomorphic.*

c) $\operatorname{Card} S = \operatorname{Card} T$.

These conditions imply:

d) *$\mathcal{C}\ell_w(S)$ and $\mathcal{C}\ell_w(T)$ are isomorphic.*

a $\Rightarrow$ b follows from Proposition 7.2.1.5 b).
b $\Rightarrow$ c follows from Proposition 7.1.2.3 h).
c $\Rightarrow$ a & d follows from Proposition 7.2.1.2 i_1), i_2). ■

Corollary 7.2.1.7 (7) *Suppose T finite. Take an element a not belonging to T and extend ρ to a function*

$$\rho' : (T \cup \{a\}) \times (T \cup \{a\}) \longrightarrow \{-1, +1\}$$

by defining

$$\rho'(a,a) := \begin{cases} +1 & \text{if } T^2 = -1 \\ -1 & \text{if } T^2 = 1 \end{cases}$$

and

$$\rho'(t,a) = \rho'(a,t) = \begin{cases} +1 & \text{if } tT = Tt \\ -1 & \text{if } tT = -Tt \end{cases}$$

for every $t \in T$. *Then there is a unique isomorphism of real* C^**-algebras*

$$u : \mathcal{C}\ell^{\mathbb{R}}(\rho') \longrightarrow \mathcal{C}\ell^{\mathbb{C}}(\rho)$$

such that

$$ut = t$$

for every $t \in T$ *and*

$$ua = iT\,.$$

Put

$$x_t := t$$

for every $t \in T$ and

$$x_a := iT\,.$$

Then

$$x_r x_s = \rho'(r,s) x_s x_r\,, \quad x_s^2 = \rho'(s,s)1$$

for all distinct $r, s \in T \cup \{a\}$. By Proposition 7.2.1.5 b), there is a unique homomorphism of involutive unital real algebras

$$u : \mathcal{C}\ell^{\mathbb{R}}(\rho') \longrightarrow \mathcal{C}\ell^{\mathbb{C}}(\rho)$$

such that

$$us = x_s$$

for every $s \in T \cup \{a\}$. It is easy to see that u is surjective. Since $\mathcal{C}\ell^{\mathbb{R}}(\rho')$ and $\mathcal{C}\ell^{\mathbb{C}}(\rho)$ have the same real dimension, u is bijective. Hence u is an isomorphism of real C^*–algebras. ■

Corollary 7.2.1.8 (7) *Suppose* T *finite. Take* $\tau \in \mathbb{K}$ *with* $\tau^2 \in \{-1, +1\}$ *and* a *an element not belonging to* T *and extend* ρ *to a function*

$$\rho' : (T \cup \{a\}) \times (T \cup \{a\}) \longrightarrow \{-1, +1\}$$

by defining

$$\rho'(a,a) := \begin{cases} \tau^2 & \text{if } T^2 = 1 \\ -\tau^2 & \text{if } T^2 = -1 \end{cases}$$

and

$$\rho'(t,a) := \rho'(a,t) := \begin{cases} +1 & \text{if } tT = Tt \\ -1 & \text{if } tT = -Tt \end{cases}$$

for every $t \in T$. *Then there is a unique isomorphism of* C^**-algebras*

$$u : \mathcal{C}\ell(\rho') \longrightarrow \mathcal{C}\ell(\rho) \times \mathcal{C}\ell(\rho)$$

such that

$$ut = (t,t)$$

for every $t \in T$ *and*

$$ua = (\tau T, -\tau T)\,.$$

Put

$$x_t := (t,t)$$

for every $t \in T$ and

$$x_a := (\tau T, -\tau T)\,.$$

Then

$$x_r x_s = \rho'(r,s) x_s x_r\,, \quad x_s^2 = \rho'(s,s)1$$

for all distinct $r, s \in T \cup \{a\}$. By Proposition 7.2.1.5 b), there is a unique homomorphism of involutive unital real algebras

$$u : \mathcal{C}\ell(\rho') \longrightarrow \mathcal{C}\ell(\rho) \times \mathcal{C}\ell(\rho)$$

such that

$$us = x_s$$

for every $s \in T \cup \{a\}$. It is easy to see that u is injective. Since $\mathcal{C}\ell(\rho')$ and $\mathcal{C}\ell(\rho) \times \mathcal{C}\ell(\rho)$ have the same dimension, u is bijective. Hence u is an isomorphism of C^*–algebras. ■

Proposition 7.2.1.9 (7) *Assume*

$$\rho(s,t) = -1$$

for all distinct $s,t \in T$. *Denote by* $\mathcal{C}\ell^+(\rho)$ *(or more precisely, by* $\mathcal{C}\ell^{\mathbb{K},+}(\rho)$*) the subalgebra of* $\mathcal{C}\ell(\rho)$ *generated by* $\{st \mid s,t \in T\}$ *and put*

$$\mathfrak{T}^+ := \{A \in \mathfrak{T} \mid \operatorname{Card} A \text{ is even}\}.$$

a) $\mathfrak{T}^+$ *is an algebraic basis of the underlying vector space of* $\mathcal{C}\ell^+(\rho)$ *and* $\mathcal{C}\ell^+(\rho)$ *is an involutive unital subalgebra of* $\mathcal{C}\ell(\rho)$. *In particular,*

$$\operatorname{Dim}\mathcal{C}\ell^+(\rho) = 2^{\operatorname{Card} T - 1}$$

if T *is finite.*

b) *For every* $t \in T$ *with* $\sigma(t) = -1$, *there is a unique isomorphism of involutive unital algebras*

$$u : \mathcal{C}\ell(\rho') \longrightarrow \mathcal{C}\ell^+(\rho)$$

such that

$$us = st$$

for every $s \in T\backslash\{t\}$, *where* ρ' *denotes the restriction of* ρ *to* $(T\backslash\{t\})^2$.

c) *The map*

$$\overset{\circ}{\overbrace{\mathcal{C}\ell^{\mathbb{R},+}(\rho)}} \longrightarrow \mathcal{C}\ell^{\mathbb{C},+}(\rho), \quad (x,y) \longmapsto x + iy$$

is an isomorphism if involutive unital complex algebras.

d) *The commutant of* $\mathcal{C}\ell^+(\rho)$ *in* $\mathcal{C}\ell_w(\rho)$ *is equal to* $\mathbb{K}1$ *if* T *is infinite and to*

$$\{\alpha 1 + \beta T \mid \alpha, \beta \in \mathbb{K}\}$$

if T *is finite.*

e) *The commutant of* $\mathcal{C}\ell(\rho)$ *in* $\mathcal{C}\ell_w(\rho)$ *is equal to* $\mathbb{K}1$ *if* $\operatorname{Card} T$ *is not an odd natural number and equal to*

$$\{\alpha 1 + \beta T \mid \alpha, \beta \in \mathbb{K}\}$$

otherwise.

a) is easy to see.

b) We have

$$(rt)(st) = srt^2 = -(st)(rt)\,,$$

$$(st)^2 = s^2 = \sigma(s)1\,,$$

$$(st)^* = t^*s^* = -\sigma(s)ts = \sigma(s)(st)$$

for all distinct $r, s \in T\backslash\{t\}$. By Proposition 7.2.1.5 b), there is a unique homomorphism of involutive unital algebras

$$u : \mathcal{C}\ell(\rho') \longrightarrow \mathcal{C}\ell^+(T)$$

such that

$$us = st$$

for all $s \in T\backslash\{t\}$ and u is injective. It is easy to see that u is also surjective.

c) is obvious.

d) Let x be an element of $\mathcal{C}\ell_w(T)$ commuting with $\mathcal{C}\ell^+(T)$. Take $A \in \mathfrak{T}\backslash\{\emptyset, T\}$, $s \in A$, and $t \in T\backslash A$. Put

$$B := (A \cup \{t\})\backslash\{s\} = A\Delta\{s,t\}\,.$$

Then

$$\begin{aligned} 0 &= (xst - stx)(B) = \\ &= f(B\Delta\{s,t\}, \{s,t\})x(B\Delta\{s,t\}) - f(\{s,t\}, B\Delta\{s,t\})x(B\Delta\{s,t\}) = \\ &= (f(A, \{s,t\}) - f(\{s,t\}, A))x(A)\,, \end{aligned}$$

where f denotes the Schur function for $\mathfrak{T}$ associated to ρ. By Proposition 7.2.1.2 i_1),

$$f(A, \{s,t\})f(\{s,t\}, A) = (-1)^{\mathrm{Card}\,(A\times\{s,t\}) - \mathrm{Card}\,(A\cap\{s,t\})} = -1$$

and so

$$x(A) = 0\,.$$

Thus $x \in \mathbb{K}1$ if T is infinite and

$$x \in \{\alpha 1 + \beta T \mid \alpha, \beta \in \mathbb{K}\}$$

if T is finite. On the other hand, any element of this form commutes with $\mathcal{C}\ell^+(T)$ (Proposition 7.2.1.2 i_1)).

e) By Proposition 7.2.1.2 i_1), T commutes with every $t \in T$ iff $\operatorname{Card} T$ is an odd natural number. Thus the assertion follows from d). ■

Remark. b) was proved by Størmer (1970) for $\operatorname{Card} T = \aleph_0$.

Proposition 7.2.1.10 (7) *Assume T is infinite. Let $t \in T$ and let E be the subalgebra of $\mathcal{C}\ell^{\mathbb{R}}(T)$ generated by*

$$\{rs \mid r, s \in T \backslash \{t\}\}.$$

a) *E is isomorphic to $\mathcal{C}\ell^{\mathbb{R},+}(T)$.*

b) *$t \in E^c$.*

c) *The map*

$$\overset{o}{E} \longrightarrow \mathcal{C}\ell^{\mathbb{R}}(T), \quad (x, y) \longmapsto x + ty$$

is an injective homomorphism of involutive unital real algebras.

d) *There is an involutive unital subalgebra of $\mathcal{C}\ell^{\mathbb{R}}(T)$ which is isomorphic to the involutive unital real algebra $\mathcal{C}\ell^{\mathbb{C}}(T)$.*

a) and b) are obvious.

c) Let the map be u. u is obviously linear and injective. By b),

$$u((x, y)^*) = u((x^*, -y^*)) = x^* - ty^* = (x + ty)^* = u((x, y))^*$$

for all $(x, y) \in \overset{o}{E}$ and

$$\begin{aligned} &u((x_1, y_1))u((x_2, y_2)) = (x_1 + ty_1)(x_2 + ty_2) = \\ &= (x_1x_2 - y_1y_2) + t(x_1y_2 + y_1x_2) = \\ &= u((x_1x_2 - y_1y_2,\ x_1y_2 + y_1x_2)) = u((x_1, y_1)(x_2, y_2)) \end{aligned}$$

for all $(x_1, y_1), (x_2, y_2) \in E$.

d) By Proposition 7.2.1.9 b),c) (and Corollary 7.2.1.6 c $\Rightarrow$ a), $\mathcal{C}\ell^{\mathbb{C}}(T)$ is isomorphic to $\overset{\circ}{\mathcal{C}\ell^{\mathbb{R},+}(\rho)}$. By a) and c), $\overset{\circ}{\mathcal{C}\ell^{\mathbb{R},+}(\rho)}$ is isomorphic as involutive unital real algebra to an involutive unital subalgebra of $\mathcal{C}\ell^{\mathbb{R}}(T)$. ■

Remark. d) was proved by G.P. Wene (1989).

Proposition 7.2.1.11 (7)

a) *For every* $A \in \mathfrak{T}$ *and* $\alpha \in \mathbb{K}$ *with*

$$A^* = \alpha^2 A\,,$$

$$\frac{1}{2}(1 \pm \alpha A) \in \mathrm{Pr}\mathcal{C}\ell_c(\rho)\,.$$

b) *If* $\mathbb{K} = \mathbb{C}$ $(\mathbb{K} = \mathbb{R})$ *then the vector subspace of* $\mathcal{C}\ell_c(\rho)$ *generated by* $\mathrm{Pr}\mathcal{C}\ell_c(\rho)$ *is dense in* $\mathcal{C}\ell_c(\rho)$ *(in* $\mathrm{Re}\mathcal{C}\ell_c(\rho)$*).*

c) *Let* $A, B \in \mathfrak{T}$ *and* $\alpha, \beta \in \mathbb{K}$ *such that*

$$A^* = \alpha^2 A\,, \quad B^* = \beta^2 B\,,$$

$$AB = -BA\,, \quad (\textit{resp. } AB = BA)\,.$$

Then the infimum of $\frac{1}{2}(1+\alpha A)$ *and* $\frac{1}{2}(1+\beta B)$ *in* $\mathcal{C}\ell_c(\rho)_+$ *is equal to* 0 *(resp. to* $\frac{1}{2}(1+\alpha A)(1+\beta B)$*).*

a) By Proposition 7.2.1.2,

$$(\alpha A)^2 = \alpha^2 A^2 = \alpha^4 1 = 1\,,$$

$$(\alpha A)^* = \overline{\alpha} A^* = \overline{\alpha}\alpha^2 A = \alpha A$$

so

$$\Big(\frac{1}{2}(1 \pm \alpha A)\Big)^2 = \frac{1}{4}(1 \pm 2\alpha A + (\alpha A)^2) = \frac{1}{4}(1 \pm 2\alpha A + 1) = \frac{1}{2}(1 \pm \alpha A)\,,$$

$$\Big(\frac{1}{2}(1 \pm \alpha A)\Big)^* = \frac{1}{2}(1 \pm (\alpha A)^*) = \frac{1}{2}(1 \pm \alpha A)\,.$$

Hence

$$\frac{1}{2}(1 \pm \alpha A) \in \mathrm{Pr}\mathcal{C}\ell_c(\rho)\,.$$

b) The complex case follows immediately from a). Let $\mathbb{K} = \mathbb{R}$ and $x \in \mathrm{Re}\mathcal{C}\ell_c(\rho)$. Then

$$(x(A))_{A\in\mathfrak{T}} = (x(A))^*_{A\in\mathfrak{T}} = (\widetilde{f}(A)x(A))_{A\in\mathfrak{T}}\,,$$

where f is the Schur function associated to ρ and we get

$$\widetilde{f}(A) \neq 1 \Longrightarrow x(A) = 0$$

for every $A \in \mathfrak{T}$. But if $\widetilde{f}(A) = 1$ for an $A \in \mathfrak{T}$, then by a),

$$\frac{1}{2}(1 \pm A) \in \Pr \mathcal{C}\ell_c(\rho)\,.$$

Hence x belongs to the closed vector subspace of $\mathcal{C}\ell_c(\rho)$ generated by $\Pr \mathcal{C}\ell_c(\rho)$.

c) The case $AB = BA$ follows immediately from a) and Corollary 4.2.7.4 a $\Rightarrow$ b, so assume $AB = -BA$. Let $x \in \mathcal{C}\ell_w(\rho)_+$ with

$$x \leq \frac{1}{2}(1 + \alpha A)\,, \quad x \leq \frac{1}{2}(1 + \beta B)\,.$$

By Proposition 4.2.7.1 d $\Rightarrow$ e,

$$x = \frac{1}{2}(1 + \alpha A)x = \frac{1}{2}x + \frac{\alpha}{2}Ax$$

and so

$$x = \alpha A x\,.$$

It follows

$$x = \alpha A x = \alpha\beta A B x = -\alpha\beta B A x = -\beta B x = -x\,,$$

$$x = 0\,. \qquad \blacksquare$$

Proposition 7.2.1.12 (7) *Let S be a subset of T and $(A_\iota)_{\iota \in I}$ a family of pairwise disjoint sets such that*

$$\bigcup_{\iota \in I} A_\iota = T \backslash S$$

and

$$\operatorname{Card} A_\iota = 4$$

for every $\iota \in I$. We set

$$p_\iota := \frac{1}{2}(1 + A_\iota)$$

for every $\iota \in I$ and denote by F, G, and H the unital C^*-subalgebras of $\mathcal{C}\ell_c(T)$ generated by S, $(p_\iota)_{\iota \in I}$, and $S \cup \{p_\iota \mid \iota \in I\}$, respectively.

a)

$$\{p_\iota \mid \iota \in I\} \subset \Pr \mathcal{C}\ell_c(T)\,,$$

$$F \cup \{p_\iota \mid \iota \in I\} \subset \{p_\iota \mid \iota \in I\}^c\,,$$

$$J \in \mathfrak{P}_f(I)\backslash\{\emptyset\} \Longrightarrow \prod_{\iota\in J} p_\iota \neq 0\,, \sum_{\iota\in J} p_\iota \neq 1,.$$

b) *G is a Gelfand C^*-algebra and the map*

$$\sigma(G) \longrightarrow \{0,1\}^I\,, \quad x' \longmapsto (x'(p_\iota))_{\iota\in I}$$

is a homeomorphism.

c) *The unital C^*-algebra $\mathcal{C}(\{0,1\}^I\,,\ \mathcal{C}\ell_c(S))$ of continuous maps of $\{0,1\}^I$ into $\mathcal{C}\ell_c(S)$ (Example 4.1.1.6) is isomorphic to H.*

d) *For every $(\alpha_\iota)_{\iota\in I} \in \mathbb{K}^I$, the family $(\alpha_\iota A_\iota)_{\iota\in I}$ is summable in $\mathcal{C}\ell_c(T)$ iff $(\alpha_\iota)_{\iota\in I} \in \ell^1(I)$. If these conditions are fulfilled and if $(\alpha_\iota)_{\iota\in I} \in \mathbb{R}_+^I$, then*

$$\Big\| \sum_{\iota\in I} \alpha_\iota p_\iota \Big\| = \sum_{\iota\in I} \alpha_\iota\,.$$

a) follows from Proposition 7.2.1.2 c), i_1), i_2), and Proposition 7.2.1.11 a).

b),c), and d) follow from a) and Proposition 4.1.2.29, since F is isomorphic to $\mathcal{C}\ell_c(S)$. ■

Corollary 7.2.1.13 (7) *Assume T infinite and let X be the Cantor set. Then the C^*-algebra of continuous maps of X into $\mathcal{C}\ell_c(T)$ is isomorphic to a unital C^*-subalgebra of $\mathcal{C}\ell_c(T)$.*

The assertion follows from Proposition 7.2.1.12 b),c) (and Corollary 7.2.1.6 c $\Rightarrow$ b)) by taking S and I such that

$$\operatorname{Card} S = \operatorname{Card} T\,, \quad \operatorname{Card} I = \aleph_0\,.$$ ■

Corollary 7.2.1.14 (7) *If T is infinite, then the C^*-algebra of convergent sequences in $\mathcal{C}\ell_c(T)$ is isomorphic to a unital C^*-subalgebra of $\mathcal{C}\ell_c(T)$.*

Let $\widetilde{\mathbb{N}}$ be the Alexandroff compactification of $\mathbb{N}$. The C^*–algebra of convergent sequences in $\mathcal{C}\ell_c(T)$ is isomorphic to the C^*–algebra $\mathcal{C}(\widetilde{\mathbb{N}}, \mathcal{C}\ell_c(T))$ of continuous maps of $\widetilde{\mathbb{N}}$ into $\mathcal{C}\ell_c(T)$. Since $\widetilde{\mathbb{N}}$ is homeomorphic to a quotient space of the Cantor set X, $\mathcal{C}\ell(\widetilde{\mathbb{N}}, \mathcal{C}\ell_c(T))$ is isomorphic to a unital C^*–subalgebra of $\mathcal{C}(X, \mathcal{C}\ell_c(T))$ and the assertion follows from Corollary 7.2.1.13. ∎

Remark. No C^*–subalgebra of $\mathcal{C}\ell_c(\mathbb{N})$ is isomorphic to ℓ^∞, since $\mathcal{C}\ell_c(\mathbb{N})$ is separable (Proposition 7.1.2.3 h)).

Corollary 7.2.1.15 (7) *Assume T infinite and let $(T_n)_{n\in\mathbb{N}}$ be a sequence of subsets of T. Then the C^*–direct sum of the family $(\mathcal{C}\ell_c(T_n))_{n\in\mathbb{N}}$ of C^*–algebras is isomorphic to a C^*–subalgebra of $\mathcal{C}\ell_c(T)$.*

In fact, this direct sum is isomorphic to a C^*–subalgebra of the C^*–algebra of convergent sequences in $\mathcal{C}\ell_c(T)$, so the assertion follows from Corollary 7.2.1.14. ∎

Proposition 7.2.1.16 (7) *Assume*

$$\rho(s,t) = -1$$

for all $s,t \in T$, $s \neq t$. Let $\mathfrak{A}$ be a sei of pairwise disjoint sets of $\mathfrak{T}$ such that

$$\operatorname{Card} A \equiv 0 \qquad (mod 4)$$

for every $A \in \mathfrak{A}$ and let $\varphi \in \mathbb{K}^T$ such that

$$\varphi^2(t) \in \{-1, +1\}$$

for every $t \in T$. We set

$$B := \bigcup_{A\in\mathfrak{A}} A,$$

$$\alpha_A := \prod_{t\in A} (\sigma(t)\varphi^2(t))$$

for every $A \in \mathfrak{A}$,

$$\sigma' : T \longrightarrow \{-1, +1\}, \quad t \longmapsto \begin{cases} -\varphi^2(t)\sigma(t)\alpha_A & \text{if } t \in A \in \mathfrak{A} \\ \varphi^2(t)\sigma(t) & \text{if } t \in T\backslash B, \end{cases}$$

$$\rho' : T \times T \longrightarrow \{-1, +1\}, \quad (s,t) \longmapsto \begin{cases} -1 & \text{if } s \neq t \\ \sigma'(t) & \text{if } s = t, \end{cases}$$

and denote for every $t \in T$ *by* t' *the element of* $\mathcal{C}\ell_c(\rho')$ *defined by*

$$t' := \begin{cases} \varphi(t)tA & \text{if } t \in A \in \mathfrak{A} \\ \varphi(t)t & \text{if } t \in T \backslash B, \end{cases}$$

where the multiplications are considered in $\mathcal{C}\ell_c(\rho')$. *Then there is a unique isomorphism of* C^**-algebras*

$$u : \mathcal{C}\ell_c(\rho) \longrightarrow \mathcal{C}\ell_c(\rho')$$

such that

$$ut = t'$$

for every $t \in T$.

Let $A \in \mathfrak{A}$. Take $t \in A$. By Proposition 7.2.1.2 i_3),

$$t'^2 = -(-1)^{w(A)} \prod_{s \in A} (-\varphi^2(s)\sigma(s)\alpha_A)\varphi^2(t)(-\varphi^2(t)\sigma(t)\alpha_A)1 = \sigma(t)1,$$

$$t'^* = \sigma(t)t'.$$

If s, t are distinct elements of A, then by Proposition 7.2.1.2 i_1),

$$s't' = \varphi(s)\varphi(t)sAtA = -\varphi(s)\varphi(t)stA^2 = -\varphi(s)\varphi(t)tAsA = -t's'.$$

If $s \in A$ and $t \in T\backslash B$, then

$$s't' = \varphi(s)\varphi(t)sAt = -\varphi(s)\varphi(t)tsA = -t's',$$

$$t'^2 = \varphi^2(t)t^2 = \varphi^2(t)\sigma'(t)1 = \sigma(t)1,$$

$$t'^* = \overline{\varphi(t)}t^* = \overline{\varphi(t)}\sigma'(t)t = \overline{\varphi(t)}\varphi^2(t)\sigma(t)t = \sigma(t)t'.$$

Let now A_1, A_2 be distinct sets of $\mathfrak{A}$ and let $t_1 \in A_1$, $t_2 \in A_2$. Then by Proposition 7.2.1.2 i_1),

$$t_1't_2' = \varphi(t_1)\varphi(t_2)t_1A_1t_2A_2 = -\varphi(t_1)\varphi(t_2)t_2A_2t_1A_1 = -t_2't_1'.$$

If s, t are distinct elements of $T\backslash B$, then

$$s't' = \varphi(s)\varphi(t)st = -\varphi(s)\varphi(t)ts = -t's' .$$

By Proposition 7.2.1.5 b), there is a unique homomorphism of C^*-algebras

$$u : \mathcal{C}\ell_c(\rho) \longrightarrow \mathcal{C}\ell_c(\rho')$$

such that

$$ut = t'$$

for every $t \in T$.

Let $A \in \mathfrak{T}$ and $t \in A$. Then

$$\prod_{s\in A} s' = \prod_{s\in A}(\varphi(s)sA) =$$

$$= \Big(\prod_{s\in A}\varphi(s)\Big)AA^n(-1)^{(1+2+\cdots+(n-1))} = \Big(\prod_{s\in A}\varphi(s)\Big)A ,$$

where $n := \operatorname{Card} A$ so

$$t'\prod_{s\in A} s' = \varphi(t)tA\Big(\prod_{s\in A}\varphi(s)\Big)A = \varphi(t)\Big(\prod_{s\in A}\varphi(s)\Big)\Big(\prod_{s\in A}\sigma(s)\Big)t .$$

This shows that u is surjective. From the definition of t' it can be seen that

$$\{t'_1t'_2\ldots t'_n \mid t_1, t_2, \ldots, t_n \in T,\ t_1 < t_2 < \ldots t_n\}$$

is linearly independent, so u is an isomorphism (Proposition 7.2.1.5 b). ■

Corollary 7.2.1.17 (7) *Let A be a subset of T such that σ is constant on A and such that either A is infinite or A is finite and*

$$\operatorname{Card} A \equiv 0 \quad (mod 4) .$$

Let further

$$\rho' : T \times T \longrightarrow \{-1, +1\}$$

such that

$$\rho(s,t) = \rho'(s,t) = -1$$

for all $s, t \in T$, $s \neq t$, and

$$\rho'(t,t) = \begin{cases} -\sigma(t) & \text{if } t \in A \\ \sigma(t) & \text{if } t \in T\backslash A . \end{cases}$$

Then $\mathcal{C}\ell_c^{\mathbb{R}}(\rho)$ and $\mathcal{C}\ell_c^{\mathbb{R}}(\rho')$ are isomorphic. ■

Corollary 7.2.1.18 (7) *Assume* $\mathbb{K} = \mathbb{C}$. *Let* $\mathfrak{A}$ *be a set of pairwise disjoint sets of* $\mathfrak{T}$ *such that*

$$\operatorname{Card} A \equiv 0 \pmod 4$$

for every $A \in \mathfrak{A}$. *We set*

$$B := \bigcup_{A \in \mathfrak{A}} A$$

and

$$x_t := \begin{cases} itA & \text{if } t \in A \in \mathfrak{A} \\ t & \text{if } t \in T \backslash B. \end{cases}$$

Then there is an isomorphism of C^**-algebras*

$$u : \mathcal{C}\ell_c(T) \longrightarrow \mathcal{C}\ell_c(T)$$

such that

$$ut = x_t$$

for every $t \in T$. ■

Proposition 7.2.1.19 (7) *Let* $u : \mathcal{C}\ell(\rho) \to \mathcal{C}\ell_w(\rho)$ *be a linear map such that:*

1) $u\emptyset = \emptyset$.

2) $A \in \mathfrak{T} \backslash \{\emptyset\} \Rightarrow \langle \emptyset | uA \rangle = 0$.

3) $A, B \in \mathfrak{T} \Rightarrow u(A^*B) = (uA)^*(uB)$.

4) $u(\mathcal{C}\ell(\rho))$ *is dense in* $\ell^2(\mathfrak{T})$.

Then u *is the restriction of a unitary operator on* $\ell^2(\mathfrak{T})$.

Take $A, B \in \mathfrak{T}$ and let f be the Schur function associated to ρ. By Propositions 7.1.1.9 c), 7.1.1.6 $d_3 \Rightarrow d_1$, and 7.2.1.2 e),f),g),

$$\langle uA | uB \rangle = \langle \emptyset | (uA)^*(uB) \rangle = \langle \emptyset | u(A^*B) \rangle =$$

$$= \langle \emptyset | f(A,B)\widetilde{f}(A)u(A \Delta B) \rangle = f(A,B)\widetilde{f}(A)\langle \emptyset | u(A \Delta B) \rangle =$$

$$= \begin{cases} 0 & \text{if } A \neq B \\ 1 & \text{if } A = B\,. \end{cases}$$

Hence $u(\mathfrak{T})$ is an orthonormal set, so by 4) it is an orthonormal basis of $\ell^2(\mathfrak{T})$.

Let $x \in \mathcal{C}\ell(\rho)$. By Pythagoras' Theorem,

$$\|ux\|^2 = \Big\|u\Big(\sum_{A\in\mathfrak{T}} x(A)A\Big)\Big\|^2 = \Big\|\sum_{A\in\mathfrak{T}} x(A)uA\Big\|^2 =$$

$$= \sum_{A\in\mathfrak{T}} |x(A)|^2 = \|x\|^2\,.$$

Hence u is the restriction of an isometry $\ell^2(\mathfrak{T}) \to \ell^2(\mathfrak{T})$ and the assertion follows from Proposition 5.5.5.9 a $\Rightarrow$ d. ■

Proposition 7.2.1.20 (7) *The following are equivalent:*

a) $\rho(s,t) = 1$ *for all distinct* $s,t \in T$.

b) $A,B = BA$ *for all* $A,B \in \mathfrak{T}$.

c) $\mathcal{C}\ell_w(\rho)$ *is commutative.*

If these conditions are fulfilled and T *is infinite, then:*

d) $\mathcal{C}\ell_c(\rho)$ *is isomorphic to a unital* C^**-subalgebra of* $\mathcal{C}\ell_c(T)$.

a $\Leftrightarrow$ b $\Leftrightarrow$ c is easy to see.

d) Let $(A_\iota)_{t\in T}$ be a disjoint family in $\mathfrak{T}$ such that

$$\operatorname{Card} A_t = \begin{cases} 2 & \text{if } \sigma(t) = -1 \\ 4 & \text{if } \sigma(t) = 1\,. \end{cases}$$

If we consider $(A_t)_{t\in T}$ as a family in $\mathcal{C}\ell_c(T)$, then given $s,t \in T$, by Proposition 7.2.1.2 $\mathrm{i}_1)$, $\mathrm{i}_2)$,

$$A_sA_t = A_tA_s\,,$$

$$A_t^2 = \begin{cases} -1 & \text{if } \sigma(t) = -1 \\ 1 & \text{if } \sigma(t) = 1 \end{cases} \qquad A_t^* = \begin{cases} -A_t & \text{if } \sigma(t) = -1 \\ A_t & \text{if } \sigma(t) = 1\,. \end{cases}$$

By Proposition 7.2.1.5 b), $\mathcal{C}\ell_c(\rho)$ is isomorphic to the C^*-subalgebra of $\mathcal{C}\ell_c(T)$ generated by $(A_t)_{t\in T}$. ■

Proposition 7.2.1.21 (7) *If* $\operatorname{Card} T = \aleph_0$ *then* $\mathcal{C}\ell_c(\rho)$ *is isomorphic to a unital* C^**-subalgebra of* $\mathcal{C}\ell_c(T)$.

We may assume $T = \mathbb{N}$. Using Proposition 7.2.1.2 i_1), it is possible to construct an injective sequence $(t_n)_{n\in\mathbb{N}}$ in T and a sequence $(A_n)_{n\in\mathbb{N}}$ in $\mathfrak{T}$ such that for all $m, n \in \mathbb{N}$, $m < n$,

$$t_n \in A_n ,$$

$$\begin{cases} \operatorname{Card} A_n \equiv 2 \pmod 4 \text{ if } \sigma(n) = -1 \\ \operatorname{Card} A_n \equiv 4 \pmod 4 \text{ if } \sigma(n) = 1 , \end{cases}$$

$$A_m \cap A_n \subset \{t_k \mid k \in \mathbb{N}_m\} ,$$

$$A_m A_n = \rho(m, n) A_n A_m ,$$

where the multiplication is taken in $\mathcal{C}\ell(T)$. By Proposition 7.2.1.5 b) (and Proposition 7.2.1.2 i_2)), $\mathcal{C}\ell_c(\rho)$ is isomorphic to a unital C^*-subalgebra of $\mathcal{C}\ell_c(T)$. ■

Proposition 7.2.1.22 (7) *Let* a, b *be two distinct elements not belonging to* T . *Take* $\delta := 1$ *(resp.* $\delta := -1$ *and* $\mathbb{K} = \mathbb{C}$*) and* $\tau \in \{-1, +1\}$. *Denote by* ρ' *the extension of* ρ *to* $(T \cup \{a, b\})^2$ *defined by*

$$\rho'(a, a) = \rho'(a, b) = \rho'(b, a) = -1 , \quad \rho'(b, b) = \delta$$

and

$$\rho'(t, a) = \rho'(a, t) = \rho'(t, b) = \rho'(b, t) = \tau$$

for all $t \in T$. *Then there is a unique isomorphism of unital* C^**-algebras*

$$\varphi : \mathcal{C}\ell_c(\rho') \longrightarrow (\mathcal{C}\ell_c(\rho))_{2,2}$$

such that

$$\varphi t = (-1)^{\frac{(1-\delta)(1-\tau)}{4}} \begin{bmatrix} t & 0 \\ 0 & \tau t \end{bmatrix}$$

for every $t \in T$ *and*

$$\varphi a = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} , \qquad \varphi b = i^{\frac{1-\delta}{2}} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} ,$$

where 1 *denotes the unit of* $\mathcal{C}\ell_c(\rho)$.

Put

$$T_1 := \mathfrak{T}\,, \quad T_2 := \mathbb{Z}_2^2\,, \quad T := T_1 \times T_2\,.$$

T is isomorphic with the group $\mathfrak{P}_f(T \cup \{a,b\})$. Let f_1 be the Schur function for $\mathfrak{T}$ associated to ρ and f_2 the Schur function for $\mathbb{Z}_2^2$ defined by

f_2	$(1,0)$	$(0,1)$	$(1,1)$
$(1,0)$	-1	1	-1
$(0,1)$	-1	δ	$-\delta$
$(1,1)$	1	δ	δ

(Corollary 7.1.3.4 a)). Put

$${T_1}^+ := \begin{cases} T_1 & \text{if } \tau = 1 \\ \{A \in \mathfrak{T} \mid \operatorname{Card} A \equiv 0 \pmod 2\} & \text{if } \tau = -1\,, \end{cases}$$

$${T_2}^+ := \{t_2 \in \mathbb{Z}_2^2 \mid t_2(1) + t_2(2) \equiv 0 \pmod 2\}\,,$$

$$\varepsilon : T \longrightarrow \{-1,+1\}\,,\ (t_1,t_2) \longmapsto \begin{cases} 1 & \text{if } t_1 \in {T_1}^+ \text{ or } t_2 \in {T_2}^+ \\ -1 & \text{if } t_1 \in {T_1}^- \text{ and } t_2 \in {T_2}^-\,, \end{cases}$$

$$f : T \times T \longrightarrow \{-1,+1\}\,,\ ((s_1,s_2),(t_1,t_2)) \longmapsto \varepsilon(t_1,s_2) f_1(s_1,t_1) f_2(s_2,t_2)\,.$$

By Corollary 7.1.3.2, f is a Schur function for T and it is easy to see that f is the Schur function associated to ρ'. By Corollary 7.1.3.4 b),d), there is a unique isomorphism of C^*–algebras

$$\varphi_1 : \mathcal{C}\ell_c(\rho') \longrightarrow \mathcal{C}\ell_c(\rho))_{2,2}$$

having the required properties. ■

7.2.2 $\mathcal{C}\ell_{p,q}$

Definition 7.2.2.1 (7) *For $p,q \in \mathbb{N} \cup \{0\}$, define*

$$\rho_{p,q} : \mathbb{N}_{p+q} \times \mathbb{N}_{p+q} \longrightarrow \{-1,+1\}$$

by

$$\rho_{p,q}(s,t) = -1$$

for all distinct $s,t \in \mathbb{N}_{p+q}$ and

$$\rho_{p,q}(t,t) = \begin{cases} -1 & \text{if } t \in \mathbb{N}_p \\ +1 & \text{if } t \in \mathbb{N}_{p+q} \backslash \mathbb{N}_p . \end{cases}$$

Denote

$$\mathcal{C}\ell_{p,q}^{\mathbb{K}} := \mathcal{C}\ell^{\mathbb{K}}(\rho_{p,q}), \quad \mathcal{C}\ell_{p,q}^{\mathbb{K},+} := \mathcal{C}\ell^{\mathbb{K},+}(\rho_{p,q}), \quad \mathcal{C}\ell_{p}^{\mathbb{K}} := \mathcal{C}\ell_{p,0}^{\mathbb{K}}$$

and cal $\mathcal{C}\ell_{p}^{\mathbb{K}}$ ***the Clifford algebra of degree*** *p (Clifford, 1878).*

$\mathcal{C}\ell_{0}^{\mathbb{K}}$ and $\mathcal{C}\ell_{0,1}^{\mathbb{K}}$ are isomorphic to $\mathbb{K}$ and $\mathbb{K} \times \mathbb{K}$, respectively. $\mathcal{C}\ell_{1}^{\mathbb{R}}$ and $\mathcal{C}\ell_{2}^{\mathbb{R}}$ are isomorphic to $\mathbb{C}$ and $\mathbb{H}$, respectively.

Proposition 7.2.2.2 (7) *Let $p,q \in \mathbb{N} \cup \{0\}$.*

a) *$\mathcal{C}\ell_{p,q}^{\mathbb{C}}$ and $\mathcal{C}\ell_{p+q}^{\mathbb{C}}$ are isomorphic.*

b) *$\mathcal{C}\ell_{p+4,q}$ and $\mathcal{C}\ell_{p,q+4}$ are isomorphic.*

c) *$\mathcal{C}\ell_{p+n,q+n}$ and $(\mathcal{C}\ell_{p,q})_{2^n,2^n}$ are isomorphic for every $n \in \mathbb{N}$.*

d) *$\mathcal{C}\ell_{p,q}$ is isomorphic to $\mathcal{C}\ell_{p+1,q}^{+}$.*

a) follows from Proposition 7.2.1.5 d).
b) follows from Corollary 7.2.1.17.
c) follows from Proposition 7.2.1.22 by complete induction on n.
d) follows from Proposition 7.2.1.9 b). ■

Proposition 7.2.2.3 (7) *Let $p,q \in \mathbb{N} \cup \{0\}$ such that $p+q$ is even.*

a) *If $\frac{p-q}{2}$ is even (odd), then the real C^*-algebra $\mathcal{C}\ell_{p,q}^{\mathbb{C}}$ is isomorphic to $\mathcal{C}\ell_{p+1,q}^{\mathbb{R}}$ (to $\mathcal{C}\ell_{p,q+1}^{\mathbb{R}}$).*

b) *If $\frac{p-q}{2}$ is even (odd), then $\mathcal{C}\ell_{p,q} \times \mathcal{C}\ell_{p,q}$ is isomorphic to $\mathcal{C}\ell_{p,q+1}$ (to $\mathcal{C}\ell_{p+1,q}$).*

Put

$$n := p + q\,, \quad T := \mathbb{N}_n\,.$$

Then (with the notation of Proposition 7.2.1.2)

$$tT = -Tt$$

for every $t \in T$ and

$$T^2 = (-1)^{\frac{n(n-1)}{2}}(-1)^p 1 = (-1)^{\frac{p-q}{2}} 1$$

(Proposition 7.2.1.2 i_1),i_2)).

a) follows from Corollary 7.2.1.7.

b) follows from Corollary 7.2.1.8. ■

Corollary 7.2.2.4 (7) *If $p, q \in \mathbb{N} \cup \{0\}$ such that $p+q$ and $\frac{p-q-1}{2}$ are odd, then $\mathcal{C}\ell_{p+2,q}$ and $\mathcal{C}\ell_{p,q+2}$ are isomorphic.*

Put

$$p' := p + 1\,, \quad q' := q\,, \quad p'' := p\,, \quad q'' := q + 1\,.$$

Then $p'+q'$, $\frac{p'-q'}{2}$, and $p''+q''$ are even and $\frac{p''-q''}{2}$ is odd. By Proposition 7.2.2.3 a), $\mathcal{C}\ell^{\mathbb{C}}_{p',q'}$ and $\mathcal{C}\ell^{\mathbb{C}}_{p'',q''}$ are isomorphic to $\mathcal{C}\ell^{\mathbb{R}}_{p'+1,q'}$ and $\mathcal{C}\ell^{\mathbb{R}}_{p'',q''+1}$, respectively, i.e. $\mathcal{C}\ell^{\mathbb{C}}_{p+1,q}$ and $\mathcal{C}\ell^{\mathbb{C}}_{p,q+1}$ are isomorphic to $\mathcal{C}\ell^{\mathbb{R}}_{p+2,q}$ and $\mathcal{C}\ell^{\mathbb{R}}_{p,q+2}$, respectively. By Proposition 7.2.2.2 a), $\mathcal{C}\ell^{\mathbb{C}}_{p+1,q}$ and $\mathcal{C}\ell^{\mathbb{C}}_{p,q+1}$ are isomorphic, so that $\mathcal{C}\ell^{\mathbb{R}}_{p+2,q}$ and $\mathcal{C}\ell^{\mathbb{R}}_{p,q+2}$ are isomorphic. Again, by Proposition 7.2.2.2 a), $\mathcal{C}\ell^{\mathbb{C}}_{p+2,q}$ and $\mathcal{C}\ell^{\mathbb{C}}_{p,q+2}$ are isomorphic. ■

Proposition 7.2.2.5 (7) *There is a unique sequence $(u_n)_{n\in\mathbb{N}}$ such that for every $n \in \mathbb{N}$:*

1) *$u_n : \mathcal{C}\ell^{\mathbb{C}}_{2n} \to \mathbb{C}_{2^n,2^n}$ is an isomorphism of complex C^*-algebras.*

2) *For every $j \in \mathbb{N}_{2n-2}$,*

$$u_n e_j = \begin{bmatrix} -u_{n-1}e_j & 0 \\ 0 & u_{n-1}e_j \end{bmatrix}\,.$$

3)

$$u_n e_{2n-1} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \qquad u_n e_{2n} = i \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\,,$$

where 1 denotes the unit of $\mathbb{C}_{2^{n-1},2^{n-1}}$.

4) $\overline{u_n e_j} = (-1)^{j+1} u_n e_j$ *for every* $j \in \mathbb{N}_{2n}$.

Define the sequence $(u_n)_{n\in\mathbb{N}}$ inductively, using 2) and 3). 1) follows from Proposition 7.2.1.22 (with $\delta = \tau = -1$). 4) follows from 2) and 3) by complete induction. ■

Corollary 7.2.2.6 (7) $\mathcal{C}\ell^{\mathbb{C}}_{2n+1}$ *is isomorphic to* $\mathbb{C}_{2^n,2^n} \times \mathbb{C}_{2^n,2^n}$ *for every* $n \in \mathbb{N} \cup \{0\}$.

By Proposition 7.2.2.2 a) and 7.2.2.3 b), $\mathcal{C}\ell^{\mathbb{C}}_{2n} \times \mathcal{C}\ell^{\mathbb{C}}_{2n}$ is isomorphic to $\mathcal{C}\ell^{\mathbb{C}}_{2n+1}$. Hence by Proposition 7.2.2.5, $\mathcal{C}\ell^{\mathbb{C}}_{2n+1}$ is isomorphic to $\mathbb{C}_{2^n,2^n} \times \mathbb{C}_{2^n,2^n}$. ■

Theorem 7.2.2.7 (7) *Let* $n, p, q \in \mathbb{N} \cup \{0\}$ *such that* $p + q = 2n$ *and let* u_n *be the isomorphism defined in Proposition 7.2.2.5. Put*

$$x_j := \begin{cases} u_n e_j & \text{if } j \in \mathbb{N}_p \\ i u_n e_j & \text{if } j \in \mathbb{N}_{2n \setminus \mathbb{N}_p} \end{cases}$$

for every $j \in \mathbb{N}_{2n}$,

$$P := \{j \in \mathbb{N}_{2n} \mid \overline{x}_j = x_j\},$$

$$A := \{2j - 1 \mid j \in \mathbb{N}_n\}, \quad B := \{2j \mid j \in \mathbb{N}_n\},$$

$$y := \begin{cases} \prod\limits_{j \in \mathbb{N}_{2n} \setminus P} x_j & \text{if } \operatorname{Card} P \text{ is even} \\ \prod\limits_{j \in P} x_j & \text{if } \operatorname{Card} P \text{ is odd,} \end{cases}$$

and

$$\tau := \begin{cases} \frac{n(n-1)+p}{2} & \text{if } p \text{ is even} \\ \frac{n(n+1)+p+1}{2} & \text{if } p \text{ is odd.} \end{cases}$$

a) $j \in A \cap \mathbb{N}_p \Rightarrow \overline{x}_j = x_j$, $x_j^* = -x_j$,
$j \in A \setminus \mathbb{N}_p \Rightarrow \overline{x_j} = -x_j$, $x_j^* = x_j$,
$j \in B \cap \mathbb{N}_p \Rightarrow \overline{x_j} = -x_j$, $x^* = -x_j$,
$j \in B \setminus \mathbb{N}_p \Rightarrow \overline{x_j} = x_j$, $x_j^* = x_j$.

b) $\operatorname{Card}(A \cap \mathbb{N}_p) = \begin{cases} \frac{p}{2} & \text{if } p \text{ is even} \\ \frac{p+1}{2} & \text{if } p \text{ is odd}, \end{cases}$

$\operatorname{Card}(B \cap \mathbb{N}_p) = \begin{cases} \frac{p}{2} & \text{if } p \text{ is even} \\ \frac{p-1}{2} & \text{if } p \text{ is odd}. \end{cases}$

c) $P = (A \cap \mathbb{N}_p) \cup (B \backslash \mathbb{N}_p)$.

d) $\operatorname{Card} P = \begin{cases} n & \text{if } p \text{ is even} \\ n+1 & \text{if } p \text{ is odd.} \end{cases}$

e) *y is unitary and $\overline{y} = y$.*

f) $y^* = (-1)^\tau y$.

g) *$y^* = \overline{y}$ if τ is even and $y^* = -\overline{y}$ if τ is odd.*

h) *$y\overline{x}_j y^* = x_j$ for every $j \in \mathbb{N}_{2n}$.*

i) *There is a unique isomorphism of complex C^*-algebras*

$$v : \mathcal{C}\ell^{\mathbb{C}}_{p,q} \longrightarrow \mathbb{C}_{2^n,2^n}$$

such that

$$ve_j = x_j$$

for every $j \in \mathbb{N}_{2n}$.

j) $v(\mathcal{C}\ell^{\mathbb{R}}_{p,q}) = \{x \in \mathbb{C}_{2^n,2^n} \mid y\overline{x}y^* = x\}$.

k) *If τ is even, then $\mathcal{C}\ell^{\mathbb{R}}_{p,q}$ is isomorphic to $\mathbb{R}_{2^n,2^n}$ and*

$$\operatorname{Dim}_{\mathbb{R}} \mathcal{C}\ell^{\mathbb{R}}_{p,q} = 2^{n-1}(2^n + 1).$$

l) *If τ is odd, then $\mathcal{C}\ell^{\mathbb{R}}_{p,q}$ is isomorphic to $\mathbb{H}_{2^{n-1},2^{n-1}}$ and*

$$\operatorname{Dim}_{\mathbb{R}} \mathcal{C}\ell^{\mathbb{R}}_{p,q} = 2^{n-1}(2^n - 1).$$

m) *$p - q$ is even and*

$$\tau \text{ even} \iff \left(\frac{p-q}{2} \equiv 0 \pmod 4\right) \text{ or } \frac{p-q}{2} \equiv 3 \pmod 4\right).$$

a) follows from Proposition 7.2.2.5 4).

b) is easy to see.

c) follows from a).

d) follows from b) and c).

e) For $j \in \mathbb{N}_{2n}$,

$$e_j^* e_j = e_j e_j^* = 1\,,$$

so that $u_n e_j$ and x_j are unitary. Hence y is unitary. The relation $\overline{y} = y$ is easy to see.

f) Case 1 p even

By d), $\operatorname{Card} P = n$. By a),b),c), and Proposition 7.2.1.2 i_2),

$$y^* = (-1)^{\frac{n(n-1)}{2}}(-1)^{\frac{p}{2}} y = (-1)^{\frac{n(n-1)+p}{2}} y\,.$$

Case 2 p odd

By d), $\operatorname{Card} P = n+1$. By a),b),c), and Proposition 7.2.1.2 i_2),

$$y^* = (-1)^{\frac{n(n+1)}{2}}(-1)^{\frac{p+1}{2}} y = (-1)^{\frac{n(n+1)+p+1}{2}} y\,.$$

g) follows from e) and f).

h) If $j \in P$ $(j \notin P)$, then by e),

$$y\overline{x}_j y^* = y x_j y^* = x_j y y^* = x_j$$

$$(y\overline{x}_j y^* = -y x_j y^* = x_j y y^* = x_j)\,.$$

i) follows from Proposition 7.2.1.5 d).

j) follows from h) and i).

k) By g), $y^* = \overline{y}$. By j) and Corollary 5.5.7.9, $\mathcal{C}\ell_{p,q}^{\mathbb{R}}$ is isomorphic to $\mathbb{R}_{2^n,2^n}$, so that (Corollary 5.6.6.9)

$$\operatorname{Dim}\operatorname{Re}\mathcal{C}\ell_{p,q}^{\mathbb{R}} = \operatorname{Dim}\operatorname{Re}\mathbb{R}_{2^n,2^n} = \frac{2^n(2^n+1)}{2} = 2^{n-1}(2^n+1)\,.$$

l) By g), $y^* = -\overline{y}$. By j) and Proposition 5.5.7.11 g), $\mathcal{C}\ell_{2n}^{\mathbb{R}}$ is isomorphic to $F(\mathbb{C}_{2^n,2^n})$. By Corollary 5.6.6.3 a), $F(\mathbb{C}_{2^n,2^n})$ is isomorphic to $\mathbb{H}_{2^{n-1},2^{n-1}}$, so that (Corollary 5.6.6.9)

$$\operatorname{Dim}\operatorname{Re}\mathcal{C}\ell_{p,q}^{\mathbb{R}} = \operatorname{Dim}\operatorname{Re}\mathbb{H}_{2^{n-1},2^{n-1}} = 2\cdot 2^{2(n-1)} - 2^{n-1} = 2^{n-1}(2^n-1)\,.$$

m) Since $p+q$ is even, $p-q$ is also even. Put $r := \frac{p-q}{2}$. Then

$$\begin{cases} p+q = 2n \\ p-q = 2r \end{cases}$$

so that

$$p = n + r\,.$$

Case 1 r is even

$$\tau = \begin{cases} \frac{n(n-1)+n+r}{2} = \frac{n^2}{2} + \frac{r}{2} & \text{if } p \text{ is even} \\ \frac{n(n+1)+n+r+1}{2} = \frac{(n+1)^2}{2} + \frac{r}{2} & \text{if } p \text{ is odd} \end{cases}$$

Since

$$p \text{ even} \iff n \text{ even}\,,$$

it follows

$$\tau \text{ even} \iff \frac{r}{2} \text{ even}\,.$$

Case 2 r is odd

$$\tau = \begin{cases} \frac{n(n-1)+n+r}{2} = \frac{(n+1)^2-2(n+1)}{2} + \frac{r+1}{2} & \text{if } p \text{ is even} \\ \frac{n(n+1)+n+r+1}{2} = \frac{n^2+2n}{2} + \frac{r+1}{2} & \text{if } p \text{ is odd}\,. \end{cases}$$

Since

$$p \text{ even} \iff n \text{ odd}\,,$$

it follows

$$\tau \text{ even} \iff \frac{r+1}{2} \text{ even}\,. \qquad \blacksquare$$

Corollary 7.2.2.8 (7) *Let $p,q,p',q' \in \mathbb{N} \cup \{0\}$, such that $p+q = p'+q'$ is even. Then $C\ell^{\mathbb{R}}_{p,q}$ and $C\ell^{\mathbb{R}}_{p',q'}$ are isomorphic iff*

$$\begin{cases} p-p' \text{ even} \implies \frac{p-p'}{2} \text{ even} \\ p-p' \text{ odd} \implies \frac{p-q'+1}{2} \text{ even.} \end{cases}$$

Put

$$n := \frac{p+q}{2} = \frac{p'+q'}{2},$$

$$\tau := \begin{cases} \frac{n(n-1)+p}{2} & \text{if } p \text{ is even} \\ \frac{n(n+1)+p+1}{2} & \text{if } p \text{ is odd} \end{cases} \qquad \tau' := \begin{cases} \frac{n(n-1)+p'}{2} & \text{if } p' \text{ is even} \\ \frac{n(n+1)+p'+1}{2} & \text{if } p' \text{ is odd} \end{cases}$$

If $p - p'$ is even, then

$$\tau - \tau' = \frac{p-p'}{2}.$$

If p is even and p' odd, then

$$\tau - \tau' + (p'+1) = \frac{n(n-1)+p}{2} - \frac{n(n+1)+p'+1}{2} + p' + 1 =$$

$$= \frac{1}{2}(n^2 - n + p - n^2 - n - p' - 1 + 2p' + 2) =$$

$$= \frac{1}{2}(p - 2n + p' + 1) = \frac{p - q' + 1}{2}.$$

If p is odd and p' even, then

$$\tau - \tau' - q' = \frac{n(n+1)+p+1}{2} - \frac{n(n-1)+p'}{2} - q' =$$

$$= \frac{1}{2}(n^2 + n + p + 1 - n^2 + n - p' - 2q') =$$

$$= \frac{1}{2}(p + 2n - p' - q' - q' + 1) = \frac{p - q' + 1}{2}.$$

Hence $\tau - \tau'$ is even iff

$$\begin{cases} p - p' \text{ even} \implies \frac{p-p'}{2} \text{ even} \\ p - p' \text{ odd} \implies \frac{p-q'+1}{2} \text{ even}. \end{cases}$$

By Theorem 7.2.2.7 k),l), $\mathcal{C}\ell^{\mathbb{R}}_{p,q}$ and $\mathcal{C}\ell^{\mathbb{R}}_{p',q'}$ are isomorphic iff $\tau - \tau'$ is even and the assertion follows from the above. ■

Corollary 7.2.2.9 (7) *If $p, q \in \mathbb{N} \cup \{0\}$, then $\mathcal{C}\ell_{p+8,q}$ is isomorphic to $(\mathcal{C}\ell_{p,q})_{16,16}$.*

Put

$$T_1 := \mathbb{Z}_2^{p+q} = \mathfrak{P}(\mathbb{N}_{p+q}),\ T_1^+(\text{ resp. } T_1^-) := \{A \subset \mathbb{N}_{p+q} \mid \operatorname{Card} A \text{ is even (odd)}\},$$

$$T_2 := \mathbb{Z}_2^8 = \mathfrak{P}(\mathbb{N}_8),\ T_2^+(\text{ resp. } T_2^-) := \{A \subset \mathbb{N}_8 \mid \operatorname{Card} A \text{ is even (odd)}\},$$

$$T := T_1 \times T_2 = \mathfrak{P}(\mathbb{N}_{8+p+q}).$$

Let ρ_1, ρ_2 be the functions on $T_1 \times T_1$ and $T_2 \times T_2$ defining $\mathcal{C}\ell_{p,q}$ and $\mathcal{C}\ell_8$, respectively, and let f_1 and f_2 be the Schur functions for T_1 and T_2, respectively, associated to the corresponding ρ_1 and ρ_2. Put

$$\rho : \mathbb{N}_{8+p+q} \times \mathbb{N}_{8+p+q} \longrightarrow \{-1,+1\},\ (j,k) \longmapsto \begin{cases} 1 & \text{if } j = k \in \mathbb{N}_{8+p+q} \backslash \mathbb{N}_{8+p} \\ -1 & \text{otherwise} \end{cases}$$

$$\varepsilon : T \longrightarrow \{-1,+1\},\quad (t_1,t_2) \longmapsto \begin{cases} 1 & \text{if } t_1 \in T_1^+ \text{ or } t_2 \in T_2^+ \\ -1 & \text{if } t_1 \in T_1^- \text{ and } t_2 \in T_2^- \end{cases},$$

$$f : T \times T \longrightarrow \{-1,+1\},\quad ((s_1,s_2),(t_1,t_2)) \longmapsto \varepsilon(t_1,s_2) f_1(s_1,t_1) f_2(s_2,t_2).$$

By Corollary 7.1.3.2, f is a Schur function for T and it is easy to see that f is the Schur function for T associated to ρ. By Proposition 7.2.4.5 and Theorem 7.2.2.7 k), $\mathcal{C}\ell_8$ is isomorphic to $\mathbb{K}_{16,16}$ so that by Theorem 7.1.3.3 c), $\mathcal{C}\ell_{p+8,q}$ is isomorphic to $(\mathcal{C}\ell_{p,q})_{16,16}$. ■

Theorem 7.2.2.10 (7) *Let $p,q \in \mathbb{N} \cup \{0\}$ such that $p+q$ is odd. Put*

$$n := \frac{p+q-1}{2}.$$

a) *If $\frac{p-q+1}{2}$ is odd, then $\mathcal{C}\ell_{p,q}^{\mathbb{R}}$ is isomorphic to $\mathbb{C}_{2^n,2^n}$.*

b) *If $\frac{p-q+1}{2}$ is even, then $\mathcal{C}\ell_{p,q}^{\mathbb{R}}$ is isomorphic to*

$$\mathbb{R}_{2^n,2^n} \times \mathbb{R}_{2^n,2^n}$$

if $\frac{p-q+1}{4}$ is even and it is isomorphic to

$$\mathbb{H}_{2^{n-1},2^{n-1}} \times \mathbb{H}_{2^{n-1},2^{n-1}}$$

if $\frac{p-q+1}{4}$ is odd.

Assume p even (odd) and put $p' := p$, $q' := q - 1$ ($p' := p - 1, q' := q$). p' and q' are even, $p' + q' = 2n$,

$$\frac{p' - q'}{2} = \frac{p - q + 1}{2} \qquad \left(\frac{p' - q'}{2} = \frac{p - q - 1}{2}\right)$$

and

$$\mathcal{C}\ell^{\mathbb{R}}_{p,q} = \mathcal{C}\ell^{\mathbb{R}}_{p',q'+1} \quad (\mathcal{C}\ell^{\mathbb{R}}_{p,q} = \mathcal{C}\ell^{\mathbb{R}}_{p'+1,q'}).$$

a) $\frac{p'-q'}{2}$ is odd (even). By Proposition 7.2.2.3 a), $\mathcal{C}\ell^{\mathbb{C}}_{p',q'}$ is isomorphic to $\mathcal{C}\ell^{\mathbb{R}}_{p,q}$. By the Propositions 7.2.2.2 a) and 7.2.2.5, $\mathcal{C}\ell^{\mathbb{C}}_{p',q'}$ is isomorphic to $\mathbb{C}_{2^n,2^n}$.

b) $\frac{p'-q'}{2}$ is even (odd), so that, by Proposition 7.2.2.3 b), $\mathcal{C}\ell^{\mathbb{R}}_{p,q}$ is isomorphic to $\mathcal{C}\ell^{\mathbb{R}}_{p',q'} \times \mathcal{C}\ell^{\mathbb{R}}_{p',q'}$. Put

$$\omega := \frac{p - q + 1}{4} \in \mathbb{Z}, \quad \tau := \frac{n(n-1) + p'}{2}.$$

Then

$$q - 1 = p - 4\omega,$$

$$n = \frac{p + q - 1}{2} = \frac{p + p - 4\omega}{2} = p - 2\omega,$$

$$\tau = \frac{(p - 2\omega)(p - 2\omega - 1) + p'}{2} = \frac{p^2 - 4p\omega + 4\omega^2 - p + 2\omega + p'}{2} =$$

$$= \omega + 2\omega(\omega - p) + \frac{p^2 - p + p'}{2}.$$

Since $\frac{p^2-p+p'}{2}$ is even, it follows that τ is even iff ω is even. By Theorem 7.2.2.7 k),l), $\mathcal{C}\ell^{\mathbb{R}}_{p',q'}$ is isomorphic to $\mathbb{R}_{2^n,2^n}$ if ω is even and to $\mathbb{H}_{2^{n-1},2^{n-1}}$ if ω is odd. ■

Remark. Since for all $p, q \in \mathbb{N} \cup \{0\}$, $\mathcal{C}\ell_{p,q+4}$ and $\mathcal{C}\ell_{p+8,q}$ are isomorphic to $\mathcal{C}\ell_{p+4,q}$ and $(\mathcal{C}\ell_{p,q})_{16,16}$, respectively (Proposition 7.2.2.2 b), and Corollary 7.2.2.9), the table below gives the complete list of isomorphic representations of $\mathcal{C}\ell^{\mathbb{R}}_{p,q}$ as product of matrix algebras.

p \ q	0	1	2	3
0	$\mathbb{R}$	$\mathbb{R} \times \mathbb{R}$	$\mathbb{R}_{2,2}$	$\mathbb{C}_{2,2}$
1	$\mathbb{C}$	$\mathbb{R}_{2,2}$	$\mathbb{R}_{2,2} \times \mathbb{R}_{2,2}$	$\mathbb{R}_{4,4}$
2	$\mathbb{H}$	$\mathbb{C}_{2,2}$	$\mathbb{R}_{4,4}$	$\mathbb{R}_{4,4} \times \mathbb{R}_{4,4}$
3	$\mathbb{H} \times \mathbb{H}$	$\mathbb{H}_{2,2}$	$\mathbb{C}_{4,4}$	$\mathbb{R}_{8,8}$
4	$\mathbb{H}_{2,2}$	$\mathbb{H}_{2,2} \times \mathbb{H}_{2,2}$	$\mathbb{H}_{4,4}$	$\mathbb{C}_{8,8}$
5	$\mathbb{C}_{4,4}$	$\mathbb{H}_{4,4}$	$\mathbb{H}_{4,4} \times \mathbb{H}_{4,4}$	$\mathbb{H}_{8,8}$
6	$\mathbb{R}_{8,8}$	$\mathbb{C}_{8,8}$	$\mathbb{H}_{8,8}$	$\mathbb{H}_{8,8} \times \mathbb{H}_{8,8}$
7	$\mathbb{R}_{8,8} \times \mathbb{R}_{8,8}$	$\mathbb{R}_{16,16}$	$\mathbb{C}_{16,16}$	$\mathbb{H}_{16,16}$

This list may be calculated using Theorem 7.2.2.7 k),l), and Theorem 7.2.2.10.

Corollary 7.2.2.11 (7) *$\mathcal{C}\ell_{p,q+1}$ and $\mathcal{C}\ell_{q,p+1}$ are isomorphic for all $p, q \in \mathbb{N} \cup \{0\}$.*

By Proposition 7.2.2.2 a), we may assume $\mathbb{K} = \mathbb{R}$.

Case 1 $p+q$ odd

$p+q+1$ is even, $p-q$ is odd, and

$$\frac{p-(p+1)+1}{2} = 0\,.$$

By Corollary 7.2.2.8, $\mathcal{C}\ell_{p,q+1}$ and $\mathcal{C}\ell_{q,p+1}$ are isomorphic.

Case 2 $p+q$ even

$p+q+1$ is odd and

$$p-(q+1)+1 = p-q = -(q-p) = -(q-(p+1)+1)$$

so that the numbers

$$\frac{p-(q+1)+1}{2}, \quad \frac{q-(p+1)+1}{2}$$

have the same parity. Moreover, if they are even, then the numbers

$$\frac{p-(q+1)+1}{4}, \quad \frac{q-(p+1)+1}{4}$$

have the same parity. By Theorem 7.2.2.10, $\mathcal{C}\ell_{p,q+1}$ and $\mathcal{C}\ell_{q,p+1}$ are isomorphic. ■

Corollary 7.2.2.12 (7) *If $p,q,p',q' \in \mathbb{N} \cup \{0\}$ such that $p+q=p'+q'$ is odd, then the following are equivalent:*

a) $\mathcal{C}\ell^{\mathbb{R}}_{p,q}$ *and* $\mathcal{C}\ell^{\mathbb{R}}_{p',q'}$ *are isomorphic.*

b) $\frac{p-q-p'+q'}{4} \in \mathbb{Z}$ *and*

$$\frac{p-q+1}{4}, \frac{p'-q'+1}{4} \in \mathbb{Z} \Longrightarrow \frac{p-q-p'+q'}{8} \in \mathbb{Z}.$$

a $\Rightarrow$ b. Put

$$n := \frac{p+q-1}{2},$$

$$E := \mathbb{R}_{2^n,2^n}, \quad F := \mathbb{C}_{2^n,2^n}, \quad G := \mathbb{H}_{2^{n-1},2^{n-1}}.$$

By Corollary 5.6.6.9, E and G are not isomorphic and E,F,G are simple. It follows that F is not isomorphic to $E \times E$ or to $G \times G$. By Proposition 4.3.5.5, $E \times E$ and $G \times G$ are not isomorphic.

By the above and Theorem 7.2.2.10, $\frac{p-q+1}{2}$ and $\frac{p'-q'+1}{2}$ are either both odd or both even and in the last case $\frac{p-q+1}{4}$ and $\frac{p'-q'+1}{4}$ have the same parity. Since

$$\frac{p-q+1}{2} - \frac{p'-q'+1}{2} = \frac{p-q-p'+q'}{2},$$

this implies b).

b $\Rightarrow$ a follows from Theorem 7.2.2.10. ■

Corollary 7.2.2.13 (7) *If $p,q \in \mathbb{N} \cup \{0\}$ such that $p+q$ and $\frac{p-q+1}{2}$ are odd, then $\mathcal{C}\ell_{p+4,q}$ is isomorphic to $(\mathcal{C}\ell_{p,q})_{4,4}$.*

By Proposition 7.2.2.2 a),c), we may assume $\mathbb{K} = \mathbb{R}$. Put

$$n := \frac{p+q-1}{2}.$$

By Theorem 7.2.2.10 a), $\mathcal{C}\ell_{p,q}$ and $\mathcal{C}\ell_{p+4,q}$ are isomorphic to $\mathbb{C}_{2^n,2^n}$ and $\mathbb{C}_{2^{n+2},2^{n+2}}$, respectively, and the assertion follows from Corollary 5.6.6.8 a). ■

Proposition 7.2.2.14 (7) *Let* $p, q \in \mathbb{N} \cup \{0\}$.

a) $\mathcal{C}\ell^{\mathbb{C}}_{p,q}$ *is simple iff* $p+q$ *is even.*

b) $\mathcal{C}\ell^{\mathbb{R}}_{p,q}$ *is not simple iff* $p+q$ *and* $\frac{p-q-1}{2}$ *are odd.*

c) $\mathcal{C}\ell^{\mathbb{R}}_{p,q}$ *is not purely real iff* $p+q$ *and* $\frac{p-q+1}{2}$ *are odd.*

d) *If* $\mathcal{C}\ell_{p,q}$ *is simple (not simple), then*

$$\mathrm{Dim}\,(\mathcal{C}\ell_{p,q})^c = 1 \ (= 2)\,.$$

a) By Proposition 7.2.2.2 a), $\mathcal{C}\ell^{\mathbb{C}}_{p,q}$ is isomorphic to $\mathcal{C}\ell^{\mathbb{C}}_{p+q}$. If $p+q$ is even, then by Proposition 7.2.2.5, $\mathcal{C}\ell^{\mathbb{C}}_{p+q}$ is isomorphic to $\mathbb{C}_{2^n,2^n}$ for $n := \frac{p+q}{2}$, so it is simple (Corollary 5.6.6.9). If $p+q$ is odd, then by Corollary 7.2.2.6, $\mathcal{C}\ell^{\mathbb{C}}_{p+q}$ is not simple.

b) and c). If $p+q$ is even, then by Theorem 7.2.2.7 k),l), $\mathcal{C}\ell^{\mathbb{R}}_{p,q}$ is isomorphic to $\mathbb{R}_{2^m,2^m}$ or to $\mathbb{H}_{2^n,2^n}$ for some $m, n \in \mathbb{N}$, hence it is simple and purely real (Corollary 5.6.6.9). So assume $p+q$ odd.

Case 1 $\frac{p-q+1}{2}$ is odd.

By Theorem 7.2.2.10 a), $\mathcal{C}\ell^{\mathbb{R}}_{p,q}$ is isomorphic to $\mathbb{C}_{2^n,2^n}$ for some $n \in \mathbb{N}$, so that it is simple (Corollary 5.6.6.9) and not purely real.

Case 2 $\frac{p-q+1}{2}$ is even.

By Theorem 7.2.2.10 b), $\mathcal{C}\ell^{\mathbb{R}}_{p,q}$ is isomorphic either to $\mathbb{R}_{2^n,2^n} \times \mathbb{R}_{2^n,2^n}$ or to $\mathbb{H}_{2^{n-1},2^{n-1}} \times \mathbb{H}_{2^{n-1},2^{n-1}}$ for some $n \in \mathbb{N}$, so that it is purely real and not simple (Corollary 5.6.6.9).

d) If $\mathcal{C}\ell_{p,q}$ is simple, then it is isomorphic to some matrix algebra and therefore $(\mathcal{C}\ell_{p,q})^c$ is one–dimensional (Corollary 5.6.6.9). If $\mathcal{C}\ell_{p,q}$ is not simple, then it is isomorphic to the product of two matrix algebras (a),b), Corollary 7.2.2.6, and Theorem 7.2.2.10 b)) and $(\mathcal{C}\ell_{p,q})^c$ is two–dimensional. ■

Corollary 7.2.2.15 (7) *Let* T *be an infinite set and*

$$\rho : T \times T \longrightarrow \{-1, +1\}$$

such that

$$\rho(s,t) = -1$$

for all distinct $s,t \in T$. *For every finite subset* S *of* T, *denote by* ρ_s *the restriction of* ρ *to* $S \times S$ *and identify* $C\ell(\rho_S)$ *canonically with an involutive unital subalgebra of* $C\ell(\rho)$. *Put*

$$E := \bigcup_{S \in \mathfrak{P}_f(T)} C\ell(\rho_S).$$

a) E *is an involutive unital subalgebra of* $C\ell(\rho)$ *and* $\{0\}$ *is its only proper ideal.*

b) $C\ell_c(\rho)$ *is simple.*

c) $C\ell_c^{\mathbb{R}}(\rho)$ *is purely real.*

a) By Proposition 7.1.2.3 a), E is an involutive unital subalgebra of $C\ell(\rho)$. By Proposition 7.2.2.14 a),b), $\{0\}$ is its only proper ideal.

b) Let F be a proper closed ideal of $C\ell_c(\rho)$. Since E is dense in $C\ell_c(\rho)$, $E \not\subset F$. By a),

$$E \cap F = \{0\}.$$

Let

$$u : C\ell_c(\rho) \longrightarrow C\ell_c(\rho)/F$$

be the quotient map. Let $S \in \mathfrak{P}_f(T)$. By Corollary 4.2.6.6, u preserves the norms of the elements of $C\ell(\rho_S)$. Since S is arbitrary, u preserves the norms of the elements of E. Hence u is an isometry, so that $F = \{0\}$. Thus $C\ell_c(\rho)$ is simple.

c) follows from b) and Proposition 4.3.5.3 a $\Rightarrow$ b, since $C\ell_c^{\mathbb{C}}(\rho)$ is the complexification of $C\ell_c^{\mathbb{R}}(\rho)$ (Proposition 7.1.2.3 c)). ■

Proposition 7.2.2.16 *Put*

$$\varphi : \mathbb{N} \longrightarrow \mathbb{N}, \quad n \longmapsto \binom{n}{0} + \binom{n}{3} + \binom{n}{4} + \binom{n}{7} + \binom{n}{8} + \dots,$$

$$\psi : \mathbb{N} \longrightarrow \mathbb{N}, \; n \longmapsto \binom{n}{0} + \binom{n}{1} + \binom{n}{4} + \binom{n}{5} + \binom{n}{8} + \binom{n}{9} + \dots,$$

and

$$\widetilde{\varphi} : \mathbb{N} \longrightarrow \mathbb{R}, \quad n \longmapsto \frac{2^n - 2\varphi(n)}{2^{\frac{n}{2}}},$$

$$\widetilde{\psi} : \mathbb{N} \longrightarrow \mathbb{R}, \quad n \longmapsto \frac{2^n - 2\psi(n)}{2^{\frac{n}{2}}}.$$

Take $n \in \mathbb{N}$ *and denote*

$$\tau : \mathbb{N}_n \cup \{0\} \longrightarrow \mathbb{N}, \quad q \longmapsto \operatorname{Dim} \operatorname{Re} \mathcal{C}\ell^{\mathbb{R}}_{n-q,q}.$$

a)
$$\begin{cases} \varphi(n) = \psi(n) & \text{if } n \equiv 0 \pmod 4 \\ \varphi(n) = 2^{n-1} & \text{if } n \equiv 1 \pmod 4 \\ \varphi(n) + \psi(n) = 2^n & \text{if } n \equiv 2 \pmod 4 \\ \psi(n) = 2^{n-1} & \text{if } n \equiv 3 \pmod 4. \end{cases}$$

b) $\tau(q+4) = \tau(q)$ *for every* $q \in \mathbb{N}_n \cup \{0\}$ *with* $q + 4 \leq n$.

c) *For every* $q \in \mathbb{N}_n \cup \{0\}$,

$$\tau(q) = 2^{n-q} \sum_{k=0}^{\infty} \binom{q+1}{4k+3} + \varphi(n-q) \sum_{k=0}^{\infty} (-1)^k \binom{q}{2k} +$$

$$+ \psi(n-q) \sum_{k=0}^{\infty} (-1)^k \binom{q}{2k+1}.$$

d)
$$\begin{aligned} &\tau(0) = \varphi(n), \\ &\tau(1) = \varphi(n-1) + \psi(n-1), \\ &\tau(2) = 2^{n-2} + 2\psi(n-2), \\ &\tau(3) = 2^{n-1} + 2(\psi(n-3) - \varphi(n-3)), \\ &\tau(4) = 5 \cdot 2^{n-3} - 4\varphi(n-4), \\ &\tau(6) = 9 \cdot 2^{n-4} - 8\psi(n-6), \end{aligned}$$
where n *is greater than the argument of* τ.

e) $\widetilde{\varphi}(n) + \widetilde{\varphi}(n+4) = \widetilde{\psi}(n) + \widetilde{\psi}(n+4) = 0$.

a) If $n \equiv 0 \pmod 4$ then

$$\psi(n) = \binom{n}{n} + \binom{n}{n-1} + \binom{n}{n-4} + \binom{n}{n-5} + \cdots + \binom{n}{4} + \binom{n}{3} + \binom{n}{0} =$$

$$= \varphi(n).$$

If $n \equiv 1 \pmod 4$, then

$$\varphi(n) = \binom{n}{n} + \binom{n}{n-3} + \binom{n}{n-4} + \cdots + \binom{n}{2} + \binom{n}{1},$$

so that

$$\varphi(n) + \varphi(n) = \sum_{k=0}^{n} \binom{n}{k} = 2^n,$$

$$\varphi(n) = 2^{n-1}.$$

If $n \equiv 2 \pmod 4$ then

$$\psi(n) = \binom{n}{n} + \binom{n}{n-1} + \binom{n}{n-4} + \binom{n}{n-5} + \cdots + \binom{n}{2} + \binom{n}{1},$$

so that

$$\varphi(n) + \psi(n) = \sum_{k=0}^{n} \binom{n}{k} = 2^n.$$

If $n \equiv 3 \pmod 4$, then

$$\psi(n) = \binom{n}{n} + \binom{n}{n-1} + \binom{n}{n-4} + \binom{n}{n-5} + \cdots + \binom{n}{3} + \binom{n}{2},$$

so that

$$\psi(n) + \psi(n) = \sum_{k=0}^{n} \binom{n}{k} = 2^n,$$

$$\psi(n) = 2^{n-1}.$$

b) By Proposition 7.2.2.2 b), $\mathcal{C}\ell_{p,n-p}$ and $\mathcal{C}\ell_{p+4,n-p-4}$ are isomorphic.

c) By Proposition 7.1.4.1 b),

$$\tau(q) = \operatorname{Card}\{A \subset \mathbb{N}_n \mid e_A^* = e_A \text{ in } \mathcal{C}\ell_{n-q,q}\}.$$

It follows

$$\begin{aligned}
\tau(q) = {} & \binom{q}{0}\left(\binom{n-q}{0} + \binom{n-q}{3} + \binom{n-q}{4} + \dots\right) + \\
& + \binom{q}{1}\left(\binom{n-q}{0} + \binom{n-q}{1} + \binom{n-q}{4} + \binom{n-q}{5} + \dots\right) + \\
& + \binom{q}{2}\left(\binom{n-q}{1} + \binom{n-q}{2} + \binom{n-q}{5} + \binom{n-q}{6} + \dots\right) + \\
& + \binom{q}{3}\left(\binom{n-q}{2} + \binom{n-q}{3} + \binom{n-q}{6} + \binom{n-q}{7} + \dots\right) + \\
& + \binom{q}{4}\left(\binom{n-q}{0} + \binom{n-q}{3} + \binom{n-q}{4} + \dots\right) + \\
& + \binom{q}{5}\left(\binom{n-q}{0} + \binom{n-q}{1} + \binom{n-q}{4} + \binom{n-q}{5} + \dots\right) + \cdots =
\end{aligned}$$

$$= \binom{q}{0}\varphi(n-q) + \binom{q}{1}\psi(n-q) + \binom{q}{2}(2^{n-q} - \varphi(n-q)) +$$

$$+ \binom{q}{3}(2^{n-q} - \psi(n-q)) + \binom{q}{4}\varphi(n-q) + \binom{q}{5}\psi(n-q) + \cdots =$$

$$= 2^{n-q}\left(\binom{q}{2} + \binom{q}{3} + \binom{q}{6} + \binom{q}{7} + \ldots\right) +$$

$$+ \varphi(n-q)\left(\binom{q}{0} - \binom{q}{2} + \binom{q}{4} - \binom{q}{6} + \ldots\right) +$$

$$+ \psi(n-q)\left(\binom{q}{1} - \binom{q}{3} + \binom{q}{5} - \binom{q}{7} + \ldots\right) =$$

$$= 2^{n-q}\sum_{k=0}^{\infty}\binom{q+1}{4k+3} + \varphi(n-q)\sum_{k=0}^{\infty}(-1)^k\binom{q}{2k} + \psi(n-q)\sum_{k=0}^{\infty}(-1)^k\binom{q}{2k+1}.$$

d) follows from c).

e) By b) and d),

$$\varphi(n) = \tau(0) = \tau(4) = 5 \cdot 2^{n-3} - 4\varphi(n-4) \text{ if } n \geq 4,$$

$$2^{n-2} + 2\psi(n-2) = \tau(2) = \tau(6) = 9 \cdot 2^{n-4} - 8\psi(n-6) \text{ if } n \geq 6.$$

It follows

$$\widetilde{\varphi}(n) + \widetilde{\varphi}(n-4) = \frac{2^n - 2\varphi(n)}{2^{\frac{n}{2}}} + \frac{2^{n-4} - 2\varphi(n-4)}{2^{\frac{n-4}{2}}} =$$

$$= \frac{4 \cdot 2^{n-3} - \varphi(n) + 2^{n-3} - 4\varphi(n-4)}{2^{\frac{n-2}{2}}} = 0$$

if $n \geq 4$ and

$$\widetilde{\psi}(n-2) + \widetilde{\psi}(n-6) = \frac{2^{n-2} - 2\psi(n-2)}{2^{\frac{n-2}{2}}} + \frac{2^{n-6} - 2\psi(n-6)}{2^{\frac{n-6}{2}}} =$$

$$= \frac{2^{n-2} - 2\psi(n-2) + 2^{n-4} - 8\psi(n-6)}{2^{\frac{n-2}{2}}} = 0$$

if $n \geq 6$. ■

Proposition 7.2.2.17 (7) *Take $n \in \mathbb{N}$ and let T be a set. Let $\ell^2(T, C\ell_n)$ be the Hilbert $C\ell_n$-module defined in Example 5.6.4.2 c). Then*

$$((\alpha_A(t))_{t\in T})_{A\subset \mathbb{N}_n} \in \ell^2(T)^{\mathfrak{P}(\mathbb{N}_n)} \Longrightarrow \left(\sum_{A\subset \mathbb{N}_n} \alpha_A(t)A\right)_{t\in T} \in \ell^2(T, C\ell_n)$$

and the map

$$\ell^2(T)^{\mathfrak{P}(\mathbb{N}_n)} \longrightarrow \ell^2(T, C\ell_n), \quad (\alpha_A)_{A\subset \mathbb{N}_n} \longmapsto \left(\sum_{A\subset \mathbb{N}_n} \alpha_A(t)A\right)_{t\in T}$$

is an isomorphism of Banach spaces.

By Proposition 7.2.1.2 e),

$$\left\| \Big(\sum_{A\subset \mathbb{N}_n} \alpha_A(t)A \Big)^* \Big(\sum_{A\subset \mathbb{N}_n} \alpha_A(t)A \Big) \right\| =$$

$$= \left\| \sum_{A\subset \mathbb{N}_n} \alpha_A(t)A \right\|^2 \leq \Big(\sum_{A\subset \mathbb{N}_n} |\alpha_A(t)| \Big)^2 \leq 2^n \sum_{A\subset \mathbb{N}_n} |\alpha_A(t)|^2$$

for every $t \in T$. Hence

$$\Big(\sum_{A\subset \mathbb{N}_n} \alpha_A(t)A \Big)_{t\in T} \in \ell^2(T, C\ell_n)$$

and

$$\left\| \sum_{A\subset \mathbb{N}_n} \alpha_A A \right\|^2 = \left\| \sum_{t\in T} \Big(\sum_{A\subset \mathbb{N}_n} \alpha_A(t)A \Big)^* \Big(\sum_{A\subset \mathbb{N}_n} \alpha_A(t)A \Big) \right\| \leq$$

$$\leq \sum_{t\in T} \left\| \Big(\sum_{A\subset \mathbb{N}_n} \alpha_A(t)A \Big)^* \Big(\sum_{A\subset \mathbb{N}_n} \alpha_A(t)A \Big) \right\| \leq 2^n \sum_{t\in T} \sum_{A\subset \mathbb{N}_n} |\alpha_A(t)|^2 =$$

$$= 2^n \sum_{A\subset \mathbb{N}_n} \|\alpha_A\|_2^2 .$$

Thus the map of the proposition is continuous. This map is obviously injective.

Take $\xi \in \ell^2(T, C\ell_n)$. By Corollary 5.5.1.28,

$$\sum_{t\in T} \|\xi(t)\|^2 = \sum_{t\in T} \|\xi(t)^*\xi(t)\| \leq 2^n \left\| \sum_{t\in T} \xi(t)^*\xi(t) \right\| = 2^n \|\xi\|^2 .$$

Given $t \in T$, let $(\alpha_A(t))_{A\subset \mathbb{N}_n}$ be a family in $\mathbb{K}$ such that

$$\xi(t) = \sum_{A \subset \mathbb{N}_n} \alpha_A(t) A \,.$$

By Proposition 7.1.1.1 e),

$$\sum_{A \subset \mathbb{N}_n} |\alpha_A(t)|^2 \leq \left\| \sum_{A \subset \mathbb{N}_n} \alpha_A(t) A \right\|^2 = \|\xi(t)\|^2$$

for every $t \in T$, so that

$$\sum_{A \subset \mathbb{N}_n} \sum_{t \in T} |\alpha_A(t)|^2 = \sum_{t \in T} \sum_{A \subset \mathbb{N}_n} |\alpha_A(t)|^2 \leq \sum_{t \in T} \|\xi(t)\|^2 \leq 2^n \|\xi\|^2 \,.$$

Hence the map

$$\ell^2(T)^{\mathfrak{P}(\mathbb{N}_n)} \longrightarrow \ell^2(T, \mathcal{C}\ell_n) \,, \quad (\alpha_A)_{A \subset \mathbb{N}_n} \longmapsto \left(\sum_{A \subset \mathbb{N}_n} \alpha_A(t) x_A \right)_{t \in T}$$

is surjective and so it is an isomorphism (Principle of Inverse Operator). ■

Proposition 7.2.2.18 (7) *Let $n \in \mathbb{N}$ and let S, T be sets. We take our notation from Proposition 7.2.2.17 and identify $\ell^2(S)^{\mathfrak{P}(\mathbb{N}_n)}$ and $\ell^2(T)^{\mathfrak{P}(\mathbb{N}_n)}$ with $\ell^2(S, \mathcal{C}\ell_n)$ and $\ell^2(T, \mathcal{C}\ell_n)$, respectively, via the isomorphism of this proposition. Given $u := (u_A)_{A \subset \mathbb{N}_n} \in \mathcal{L}(\ell^2(S)\,,\, \ell^2(T))^{\mathfrak{P}(\mathbb{N}_n)}$, define*

$$\widetilde{u} : \ell^2(S, \mathcal{C}\ell_n) \longrightarrow \ell^2(T, \mathcal{C}\ell_n) \,, \quad \alpha \longmapsto \sum_{A,B \subset \mathbb{N}_n} (u_A \alpha_B) AB \,.$$

Then

$$\widetilde{u} \in \mathcal{L}_{\mathcal{C}\ell_n}(\ell^2(S, \mathcal{C}\ell_n)\,,\, \ell^2(T, \mathcal{C}\ell_n)) \,,$$

$$\widetilde{u^*} = \widetilde{((-1)^{\omega(A)} u_A^*)_{A \subset \mathbb{N}_n}}$$

for every

$$u := (u_A)_{A \subset \mathbb{N}_n} \in \mathcal{L}(\ell^2(S)\,,\, \ell^2(T))^{\mathfrak{P}(\mathbb{N}_n)}$$

and the map

$$\mathcal{L}(\ell^2(S)\,,\, \ell^2(T))^{\mathfrak{P}(\mathbb{N}_n)} \longrightarrow \mathcal{L}_{\mathcal{C}\ell_n}(\ell^2(S, \mathcal{C}\ell_n)\,,\, \ell^2(T, \mathcal{C}\ell_n)) \,, \quad u \longmapsto \widetilde{u}$$

is an isomorphism of Banach spaces. The following are equivalent for every

$$u := (u_A)_{A \subset \mathbb{N}_n} \in \mathcal{L}(\ell^2(S)\,,\, \ell^2(T))^{\mathfrak{P}(\mathbb{N}_n)} \,.$$

a) $\{u_A \mid A \subset \mathbb{N}_n\} \subset \mathcal{K}(\ell^2(S)\,,\,\ell^2(T))\,.$

b) $\widetilde{u}$ *is compact.*

Let

$$u := (u_A)_{A\subset\mathbb{N}_n} \in \mathcal{L}(\ell^2(S)\,,\,\ell^2(T))^{\mathfrak{P}(\mathbb{N}_n)}\,.$$

Take $\alpha \in \ell^2(S, \mathcal{C}\ell_n)$ and $\beta \in \ell^2(T, \mathcal{C}\ell_n)$. Then

$$\langle \widetilde{u}\alpha|\beta\rangle = \left\langle \sum_{A,B\subset\mathbb{N}_n} (u_A\alpha_B)AB \,\middle|\, \sum_{C\subset\mathbb{N}_n} \beta_C C \right\rangle =$$

$$= \sum_{A,B,C\subset\mathbb{N}_n} \langle u_A\alpha_B \mid \beta_C\rangle C^*AB =$$

$$= \sum_{A,B,C\subset\mathbb{N}_n} \langle \alpha_B \mid u_A^*\beta_C\rangle (A^*C)^*B =$$

$$= \left\langle \sum_{B\subset\mathbb{N}_n} \alpha_B B \,\middle|\, \sum_{A,C\subset\mathbb{N}_n} (u_A^*\beta_C)A^*C \right\rangle =$$

$$= \left\langle \alpha \,\middle|\, \sum_{A,C\subset\mathbb{N}_n} (-1)^{\omega(A)}(u_A^*\beta_C)AC \right\rangle$$

(Proposition 7.2.1.2 i_2)). Hence

$$\widetilde{u} \in \mathcal{L}_{\mathcal{C}\ell_n}(\ell^2(S, \mathcal{C}\ell_n)\,,\,\ell^2(T, \mathcal{C}\ell_n))\,,$$

$$\widetilde{u}^* = \widetilde{((-1)^{\omega(A)}u_A^*)_{A\subset\mathbb{N}_n}}\,.$$

It is easy to see that the map

$$\mathcal{L}(\ell^2(S), \ell^2(T))^{\mathfrak{P}(\mathbb{N}_n)} \longrightarrow \mathcal{L}_{\mathcal{C}\ell_n}(\ell^2(S, \mathcal{C}\ell_n)\,,\,\ell^2(T, \mathcal{C}\ell_n))\,, \quad u \longmapsto \widetilde{u}$$

is linear, injective, and continuous.

Take $v \in \mathcal{L}_{\mathcal{C}\ell_n}(\ell^2(S, \mathcal{C}\ell_n), \ell^2(T, \mathcal{C}\ell_n))$. There is a family $u := (u_A)_{A\subset\mathbb{N}_n}$ in $\mathcal{L}(\ell^2(S), \ell^2(T))$ such that

$$v(\alpha 1) = \sum_{A\subset\mathbb{N}_n} (u_A\alpha)A$$

for every $\alpha \in \ell^2(S)$. We get

$$\widetilde{u}(\alpha 1) = v(\alpha 1)$$

for every $\alpha \in \ell^2(S)$. By Proposition 5.6.1.8 d),

$$\widetilde{u}\Big(\sum_{B\subset\mathbb{N}_n} \alpha_B B\Big) = \sum_{A,B\subset\mathbb{N}_n} (u_A\alpha_B)AB =$$

$$= \sum_{B\subset\mathbb{N}_n} \Big(\sum_{A\subset\mathbb{N}_n} (u_A\alpha_B)A\Big)B = \sum_{B\subset\mathbb{N}_n} v(\alpha_B 1)B =$$

$$= v\Big(\sum_{B\subset\mathbb{N}_n} \alpha_B B\Big)$$

for all $\sum_{B\subset\mathbb{N}_n} \alpha_B B \in \ell^2(S, \mathcal{C}\ell_n)$, so that $\widetilde{u} = v$ and the map

$$\mathcal{L}(\ell^2(S), \ell^2(T))^{\mathfrak{P}(\mathbb{N}_n)} \longrightarrow \mathcal{L}_{\mathcal{C}\ell_n}(\ell^2(S, \mathcal{C}\ell_n), \ell^2(T, \mathcal{C}\ell_n)), \quad u \longmapsto \widetilde{u}$$

is surjective. By the above considerations (and the Principle of Inverse Operator), it is an isomorphism of Banach spaces.

a $\Rightarrow$ b. Let $(\xi_k)_{k\in\mathbb{N}}$ be a bounded sequence in $\ell^2(S, \mathcal{C}\ell_n)$. We set

$$\sum_{A\subset\mathbb{N}_n} \xi_{k,A} A := \xi_k$$

for every $k \in \mathbb{N}$. Then $(\xi_{k,A})_{k\in\mathbb{N}}$ is a bounded sequence in $\ell^2(S)$ for every $A \subset \mathbb{N}_n$ (Proposition 7.2.2.17). By Proposition 3.1.1.19 a $\Rightarrow$ b, there is a subsequence $(\xi_{k_j})_{j\in\mathbb{N}}$ of $(\xi_k)_{k\in\mathbb{N}}$ such that $(u_A\xi_{k_j,A})_{j\in\mathbb{N}}$ converges for every $A \subset \mathbb{N}_n$. Hence $(\widetilde{u}\xi_{k_j})_{j\in\mathbb{N}}$ converges. By Proposition 3.1.1.19 b $\Rightarrow$ a, $\widetilde{u}$ is compact.

b $\Rightarrow$ a. Assume there is an $A \subset \mathbb{N}_n$ such that u_A is not compact. By Proposition 3.1.1.19 b $\Rightarrow$ a, there is a bounded sequence $(\eta_k)_{k\in\mathbb{N}}$ in $\ell^2(S)$ such that $(u_A\eta_k)_{k\in\mathbb{N}}$ posseses no convergent subsequence. Then $(\eta_k 1)_{k\in\mathbb{N}}$ is a bounded sequence in $\ell^2(S, \mathcal{C}\ell_n)$ and from

$$k \in \mathbb{N} \Longrightarrow \widetilde{u}(\eta_k 1) = \sum_{B\subset A} (u_B\eta_k)B$$

it follows that $(\widetilde{u}(\eta_k 1))_{k\in\mathbb{N}}$ posseses no convergent subsequence. By Proposition 3.1.1.19 a $\Rightarrow$ b, $\widetilde{u}$ is not compact. ■

7.2.3 $\mathcal{C}\ell(\mathbb{N})$

Throughout this subsection we assume $\mathbb{K} = \mathbb{C}$ and use the following notation. We put

$$\pi_0 := x_0 := 1 \in \mathbb{C}$$

and define the sequence $(x_n)_{n\in\mathbb{N}}$ of matrices and the sequence $((x_{n,j})_{j\in\mathbb{N}_{2n}})_{n\in\mathbb{N}}$ of families of matrices inductively:

$$x_n := \begin{bmatrix} 0 & x_{n-1} \\ (-1)^n x_{n-1} & 0 \end{bmatrix} \in \mathbb{C}_{2^n,2^n},$$

and for every $j \in \mathbb{N}_{2n-2}$ and $p \in \{0,1\}$,

$$x_{n,j} := \begin{bmatrix} x_{n-1,j} & 0 \\ 0 & x_{n-1,j} \end{bmatrix} \in \mathbb{C}_{2^n,2^n},$$

$$x_{n,2n-p} := \begin{bmatrix} 0 & (-1)^{n-1} i^{1+p} \pi_{n-1} \\ i^{1-p}\pi_{n-1} & 0 \end{bmatrix} \in \mathbb{C}_{2^n,2^n}$$

(matrices of matrices!), where

$$\pi_n := \prod_{j=1}^{2n} x_{n,j}.$$

For every $n \in \mathbb{N}$ and $A \subset \mathbb{N}_{2n}$, define

$$x_{n,A} := x_{n,j_1}\, x_{n,j_2} \ldots x_{n,j_k},$$

$$\varphi(A) := (-1)^{j_1+j_2+\ldots j_k},$$

where $A = \{j_1, j_2, \ldots, j_k\}$, $j_1 < j_2 < \cdots < j_k$ $(x_{n,\emptyset} := 1,\ \varphi(\emptyset) := 1)$. Finally, for every $n \in \mathbb{N}$, $\xi \in \mathbb{C}^n$, and $x \in \mathbb{C}_{n,n}$, put

$$\widetilde{\xi} : \mathbb{N} \longrightarrow \mathbb{C}, \quad j \longmapsto \begin{cases} \xi_j & \text{if } j \in \mathbb{N}_n \\ 0 & \text{if } j \notin \mathbb{N}_n, \end{cases}$$

$$\widetilde{x} : \ell^2 \longrightarrow \ell^2, \quad \xi \longmapsto \widetilde{x(\xi|\mathbb{N}_n)}.$$

Proposition 7.2.3.1 (7) *Let* $n \in \mathbb{N}$.

a)

$$\pi_n = \begin{bmatrix} -i\pi_{n-1} & 0 \\ 0 & i\pi_{n-1} \end{bmatrix},$$

$$\pi_n^2 = (-1)^n 1, \quad \pi_n^* = \overline{\pi}_n = (-1)^n \pi_n,$$

$$j \in \mathbb{N}_{2n} \Longrightarrow \begin{cases} x_{n,j}\pi_n = -\pi_n x_{n,j} \\ \overline{x}_{n,j} = (-1)^{\frac{1}{2}(j-\sin^2 \frac{j\pi}{2})} x_{n,j}, \end{cases}$$

$$p, q \in \{0,1\} \Longrightarrow x_{n,2n-p} x_{n,2n-q} = -i^{p-q} \begin{bmatrix} 1 & 0 \\ 0 & (-1)^{p-q}1 \end{bmatrix}.$$

b) $j, k \in \mathbb{N}_{2n} \Rightarrow x_{n,j}x_{n,k} + x_{n,k}x_{n,j} = -2\delta_{jk}1, \quad x_{n,j}^* = -x_{n,j}$.

c) *For every* $A \subset \mathbb{N}_{2n}$,

$$x_{n,A}\pi_n = \varphi(A) x_{n,\mathbb{N}_{2n}\setminus A}.$$

d) *For every* $j \in \mathbb{N}_{2n}$,

$$\overline{x_{n,j}x_{n,j+(-1)^{j+1}}} = (x_{n,j}x_{n,j+(-1)^{j+1}})^* = -x_{n,j}x_{n,j+(-1)^{j+1}}.$$

e) *Given subsets* A, B *of* $\mathbb{N}_{2n}$, *there is an* $\varepsilon \in \{1,-1\}$ *such that for every* $k \in \mathbb{N}$,

$$x_{n+k,A}\ x_{n+k,B} = \varepsilon x_{n+k,A\triangle B}.$$

f) x_n *is unitary,*

$$x_n^* = (\cos\frac{n\pi}{2} - \sin\frac{n\pi}{2})\overline{x_n},$$

and

$$x_n \overline{x}_{n,j} x_n^* = x_{n,j}$$

for every $j \in \mathbb{N}_{2n}$.

g) $n \in \mathbb{N}$, $A \subset \mathbb{N}_{2n} \Rightarrow \|x_{n,A}\| \leq 1$, $\|\widetilde{x}_{n,A}\| \leq 1$.

a) We prove the assertion by induction on n. So assume the assertion holds for $n-1$. We have

$$x_{n,2n-p}x_{n,2n-q} =$$

$$= \begin{bmatrix} 0 & (-1)^{n-1}i^{1+p}\pi_{n-1} \\ i^{1-p}\pi_{n-1} & 0 \end{bmatrix} \begin{bmatrix} 0 & (-1)^{n-1}i^{1+q}\pi_{n-1} \\ i^{1-q}\pi_{n-1} & 0 \end{bmatrix} =$$

$$= \begin{bmatrix} (-1)^n i^{p-q}\pi_{n-1}^2 & 0 \\ 0 & (-1)^n i^{q-p}\pi_{n-1}^2 \end{bmatrix} = -i^{p-q} \begin{bmatrix} 1 & 0 \\ 0 & (-1)^{p-q}1 \end{bmatrix}.$$

It follows that

$$\pi_n = \left(\prod_{j=1}^{2(n-1)} x_{n,j} \right) (x_{n,2n-1}x_{2,2n}) =$$

$$= \begin{bmatrix} \pi_{n-1} & 0 \\ 0 & \pi_{n-1} \end{bmatrix} \begin{bmatrix} -i & 0 \\ 0 & i \end{bmatrix} = \begin{bmatrix} -i\pi_{n-1} & 0 \\ 0 & i\pi_{n-1} \end{bmatrix},$$

$$\pi_n^2 = \begin{bmatrix} -\pi_{n-1}^2 & 0 \\ 0 & -\pi_{n-1}^2 \end{bmatrix} = \begin{bmatrix} (-1)^n 1 & 0 \\ 0 & (-1)^n 1 \end{bmatrix} = (-1)^n 1,$$

$$\pi_n^* = \begin{bmatrix} i\pi_{n-1}^* & 0 \\ 0 & -i\pi_{n-1}^* \end{bmatrix} = \begin{bmatrix} (-1)^{n-1}i\pi_{n-1} & 0 \\ 0 & (-1)^n i\pi_{n-1} \end{bmatrix} = (-1)^n \pi_n,$$

$$\overline{\pi}_n = \begin{bmatrix} i\overline{\pi}_{n-1} & 0 \\ 0 & -i\overline{\pi}_{n-1} \end{bmatrix} = \begin{bmatrix} i\pi_{n-1}^* & 0 \\ 0 & -i\pi_{n-1}^* \end{bmatrix} = \pi_n^*,$$

$$x_{n,j}\pi_n = \begin{bmatrix} x_{n-1,j} & 0 \\ 0 & x_{n-1,j} \end{bmatrix} \begin{bmatrix} -i\pi_{n-1} & 0 \\ 0 & i\pi_{n-1} \end{bmatrix} =$$

$$= \begin{bmatrix} -ix_{n-1,j}\pi_{n-1} & 0 \\ 0 & ix_{n-1,j}\pi_{n-1} \end{bmatrix} = -\begin{bmatrix} -i\pi_{n-1}x_{n-1,j} & 0 \\ 0 & i\pi_{n-1}x_{n-1,j} \end{bmatrix} =$$

$$= -\begin{bmatrix} -i\pi_{n-1} & 0 \\ 0 & i\pi_{n-1} \end{bmatrix}\begin{bmatrix} x_{n-1,j} & 0 \\ 0 & x_{n-1,j} \end{bmatrix} = -\pi_n x_{n,j}\,.$$

If $j \in \mathbb{N}_{2n-2}$, then

$$\overline{x}_{n,j} = \begin{bmatrix} \overline{x}_{n-1,j} & 0 \\ 0 & \overline{x}_{n-1,j} \end{bmatrix} = (-1)^{\frac{1}{2}(j-\sin^2 j\frac{\pi}{2})}\begin{bmatrix} x_{n-1,j} & 0 \\ 0 & x_{n-1,j} \end{bmatrix} =$$

$$= (-1)^{\frac{1}{2}(j-\sin^2 \frac{j\pi}{2})}x_{n,j}\,.$$

By the above result,

$$\overline{x}_{n,2n-1} = \begin{bmatrix} 0 & (-1)^n\overline{\pi}_{n-1} \\ \overline{\pi}_{n-1} & 0 \end{bmatrix} =$$

$$= (-1)^{n-1}\begin{bmatrix} 0 & (-1)^n\pi_{n-1} \\ \pi_{n-1} & 0 \end{bmatrix} = (-1)^{\frac{1}{2}(2n-1-\sin^2\frac{(2n-1)\pi}{2})}x_{n,2n-1}\,,$$

$$\overline{x}_{n,2n} = \begin{bmatrix} 0 & -(-1)^{n-1}i\overline{\pi}_{n-1} \\ -i\overline{\pi}_{n-1} & 0 \end{bmatrix} =$$

$$= (-1)^n\begin{bmatrix} 0 & (-1)^{n-1}i\,\pi_{n-1} \\ i\,\pi_{n-1} & 0 \end{bmatrix} = (-1)^{\frac{1}{2}(2n-\sin^2\frac{2n\pi}{2})}x_{n,2n}\,.$$

b) Assume the assertion holds for $n-1$. Take $j,k \in \mathbb{N}_{2n-2}$ and $p,q \in \{0,1\}$. Then, by a),

$$x^*_{n,j} = \begin{bmatrix} x^*_{n-1,j} & 0 \\ 0 & x^*_{n-1,j} \end{bmatrix} =$$

$$= \begin{bmatrix} -x_{n-1,j} & 0 \\ 0 & -x_{n-1,j} \end{bmatrix} = -x_{n,j} ,$$

$$x_{n,j}x_{n,k} =$$

$$= \begin{bmatrix} x_{n-1,j} & 0 \\ 0 & x_{n-1,j} \end{bmatrix} \begin{bmatrix} x_{n-1,k} & 0 \\ 0 & x_{n-1,k} \end{bmatrix} = \begin{bmatrix} x_{n-1,j}x_{n-1,k} & 0 \\ 0 & x_{n-1,j}x_{n-1,k} \end{bmatrix} ,$$

$$x_{n,j}x_{nk} + x_{n,k}x_{nj} =$$

$$= \begin{bmatrix} x_{n-1,j}x_{n-1,k} + x_{n-1,k}x_{n-1,j} & 0 \\ 0 & x_{n-1,j}x_{n-1,k} + x_{n-1,k}x_{n-1,j} \end{bmatrix} = -2\delta_{jk}1 ,$$

$$x_{n,j}x_{n,2n-p} = \begin{bmatrix} x_{n-1,j} & 0 \\ 0 & x_{n-1,j} \end{bmatrix} \begin{bmatrix} 0 & (-1)^{n-1}i^{1+p}\pi_{n-1} \\ i^{1-p}\pi_{n-1} & 0 \end{bmatrix} =$$

$$= \begin{bmatrix} 0 & (-1)^{n-1}i^{1+p}x_{n-1,j}\pi_{n-1} \\ i^{1-p}x_{n-1,j}\pi_{n-1} & 0 \end{bmatrix} =$$

$$= -\begin{bmatrix} 0 & (-1)^{n-1}i^{1+p}\pi_{n-1}x_{n-1,j} \\ i^{1-p}\pi_{n-1}x_{x-1,j} & 0 \end{bmatrix} ,$$

$$x_{n,2n-p}x_{n,j} = \begin{bmatrix} 0 & (-1)^{n-1}i^{1+p}\pi_{n-1} \\ i^{1-p}\pi_{n-1} & 0 \end{bmatrix} \begin{bmatrix} x_{n-1,j} & 0 \\ 0 & x_{n-1,j} \end{bmatrix} =$$

$$= \begin{bmatrix} 0 & (-1)^{n-1}i^{1+p}\pi_{n-1}x_{n-1,j} \\ i^{1-p}\pi_{n-1}x_{n-1,j} & 0 \end{bmatrix} = -x_{n,j}x_{n,2n-p} ,$$

$$x^*_{n,2n-p} = \begin{bmatrix} 0 & i^{p-1}\pi^*_{n-1} \\ (-1)^{n-1}i^{-1-p}\pi^*_{n-1} & 0 \end{bmatrix} =$$

$$= \begin{bmatrix} 0 & -(-1)^{n-1} i^{1+p} \pi_{n-1} \\ -i^{1-p} \pi_{n-1} & 0 \end{bmatrix} = -x_{n,2n-p} ,$$

$$x_{n,2n-p} x_{n,2n-q} + x_{n,2n-q} x_{n,2n-p} =$$

$$= -i^{p-q} \begin{bmatrix} 1 & 0 \\ 0 & (-1)^{p-q} 1 \end{bmatrix} - i^{q-p} \begin{bmatrix} 1 & 0 \\ 0 & (-1)^{q-p} 1 \end{bmatrix} = -2\delta_{p,q} 1 .$$

c) Let $A := \{j_1, j_2, \ldots, j_k\}$, where $j_1 < j_2 < \cdots < j_k$. Then by b),

$$x_{n,A} \pi_n = x_{n,j_1} x_{n,j_2} \ldots x_{n,j_k} x_{n,1} x_{n,2} \ldots x_{n,2n} =$$

$$= (-1)^{j_k + j_{k-1} + \cdots + j_1} x_{n,\mathbb{N}_{2n} \setminus A} = \varphi(A) x_{n,\mathbb{N}_{2n} \setminus A} .$$

d) For $j \in \mathbb{N}_{2n}$, put

$$j' := j + (-1)^{j+1} .$$

By b),

$$(x_{n,j} x_{n,j'})^* = x^*_{n,j'} x^*_{n,j} = x_{n,j'} x_{n,j} = -x_{n,j} x_{n,j'} .$$

We prove the other relation by induction. Assume it holds for $n-1$. Then for $j \in \mathbb{N}_{2n-2}$,

$$\overline{x_{n,j} x_{n,j'}} = \overline{x}_{n,j} \overline{x}_{n,j'} = \begin{bmatrix} \overline{x}_{n-1,j} & 0 \\ 0 & \overline{x}_{n-1,j} \end{bmatrix} \begin{bmatrix} \overline{x}_{n-1,j'} & 0 \\ 0 & \overline{x}_{n-1,j'} \end{bmatrix} =$$

$$= \begin{bmatrix} \overline{x_{n-1,j} x_{n-1,j'}} & 0 \\ 0 & \overline{x_{n-1,j} x_{n-1,j'}} \end{bmatrix} =$$

$$= \begin{bmatrix} -x_{n-1,j} x_{n-1,j'} & 0 \\ 0 & -x_{n-1,j} x_{n-1,j'} \end{bmatrix} =$$

$$= - \begin{bmatrix} x_{n-1,j} & 0 \\ 0 & x_{n-1,j} \end{bmatrix} \begin{bmatrix} x_{n-1,j'} & 0 \\ 0 & x_{n-1,j'} \end{bmatrix} = -x_{n,j}x_{n,j'} .$$

Now let $j \in \{2n-1, 2n\}$ and put

$$p := 2n - j , \quad q := 2n - j' .$$

By a),

$$\overline{x_{n,j}x_{n,j'}} = \overline{x_{n,2n-p}x_{n,2n-q}} =$$

$$= -x_{n,2n-p}x_{2n-q} = -x_{n,j}x_{n,j'} .$$

e) By b), there is a $\varepsilon \in \{1, -1\}$ such that

$$x_{n,A}x_{n,B} = \varepsilon x_{n,A\triangle B} .$$

It follows that

$$x_{n+1,A}x_{n+1,B} = \begin{bmatrix} x_{n,A} & 0 \\ 0 & x_{n,A} \end{bmatrix} \begin{bmatrix} x_{n,B} & 0 \\ 0 & x_{n,B} \end{bmatrix} =$$

$$= \begin{bmatrix} x_{n,A}x_{n,B} & 0 \\ 0 & x_{n,A}x_{n,B} \end{bmatrix} = \begin{bmatrix} \varepsilon x_{n,A\triangle B} & 0 \\ 0 & \varepsilon x_{n,A\triangle B} \end{bmatrix} = \varepsilon x_{n+1,A\triangle B} .$$

By induction,

$$x_{n+k,A}x_{n+k,B} = \varepsilon x_{n+k,A\triangle B} .$$

f) We prove the assertion by induction on n. Assume the assertion holds for $n-1$. Then

$$x_n^* x_n = \begin{bmatrix} 0 & (-1)^n x_{n-1}^* \\ x_{n-1}^* & 0 \end{bmatrix} \begin{bmatrix} 0 & x_{n-1} \\ (-1)^n x_{n-1} & 0 \end{bmatrix} =$$

$$= \begin{bmatrix} x_{n-1}^* x_{n-1} & 0 \\ 0 & x_{n-1}^* x_{n-1} \end{bmatrix} = 1 ,$$

$$x_n^* = \begin{bmatrix} 0 & (-1)^n x_{n-1}^* \\ x_{n-1}^* & 0 \end{bmatrix} =$$

$$= (-1)^n \left(\cos \frac{(n-1)\pi}{2} - \sin \frac{(n-1)\pi}{2} \right) \begin{bmatrix} 0 & \overline{x}_{n-1} \\ (-1)^n \overline{x}_{n-1} & 0 \end{bmatrix} =$$

$$= \left(\cos \frac{n\pi}{2} - \sin \frac{n\pi}{2} \right) \overline{x}_n .$$

For $j \in \mathbb{N}_{2n-2}$,

$$x_n \overline{x}_{n,j} x_n^* =$$

$$= \begin{bmatrix} 0 & x_{n-1} \\ (-1)^n x_{n-1} & 0 \end{bmatrix} \begin{bmatrix} \overline{x}_{n-1,j} & 0 \\ 0 & \overline{x}_{n-1,j} \end{bmatrix} \begin{bmatrix} 0 & (-1)^n x_{n-1}^* \\ x_{n-1}^* & 0 \end{bmatrix} =$$

$$= \begin{bmatrix} 0 & x_{n-1} \overline{x}_{n-1,j} \\ (-1)^n x_{n-1} \overline{x}_{n-1,j} & 0 \end{bmatrix} \begin{bmatrix} 0 & (-1)^n x_{n-1}^* \\ x_{n-1}^* & 0 \end{bmatrix} =$$

$$= \begin{bmatrix} x_{n-1} \overline{x}_{n-1,j} x_{n-1}^* & 0 \\ 0 & x_{n-1} \overline{x}_{n-1,j} x_{n-1}^* \end{bmatrix} =$$

$$= \begin{bmatrix} x_{n-1,j} & 0 \\ 0 & x_{n-1,j} \end{bmatrix} = x_{n,j} .$$

Take $p \in \{0, 1\}$. Then

$$x_n \overline{x}_{n,2n-p} x_n^* =$$

$$= \begin{bmatrix} 0 & x_{n-1} \\ (-1)^n x_{n-1} & 0 \end{bmatrix} \begin{bmatrix} 0 & (-1)^{n-1} i^{-1-p} \overline{\pi}_{n-1} \\ i^{p-1} \overline{\pi}_{n-1} & 0 \end{bmatrix} \times$$

$$\times \begin{bmatrix} 0 & (-1)^n x_{n-1}^* \\ x_{n-1}^* & 0 \end{bmatrix} =$$

$$= \begin{bmatrix} i^{p-1} x_{n-1} \overline{\pi}_{n-1} & 0 \\ 0 & -i^{-1-p} x_{n-1} \overline{\pi}_{n-1} \end{bmatrix} \begin{bmatrix} 0 & (-1)^n x_{n-1}^* \\ x_{n-1}^* & 0 \end{bmatrix} =$$

$$= \begin{bmatrix} 0 & (-1)^n i^{p-1} x_{n-1} \overline{\pi}_{n-1} x_{n-1}^* \\ -i^{-1-p} x_{n-1} \overline{\pi}_{n-1} x_{n-1}^* & 0 \end{bmatrix} =$$

$$= \begin{bmatrix} 0 & (-1)^{n-1} i^{p+1} \pi_{n-1} \\ i^{1-p} \pi_{n-1} & 0 \end{bmatrix} = x_{n,2n-p} \,.$$

g) follows by induction. ■

Proposition 7.2.3.2 (7) *Take* $n \in \mathbb{N}$ *and let* $(\alpha_A)_{A \subset \mathbb{N}_{2n}}$ *be a family in* $\mathbb{C}$. *Given* $A \subset \mathbb{N}_{2n-2}$, *define*

$$\beta_A := \alpha_{A \cup \{2n-1\}}, \quad \gamma_A := \alpha_{A \cup \{2n\}}, \quad \delta_A := \alpha_{A \cup \{2n-1,2n\}},$$

$$\tilde{A} := \mathbb{N}_{2n-2} \backslash A \,.$$

a) $\sum_{A \subset \mathbb{N}_{2n}} \alpha_A x_{n,A} =$

$$= \begin{bmatrix} \sum_{A \subset \mathbb{N}_{2n-2}} (\alpha_A - i\delta_A) x_{n-1,A} & (-1)^n \sum_{A \subset \mathbb{N}_{2n-2}} \varphi(A)(\beta_A - i\gamma_A) x_{n-1,\tilde{A}} \\ \sum_{A \subset \mathbb{N}_{2n-2}} \varphi(A)(\beta_A + i\gamma_A) x_{n-1,\tilde{A}} & \sum_{A \subset \mathbb{N}_{2n-2}} (\alpha_A + i\delta_A) x_{n-1,A} \end{bmatrix} .$$

b) $\left\| \sum_{A \subset \mathbb{N}_{2n-2}} \alpha_A x_{n-1,A} \right\| \leq \left\| \sum_{A \subset \mathbb{N}_{2n}} \alpha_A x_{n,A} \right\|,$

$\left\| \sum_{A \subset \mathbb{N}_{2n-2}} \varphi(A) \beta_A x_{n-1,\tilde{A}} \right\| \leq \left\| \sum_{A \subset \mathbb{N}_{2n}} \alpha_A x_{n,A} \right\|,$

$\left\| \sum_{A \subset \mathbb{N}_{2n-2}} \varphi(A) \gamma_A x_{n-1,\tilde{A}} \right\| \leq \left\| \sum_{A \subset \mathbb{N}_{2n}} \alpha_A x_{n,A} \right\|,$

$\left\| \sum_{A \subset \mathbb{N}_{2n-2}} \delta_A x_{n-1,A} \right\| \leq \left\| \sum_{A \subset \mathbb{N}_{2n}} \alpha_A x_{n,A} \right\|.$

c) $\left\| \sum_{A \subset \mathbb{N}_{2n}} \alpha_A x_{n,A} \right\| = \left\| \sum_{A \subset \mathbb{N}_{2n}} \alpha_A x_{n+k,A} \right\|$ *for every* $k \in \mathbb{N}$.

d) *There is a* $\xi \in \mathbb{C}^{2^{n+1}}$ *such that*

$$\|\xi\| = 1\,, \quad \xi e_{\mathbb{N}_{2^n}} = 0\,,$$

$$\Big\|\Big(\sum_{A\subset\mathbb{N}_{2n}} \alpha_A x_{n+1,A}\Big)\xi\Big\| = \Big\|\sum_{A\subset\mathbb{N}_{2n}} \alpha_A x_{n,A}\Big\|\,.$$

a) Take $A \subset \mathbb{N}_{2n-2}$. Then by Proposition 7.2.3.1 c), for $p \in \{0,1\}$,

$$x_{n,A} = \begin{bmatrix} x_{n-1,A} & 0 \\ 0 & x_{n-1,A} \end{bmatrix},$$

$$x_{n,A\cup\{2n-p\}} = x_{n,A}x_{n,2n-p} =$$

$$= \begin{bmatrix} x_{n-1,A} & 0 \\ 0 & x_{n-1,A} \end{bmatrix} \begin{bmatrix} 0 & (-1)^{n-1}i^{1+p}\pi_{n-1} \\ i^{1-p}\pi_{n-1} & 0 \end{bmatrix} =$$

$$= \begin{bmatrix} 0 & (-1)^{n-1}i^{1+p}x_{n-1,A}\pi_{n-1} \\ i^{1-p}x_{n-1,A}\pi_{n-1} & 0 \end{bmatrix} =$$

$$= \varphi(A) \begin{bmatrix} 0 & (-1)^{n-1}i^{1+p}x_{n-1,\tilde{A}} \\ i^{1-p}x_{n-1,\tilde{A}} & 0 \end{bmatrix}.$$

By Proposition 7.2.3.1 a),

$$x_{n,A\cup\{2n-1,2n\}} = x_{n,A}x_{n,2n-1}x_{n,2n} =$$

$$= \begin{bmatrix} x_{n-1,A} & 0 \\ 0 & x_{n-1,A} \end{bmatrix} \begin{bmatrix} -i & 0 \\ 0 & i \end{bmatrix} = \begin{bmatrix} -ix_{n-1,A} & 0 \\ 0 & ix_{n-1,A} \end{bmatrix}.$$

Hence

$$\sum_{A\subset\mathbb{N}_{2n}} \alpha_A x_{n,A} =$$

$$= \begin{bmatrix} \sum\limits_{A\subset\mathbb{N}_{2n-2}} (\alpha_A - i\delta_A)x_{n-1,A} & (-1)^n \sum\limits_{A\subset\mathbb{N}_{2n-2}} \varphi(A)(\beta_A - i\gamma_A)x_{n-1,\tilde{A}} \\ \sum\limits_{A\subset\mathbb{N}_{2n-2}} \varphi(A)(\beta_A + i\gamma_A)x_{n-1,\tilde{A}} & \sum\limits_{A\subset\mathbb{N}_{2n-2}} (\alpha_A + i\delta_A)x_{n-1,A} \end{bmatrix}.$$

b) By a) and Theorem 5.6.6.1 a),

$$\Big\| \sum_{A\subset\mathbb{N}_{2n-2}} (\alpha_A \pm i\delta_A)x_{n-1,A} \Big\| \le \Big\| \sum_{A\subset\mathbb{N}_{2n}} \alpha_A x_{n,A} \Big\|,$$

$$\Big\| \sum_{A\subset\mathbb{N}_{2n-2}} \varphi(A)(\gamma_A \pm i\beta_A)x_{n-1,\tilde{A}} \Big\| \le$$

$$\le \Big\| \sum_{A\subset\mathbb{N}_{2n}} \alpha_A x_{n,A} \Big\|.$$

It follows that

$$\Big\| \sum_{A\subset\mathbb{N}_{2n-2}} \alpha_A x_{n-1,A} \Big\| =$$

$$= \Big\| \frac{1}{2} \sum_{A\subset\mathbb{N}_{2n-2}} (\alpha_A + i\delta_A)x_{n-1,A} + \frac{1}{2} \sum_{A\subset\mathbb{N}_{2n-2}} (\alpha_A - i\delta_A)x_{n-1,A} \Big\| \le \Big\| \sum_{A\subset\mathbb{N}_{2n}} \alpha_A x_{n,A} \Big\|.$$

The other three inequalities are proved similarly.

c) For any $j \in \mathbb{N}$,

$$\Big\| \sum_{A\subset\mathbb{N}_{2n}} \alpha_A x_{n+j,A} \Big\| = \left\| \begin{bmatrix} \sum\limits_{A\subset\mathbb{N}_{2n}} \alpha_A x_{n+j-1,A} & 0 \\ 0 & \sum\limits_{A\subset\mathbb{N}_{2n}} \alpha_A x_{n+j-1,A} \end{bmatrix} \right\| =$$

$$= \Big\| \sum_{A\subset\mathbb{N}_{2n}} \alpha_A x_{n+j-1,A} \Big\|.$$

The assertion follows by induction.

d) There is an $\eta \in \mathbb{C}^{2^n}$ such that $\|\eta\| = 1$ and

$$\Big\| \Big(\sum_{A\subset\mathbb{N}_{2n}} \alpha_A x_{n,A} \Big)\eta \Big\| = \Big\| \sum_{A\subset\mathbb{N}_{2n}} \alpha_A x_{n,A} \Big\|.$$

Put

$$\xi := \begin{bmatrix} 0 \\ \eta \end{bmatrix} \in \mathbb{C}^{2^{n+1}} .$$

Then

$$\|\xi\| = 1 , \quad \xi e_{\mathbb{N}_{2n}} = 0 ,$$

and

$$\Big\| \Big(\sum_{A \subset \mathbb{N}_{2n}} \alpha_A x_{n+1,A} \Big) \xi \Big\| = \left\| \begin{bmatrix} \sum\limits_{A \subset \mathbb{N}_{2n}} \alpha_A x_{n,A} & 0 \\ 0 & \sum\limits_{A \subset \mathbb{N}_{2n}} \alpha_A x_{n,A} \end{bmatrix} \begin{bmatrix} 0 \\ \eta \end{bmatrix} \right\| =$$

$$= \left\| \begin{bmatrix} 0 \\ \Big(\sum\limits_{A \subset \mathbb{N}_{2n}} \alpha_A x_{n,A} \Big) \eta \end{bmatrix} \right\| = \Big\| \Big(\sum_{A \subset \mathbb{N}_{2n}} \alpha_A x_{n,A} \Big) \eta \Big\| =$$

$$= \Big\| \sum_{A \subset \mathbb{N}_{2n}} \alpha_A x_{n,A} \Big\| . \qquad \blacksquare$$

Proposition 7.2.3.3 (7)

a) *If* $n, p \in \mathbb{N}$, $n < p$, $A \subset \mathbb{N}_{2n}$, *and* $\xi \in \ell^2$, *then*

$$\widetilde{x}_{n,A}\Big(\xi e_{\mathbb{N}_{2n}} \Big) = \widetilde{x}_{p,A}(\xi e_{\mathbb{N}_{2n}}) .$$

b) *For every* $n \in \mathbb{N}$ *and* $A \subset \mathbb{N}_{2n}$, *the sequence* $(\widetilde{x}_{n+k,A})_{k \in \mathbb{N}}$ *converges pointwise to an* $x_{\infty,A} \in \mathcal{L}(\ell^2)$. *Given* $j \in \mathbb{N}$, *put*

$$x_{\infty,j} := x_{\infty,\{j\}} .$$

c) *For all* $\xi \in \ell^2$ *and* $A, B \in \mathfrak{P}_f(\mathbb{N})$,

$$\lim_{\substack{n \to \infty \\ p \to \infty}} \widetilde{x}_{n,A} \widetilde{x}_{p,B} \xi = x_{\infty,A} x_{\infty,B} \xi .$$

d) *If* $n \in \mathbb{N}$, $(j_k)_{k \in \mathbb{N}_n}$ *is a strictly increasing family in* $\mathbb{N}$, *and*

$$A := \{ j_k | k \in \mathbb{N}_n \}$$

then

$$x_{\infty,A} = \prod_{k=1}^{n} x_{\infty,j_k} .$$

e) *For all* $j,k \in \mathbb{N}$,

$$x_{\infty,j}^* = -x_{\infty,j}\,, \quad \overline{x}_{\infty,j} = (-1)^{\frac{1}{2}(j-\sin^2\frac{\pi j}{2})} x_{\infty,j}\,,$$

$$x_{\infty,j}x_{\infty,k} + x_{\infty,k}x_{\infty,j} = -2\delta_{j,k}1\,.$$

a) For every $k \in \mathbb{N}_{p-n}$,

$$x_{n+k,A}\Big((\xi e_{\mathbb{N}_{2n}})|\mathbb{N}_{2(n+k)}\Big) =$$

$$= \begin{bmatrix} x_{n+k-1,A} & 0 \\ 0 & x_{n+k-1,A} \end{bmatrix} \begin{bmatrix} (\xi e_{\mathbb{N}_{2n}})|\mathbb{N}_{2(n+k-1)} \\ 0 \end{bmatrix} =$$

$$= \begin{bmatrix} x_{n+k-1,A}\Big((\xi e_{\mathbb{N}_{2n}})|\mathbb{N}_{2(n+k-1)}\Big) \\ 0 \end{bmatrix},$$

and so

$$\widetilde{x}_{n+k,A}(\xi e_{\mathbb{N}_{2n}}) = \widetilde{x_{n+k,A}\Big((\xi e_{\mathbb{N}_{2n}})|\mathbb{N}_{2(n+k)}\Big)} =$$

$$= \widetilde{x_{n+k-1,A}\Big((\xi e_{\mathbb{N}_{2n}})|\mathbb{N}_{2(n+k-1)}\Big)} = \widetilde{x}_{n+k-1}(\xi e_{\mathbb{N}_{2n}})\,.$$

Since k is arbitrary, we get

$$\widetilde{x}_{n,A}(\xi e_{\mathbb{N}_{2n}}) = \widetilde{x}_{p,A}(\xi e_{\mathbb{N}_{2n}})\,.$$

b) Take $\xi \in \ell^2$ and $p,q \in \mathbb{N}$, with $n \le p \le q$. By a) (and Proposition 7.2.3.1 g)),

$$\|\widetilde{x}_{q,A}\xi - \widetilde{x}_{p,A}\xi\| = \|\widetilde{x}_{q,A}(\xi e_{\mathbb{N}\setminus\mathbb{N}_{2p}}) - \widetilde{x}_{p,A}(\xi e_{\mathbb{N}\setminus\mathbb{N}_{2p}})\| \le$$

$$\le 2\|\xi e_{\mathbb{N}\setminus\mathbb{N}_{2p}}\| = 2\Big(\sum_{k=2p+1}^{\infty} |\xi_k|^2\Big)^{\frac{1}{2}}.$$

Hence $(\widetilde{x}_{n+k,A}\xi)_{k\in\mathbb{N}}$ converges. By Corollary 1.4.1.3, the pointwise limit of $(\widetilde{x}_{n+k,A})_{k\in\mathbb{N}}$ belongs to $\mathcal{L}(\ell^2)$.

c) By Proposition 7.2.3.1 g), for sufficiently large natural numbers n, p,

$$\|\widetilde{x}_{n,A}\widetilde{x}_{p,B}\xi - x_{\infty,A}x_{\infty,B}\xi\| \leq$$

$$\leq \|\widetilde{x}_{n,A}\widetilde{x}_{p,B}\xi - \widetilde{x}_{n,A}x_{\infty,B}\xi\| + \|\widetilde{x}_{n,A}x_{\infty,B}\xi - x_{\infty,A}x_{\infty,B}\xi\| \leq$$

$$\leq \|\widetilde{x}_{p,B}\xi - x_{\infty,B}\xi\| + \|\widetilde{x}_{n,A}x_{\infty,B}\xi - x_{\infty,A}x_{\infty,B}\xi\| .$$

Hence

$$\lim_{\substack{n\to\infty \\ p\to\infty}} \widetilde{x}_{n,A}\widetilde{x}_{p,B}\xi = x_{\infty,A}x_{\infty,B}\xi .$$

d) follows from c).

e) The relations

$$x_{\infty,j}x_{\infty,k} + x_{\infty,k}x_{\infty,j} = -2\delta_{jk}1 , \quad \overline{x}_{\infty,j} = (-1)^{\frac{1}{2}(j-\sin^2\frac{\pi j}{2})}x_{\infty,j}$$

follow immediately from c) and Proposition 7.2.3.1 a),b). Let $\xi, \eta \in \ell^2$. Then by Proposition 7.2.3.1 b),

$$\langle x_{\infty,j}\xi|\eta\rangle = \lim_{n\to\infty}\langle\widetilde{x}_{n,j}\xi|\eta\rangle = \lim_{n\to\infty}\langle\xi| - \widetilde{x}_{n,j}\eta\rangle =$$

$$= \langle\xi| - x_{\infty,j}\eta\rangle ,$$

so

$$x^*_{\infty,j} = -x_{\infty,j} . \qquad \blacksquare$$

Proposition 7.2.3.4 (7)

a) *There is a unique injective homomorphism of unital C^*-algebras*

$$u : \mathcal{C}\ell_c(\mathbb{N}) \longrightarrow \mathcal{L}(\ell^2)$$

such that

$$ue_j = x_{\infty,j}$$

for every $j \in \mathbb{N}$.

b) *$u(\mathcal{C}\ell_c(\mathbb{N}))$ is the unital C^*- -subalgebra of $\mathcal{L}(\ell^2)$ generated by $\{x_{\infty,j}|j \in \mathbb{N}\}$.*

c) *$u(\mathcal{C}\ell_c(\mathbb{N})) \cap \mathcal{K}(\ell^2) = \{0\}$.*

d) *If*

$$q : \mathcal{L}(\ell^2) \longrightarrow \mathcal{L}(\ell^2)/\mathcal{K}(\ell^2)$$

denotes the quotient map, then the restriction of q to $u(\mathcal{C}\ell_c(\mathbb{N}))$ is injective but not surjective.

a) follows from Proposition 7.2.3.3 d),e) and Proposition 7.2.1.5 b).

b) follows from a).

c) Take $x \in \mathcal{C}\ell_c(\mathbb{N})\backslash\{0\}$ and let $y \in \ell^2(\mathfrak{N})$ such that (with the notation of Definition 7.2.1.3) $x = \vec{y}$. By Proposition 7.1.1.3 $b_2 \Rightarrow b_1$, there is an $n_0 \in \mathbb{N}$ such that

$$\|\vec{y} - \vec{y}_{\mathbb{N}_{2n}}\| < \frac{1}{3}\|x\|$$

for all $n \in \mathbb{N}$, with $n \geq n_0$. Take $n \in \mathbb{N}$, with $n > n_0$. By a) (and Proposition 7.2.3.3 d)),

$$\Big\| \sum_{A \subset \mathbb{N}_{2n}} y(A) x_{\infty,A} \Big\| = \|\vec{y}_{\mathbb{N}_{2n}}\| \geq \|x\| - \|x - \vec{y}_{\mathbb{N}_{2n}}\| > \frac{2}{3}\|x\| .$$

By Proposition 7.2.3.2 d), there is a $\xi_n \in \mathbb{C}^{2^n}$ such that

$$\|\xi_n\| = 1 , \quad \xi_n e_{\mathbb{N}_{2^n-1}} = 0 ,$$

and

$$\Big\| \Big(\sum_{A \subset \mathbb{N}_{2n}} y(A) x_{n,A} \Big) \xi_n \Big\| = \Big\| \sum_{A \subset \mathbb{N}_{2(n-1)}} y(A) x_{n-1,A} \Big\| .$$

By Proposition 7.2.3.2 c),

$$\Big\| \sum_{A \subset \mathbb{N}_{2(n-1)}} y(A) x_{n-1,A} \Big\| = \Big\| \sum_{A \subset \mathbb{N}_{2(n-1)}} y(A) x_{\infty,A} \Big\| > \frac{2}{3}\|x\| ,$$

so that

$$\Big\| \Big(\sum_{A \subset \mathbb{N}_{2n}} y(A) x_{n,A} \Big) \xi_n \Big\| > \frac{2}{3}\|x\| .$$

We get (Proposition 7.2.3.3 a))

$$\frac{2}{3}\|x\| < \Big\| \Big(\sum_{A \subset \mathbb{N}_{2(n-1)}} y(A) \widetilde{x}_{n,A} \Big) \widetilde{\xi}_n \Big\| = \Big\| \Big(\sum_{A \subset \mathbb{N}_{2(n-1)}} y(A) x_{\infty,A} \Big) \widetilde{\xi}_n \Big\| \leq$$

$$\leq \|x\tilde{\xi}_n\| + \Big\|\Big(x - \sum_{A\subset\mathbb{N}_{2(n-1)}} y(A)x_{\infty,A}\Big)\tilde{\xi}_n\Big\| \leq \|x\tilde{\xi}_n\| + \frac{1}{3}\|x\|\,,$$

$$\|x\tilde{\xi}_n\| > \frac{1}{3}\|x\|\,.$$

Hence $(x\tilde{\xi}_n)_{n\in\mathbb{N}}$ does not converge to 0. Since $(\tilde{\xi}_n)_{n\in\mathbb{N}}$ converges weakly to 0, x is not compact (Theorem 5.5.5.1 a $\Rightarrow$ c).

d) By c), the restriction of q to $u(\mathcal{C}\ell_c(\mathbb{N}))$ is injective. Since $\mathcal{C}\ell_c(\mathbb{N})$ is separable (Proposition 7.1.2.3 h)) and $\mathcal{L}(\ell^2)/\mathcal{K}(\ell^2)$ is not separable (Proposition 5.5.2.12 e)), it follows that the restriction of q to $u(\mathcal{C}\ell_c(\mathbb{N}))$ is not surjective. ■

Proposition 7.2.3.5 (7) *Let u be the representation of $\mathcal{C}\ell_c(\mathbb{N})$ defined in Proposition 7.2.3.4 a) and put*

$$E := \{x \in u(\mathcal{C}\ell_c(\mathbb{N}))|\overline{x} = x\}\,.$$

a) *E is a real C^*-subalgebra of $u(\mathcal{C}\ell_c(\mathbb{N}))$ such that the map*

$$\overset{o}{E} \longrightarrow u(\mathcal{C}\ell_c(\mathbb{N}))\,, \quad (x,y) \longmapsto x + iy$$

is an isomorphism of complex C^*-algebras.

b) *E is isomorphic to $\mathcal{C}\ell_c^{\mathbb{R}}(\mathbb{N})$.*

a) is easy to see.

b) We denote

$$B := \{n \in \mathbb{N}|n \equiv 2 \pmod 4 \text{ or } n \equiv 3 \pmod 4\}$$

and denote by $\mathfrak{A}$ a partition of B such that A is finite and

$$\operatorname{Card} A \equiv 0 \pmod 4$$

for every $A \in \mathfrak{A}$. By Proposition 7.2.1.18, there is an isomorphism of C^*-algebras

$$v : \mathcal{C}\ell_c(\mathbb{N}) \longrightarrow u(\mathcal{C}\ell_c(\mathbb{N}))$$

such that

$$ve_n = \begin{cases} ix_{\infty,n}x_{\infty,A} & \text{if } n \in A \in \mathfrak{A} \\ x_{\infty,n} & \text{if } n \in \mathbb{N}\backslash B\,. \end{cases}$$

By Proposition 7.2.3.3 e),

$$\overline{ve}_n = ve_n$$

for every $n \in \mathbb{N}$. Hence

$$v(\mathcal{C}\ell_c^{\mathbb{R}}(\mathbb{N})) = E$$

and the map

$$\mathcal{C}\ell_c^{\mathbb{R}}(\mathbb{N}) \longrightarrow E\,, \quad x \longmapsto vx$$

is an isomorphism of real C^*-algebras. ■

Proposition 7.2.3.6 (7) *We define a map*

$$\psi : \mathbb{N} \longrightarrow \mathbb{N} \cup \{0\}$$

in the following way: if $n \in \mathbb{N}$ *and*

$$\sum_{j=1}^{p} 2^{k_j}$$

is the binary representation of $n-1$ *(where* $(k_j)_{j\in\mathbb{N}_p}$ *is a strictly increasing family in* $\mathbb{N} \cup \{0\}$ *), then we set* $\psi(n) := p$.

a) *The sequence* $\left(i^n \prod_{j=1}^{2n} x_{\infty,j}\right)_{n\in\mathbb{N}}$ *converges pointwise to the map*

$$\pi_\infty : \ell^2 \longrightarrow \ell^2\,, \quad \xi \longmapsto \left((-1)^{\psi(n)}\xi_n\right)_{n\in\mathbb{N}}.$$

b) $\pi_\infty^* = \overline{\pi}_\infty = \pi_\infty\,, \quad \pi_\infty^2 = 1\,.$

c) $j \in \mathbb{N} \Rightarrow x_{\infty,j}\pi_\infty = -\pi_\infty x_{\infty,j}\,.$

d) *If* u *denotes the map defined in Proposition 7.2.3.4 a) and* E *denotes the real (complex)* C^**-subalgebra of* $\mathcal{L}(\ell^2)$ *generated by*

$$u(\mathcal{C}\ell_c^{\mathbb{R}}(\mathbb{N})) \cup \{i\pi_\infty\} \quad (u(\mathcal{C}\ell_c^{\mathbb{C}}(\mathbb{N})) \cup \{\pi_\infty\})$$

then there is a unique isomorphism of C^**-algebras*

$$v : \mathcal{C}\ell_c(\mathbb{N}) \longrightarrow E$$

such that

$$ve_j = x_{\infty,j-1}$$

for all $j \in \mathbb{N}\backslash\{1\}$ *and*

$$ve_1 = i\pi_\infty\,.$$

a) By Proposition 7.2.3.1 a),

$$i\pi_n = \begin{bmatrix} \pi_{n-1} & 0 \\ 0 & -\pi_{n-1} \end{bmatrix}$$

for every $n \in \mathbb{N}$. We get by induction,

$$i^n \pi_n \xi = ((-1)^{\psi(k)} \xi_k)_{k \in \mathbb{N}_{2n}}$$

for every $n \in \mathbb{N}$ and $\xi \in \mathbb{C}^{2^n}$. The assertion follows from Proposition 7.2.3.3.

b) follows from a) and Proposition 7.2.3.3 e).

c) We have

$$x_{\infty,j} \left(i^n \prod_{k=1}^{2n} x_{\infty,k} \right) = - \left(i^n \prod_{k=1}^{2n} x_{\infty,k} \right) x_{\infty,j}$$

for every $n \in \mathbb{N}$, $j \leq 2n$ (Proposition 7.2.3.3 e)), so

$$x_{\infty,j} \pi_\infty = -\pi_\infty x_{\infty,j} .$$

d) follows from b),c), and Proposition 7.2.1.5 b). ■

Proposition 7.2.3.7 (7) *Let π_∞ be the operator defined in Proposition 7.2.3.6 a) and u the representation of $\mathcal{C}\ell_c(\mathbb{N})$ defined in Proposition 7.2.3.4 a).*

a) *The commutant of $u(\mathcal{C}\ell^+(\mathbb{N}))$ in $\mathcal{L}(\ell^2)$ is equal to*

$$\{\alpha 1 + \beta \pi_\infty | \alpha, \beta \in \mathbb{C}\} .$$

b) *A closed vector subspace H of ℓ^2 is $u(\mathcal{C}\ell^+(\mathbb{N}))$-invariant iff*

$$\pi_H \in \{0, 1, \frac{1}{2}(1 \pm \pi_\infty)\} .$$

c) *The commutant of $u(\mathcal{C}\ell_c(\mathbb{N}))$ in $\mathcal{L}(\ell^2)$ is equal to $\mathbb{C}1$.*

d) *$u(\mathcal{C}\ell_c(\mathbb{N}))$ acts irreducibly on ℓ^2.*

a) For $n \in \mathbb{N}$, let H_n be the vector subspace of ℓ^2 generated by $(e_j)_{j \in \mathbb{N}_{2n}}$ and set

$$z_n : H_n \longrightarrow H_n , \quad \xi \longmapsto (z\tilde{\xi}) | \mathbb{N}_{2n}$$

for every $z \in \mathcal{L}(\ell^2)$. Then for every $n \in \mathbb{N}$, $z \in \mathcal{L}(\ell^2)$, and $\xi, \eta \in H_n$,

$$\langle z_n\xi|\eta\rangle = \langle z\widetilde{\xi}|\widetilde{\eta}\rangle = \langle\widetilde{\xi}|z^*\widetilde{\eta}\rangle = \langle\xi|(z^*)_n\eta\rangle\,.$$

Thus

$$(z_n)^* = (z^*)_n\,.$$

Let z be an element of $\mathcal{L}(\ell^2)$ commuting with $\mathcal{C}\ell^+(\mathbb{N})$. Take $n \in \mathbb{N}$. Let

$$\mathfrak{A} := \{A \subset \mathbb{N}_{2n} | \operatorname{Card} A \equiv 0 \pmod 2\}\,,$$

and take $(\alpha_A)_{A\in\mathfrak{A}} \in \mathbb{C}^{\mathfrak{A}}$. Put

$$y := \sum_{A\in\mathfrak{A}} \alpha_A x_{\infty,A}\,.$$

By Proposition 7.2.1.9 a), $y \in u(\mathcal{C}\ell^+(\mathbb{N}))$. By Proposition 7.2.3.3 a),

$$y_n = \sum_{A\in\mathfrak{A}} \alpha_A x_{n,A}\,.$$

Then for all $\xi, \eta \in H_n$,

$$\langle y_n z_n\xi|\eta\rangle = \langle z_n\xi|(y_n)^*\eta\rangle = \langle z_n\xi|(y^*)_n\eta\rangle =$$

$$= \langle z\widetilde{\xi}|y^*\widetilde{\eta}\rangle = \langle yz\widetilde{\xi}|\widetilde{\eta}\rangle = \langle zy\widetilde{\xi}|\widetilde{\eta}\rangle =$$

$$= \langle y\widetilde{\xi}|z^*\widetilde{\eta}\rangle = \langle y_n\xi|(z^*)_n\eta\rangle = \langle y_n\xi|(z_n)^*\eta\rangle =$$

$$= \langle z_n y_n\xi|\eta\rangle\,,$$

so that

$$z_n y_n = y_n z_n\,.$$

By Proposition 7.2.1.9 a),d), there are $\alpha_n, \beta_n \in \mathbb{C}$ such that

$$z_n = \alpha_n 1 + \beta_n i^n \pi_n\,.$$

By Proposition 7.2.3.1 a),

$$z_{n+1} = \alpha_{n+1}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \beta_{n+1} i^n \begin{bmatrix} \pi_n & 0 \\ 0 & -\pi_n \end{bmatrix} =$$

$$= \begin{bmatrix} \alpha_{n+1}1 + \beta_{n+1}i^n\pi_n & 0 \\ 0 & \alpha_n 1 - \beta_{n+1}i^n\pi_n \end{bmatrix} .$$

Thus

$$\alpha_{n+1} = \alpha_n , \quad \beta_{n+1} = \beta_n .$$

We get

$$z_n\xi = \alpha_1\xi + \beta_1 i^n \pi_n \xi$$

for every $\xi \in \ell^2(\mathbb{N}_{2n})$.

Take $\xi \in \ell^2$. We set

$$\xi_n := \xi|\mathbb{N}_{2n}$$

for every $n \in \mathbb{N}$. Then

$$z\widetilde{\xi_n} = \widetilde{z_n\xi_n} = \alpha_1\widetilde{\xi_n} + \beta_1 i^n \widetilde{\pi_n\xi_n} ,$$

$$\left\| \widetilde{\pi_n\xi_n} - \left(\prod_{j=1}^{2n} x_{\infty,j} \right) \xi \right\| \leq \left\| \xi - \widetilde{\xi_n} \right\|$$

for every $n \in \mathbb{N}$, so that

$$z\xi = \lim_{n\to\infty} z\widetilde{\xi_n} = \alpha_1\xi + \beta_1 \lim_{n\to\infty} \left(i^n \prod_{j=1}^{2n} x_{\infty,j} \right) \xi =$$

$$= \alpha_1\xi + \beta_1\pi_\infty\xi .$$

Hence

$$z = \alpha_1 1 + \beta_1\pi_\infty .$$

Since π_∞ commutes with $u(\mathcal{C}\ell^+(\mathbb{N}))$ (Proposition 7.2.3.6 c)) it follows that

$$\{\alpha 1 + \beta\pi_\infty | \alpha, \beta \in \mathbb{C}\}$$

is the commutant of $u(\mathcal{C}\ell^+(\mathbb{N}))$.

b) By a) and Proposition 5.3.2.9, H is $u(\mathcal{C}\ell^+(\mathbb{N}))$–invariant iff there are $\alpha, \beta \in \mathbb{C}$ such that

$$\pi_H = \alpha 1 + \beta\pi_\infty .$$

By Proposition 7.2.3.6 b),

$$\alpha 1 + \beta\pi_\infty = (\alpha 1 + \beta\pi_\infty)^2 = (\alpha^2 + \beta^2)1 + 2\alpha\beta\pi_\infty ,$$

$$\alpha 1 + \beta\pi_\infty = (\alpha + \beta\pi_\infty)^* = \overline{\alpha} 1 + \overline{\beta}\pi_\infty ,$$

whence

$$\alpha = \overline{\alpha} = \alpha^2 + \beta^2 ,$$

$$\beta = \overline{\beta} = 2\alpha\beta .$$

We get $\alpha = \beta = 0$ or $\alpha = 1$ and $\beta = 0$ or $\alpha = \frac{1}{2}$ and $\beta = \pm\frac{1}{2}$. Hence H is $u(\mathcal{C}\ell^+(\mathbb{N}))$-invariant iff

$$\pi_H \in \{1, 0, \frac{1}{2}(1 \pm \pi_\infty)\} .$$

c) follows from b) and Proposition 7.2.3.6 c).

d) follows from c) and Proposition 5.3.2.9. ■

Corollary 7.2.3.8 (7) *Let u be the representation of $\mathcal{C}\ell_c(\mathbb{N})$ defined in Proposition 7.2.3.4 a). Put*

$$p := \frac{1}{2}(1 + \pi_\infty) ,$$

where π_∞ was defined in Proposition 7.2.3.6 a).

a) *$\mathcal{L}(\ell^2)$ is the von Neumann algebra on ℓ^2 generated by $u(\mathcal{C}\ell_c(\mathbb{N}))$.*

b) *$p \in \Pr\mathcal{L}(\ell^2)$ and $p\mathcal{L}(\ell^2)p + (1-p)\mathcal{L}(\ell^2)(1-p)$ is the von Neumann algebra on ℓ^2 generated by $u(\mathcal{C}\ell^+(\mathbb{N}))$.*

a) By Theorem 7.2.3.7 c),

$$u(\mathcal{C}\ell_c(\mathbb{N}))^c = \mathbb{C}1 ,$$

so that

$$u(\mathcal{C}\ell_c(\mathbb{N}))^{cc} = \mathcal{L}(\ell^2)$$

and the assertion follows from Corollary 6.3.5.6.

b) By Theorem 7.2.3.7 a),

$$u(\mathcal{C}\ell^+(\mathbb{N}))^c = \{\alpha p + \beta(1-p) | \alpha, \beta \in \mathbb{C}\} .$$

By Proposition 7.2.3.6 b), $p \in \Pr\mathcal{L}(\ell^2)$ and by Proposition 2.1.3.25,

$$u(\mathcal{C}\ell^+(\mathbb{N}))^{cc} = p\mathcal{L}(\ell^2)p + (1-p)\mathcal{L}(\ell^2)(1-p) .$$

The assertion now follows from Corollary 6.3.5.6. ■

Name Index

Alaoglu, L. 1.2.8.1
Arens, R.F. 1.5.2.10, 2.2.7.13
Arzelà, C. 1.1.2.16
Ascoli, G. 1.1.2.14, 1.1.2.16
Atkinson, F.V. 3.1.3.7, 3.1.3.11, 3.1.3.12, 3.1.3.21, 5.3.3.16
Autonne, L. 2.3.1.3
Banach, S. 1.1.1.2, 1.2.8.2, 1.3.1.2, 1.3.2, 1.3.3.1,
1.3.4.1, 1.3.4.10, 1.4.1.2, 1.4.2.3, 1.4.2.19
Beurling, A. 2.2.5.4
Bourbaki, N. 1.2.8.1
Branges, L. de 1.3.5.14
Calkin, J.W. 5.4.3.5
Carleman, T. 3.1.3.1, 6.1.4.1
Cauchy, A. 1.3.10.6
Choquet, G. 5.4.3.5
Clifford, W.K. 7.2.2.1
Dedekind, R. 1.7.2.1
Dieudonné, J. 1.2.8.2, 3.1.3.9
Dixmier, J. 4.4.4.4
Drewnowski, L. 4.3.2.13
Dworetzky, A. 1.1.6.14
Dye, H.A. 4.1.3.7
Eberlein, W.F. 1.3.7.15
Effros, E.G. 4.2.4.15
Enflo, P. 3.1.1.7
Ford, J.W.M. 2.4.2.4
Fourier, J.-B.-J. 2.4.6.2
Fréchet, M. 1.1.1.2, 1.1.2.13, 5.2.5.2
Fredholm, E. 3.1.6.23
Frobenius, G.F. 2.1.4.21
Fuglede, B. 4.1.4.1
Fukamiya, M. 4.2.1.1
Gelfand, I.M. 1.4.1.9, 2.2.5.4, 2.2.5.5, 2.4.1.2, 2.4.5.7, 4.1.1.1, 4.1.2.5
4.1.3.1, 4.2.6.6, 5.4.1.2, 5.4.2.5
Gohberg, I. 3.1.3.12

Goldstine, H.H. 1.3.6.8
Goodearl, K.R. 2.2.1.19, 4.1.1.1, 5.4.2.13
Gowers, W.T. 1.2.1.12
Gram, J.P. 5.5.1.18
Grothendieck, A. 1.6.1.1, 3.1.6.25, 4.2.8.13
Hahn, H. 1.1.1.2, 1.2.1.3, 1.3.3.1, 1.3.6.1, 1.3.6.3, 1.3.8.1, 1.4.1.4
Hamilton, W.R. 2.1.4.17
Hellinger, E. 5.2.5.4
Helly, E. 1.1.1.2, 1.3.3.13
Hilbert, D. 2.1.3.1
Hirschfeld, R.A. 2.2.5.6
Jacobson, N. 2.1.3.10
James, R.C. 1.3.8.1
Jordan, C. 5.1.1.6
Kadison, R.K. 4.3.3.20, 6.3.6.1
Kaplanski, I. 4.1.2.1, 4.2.6.5, 4.4.2.24, 5.6.1.1, 6.3.1.10
Kelley, J.L. 4.2.1.1
Kojima, ?? 1.2.3.11
Kolmogoroff, A. 1.1.1.2
Kottman, C.A. E 1.3.5
Krein, M.G. 1.3.1.10, 1.3.7.3
Laguerre, E.N. 2.2.3.5
Laurent, P.A. 1.3.10.8
Lax, P.D. 5.3.1.3
Le Page, C. 2.2.3.8, 2.2.4.3, 2.2.5.6
Lindenstrauss, J. 1.2.5.13
Liouville, J. 1.3.10.6, 6.2.2.5
Lomonsov, V.I. 3.1.5.10
Löwing, H. 5.1.1.1
Mackey, G.W. 1.3.7.2
Mazur, S. 2.2.5.5
Mercer, J. 6.1.9.4
Mihlin, S.G. 3.1.3.12
Milgram, A.N. 5.3.1.3
Milman, D.P. 1.3.1.10
Minkowski, H. 1.1.1.2, 1.1.3.4
Murray, F.J. 1.2.5.8
Nagumo, M. 2.2.1.1

Naimark, M.A. 4.1.1.1, 4.1.2.5, 4.1.3.1, 4.2.6.6, 5.4.1.2, 5.4.2.5, 5.5.1.24
Neumann, C. 2.2.3.5
Neumann, J. von 1.1.1.2, 3.1.3.1, 5.1.1.1, 5.1.1.6, 5.2.4, 6.1.2.1, 6.3.5.5
Nikodym, O. 4.3.2.14, 5.2.1.2
Noether, F. 3.1.3.1
Palmer, T.W. 4.1.1.1
Paschke, W.L. 5.6.1.1, 5.6.2.2, 5.6.2.6, 5.6.2.11, 5.6.3.3, 5.6.3.5
Pedersen, G.K. 4.2.4.16, 6.3.4.3
Peter, F. 2.2.1.15
Pettis, P.J. 1.3.8.4, 1.3.8.5
Phillips, R.S. 1.2.5.14, E 1.3.3, 2.1.4.9
Pierce, B. 2.1.1.1, 2.1.3.6
Plancherel, M. 5.5.4.1
Putnam, I.F. 4.1.4.1
Rellich, F. 5.1.1.1
Rickart, C.E. 4.1.1.20, 4.1.2.12
Rieffel, M.A. 5.6.1.1
Riesz, F. 1.1.1.2, 1.1.4.4, 1.2.1.1, 1.2.2.5, 2.2.5.1
3.1.1.1, 3.1.3.8, 3.1.3.17, 3.1.5.1, 5.2.1.2, 5.2.5.2, 5.3.3.20
Rogers, C.A. 1.1.6.14
Rosenberg, A. 6.3.6.12
Rosenblum, M. 4.1.4.1
Russo, B. 4.1.3.7
Sakai, S. 4.4, 4.4.1.1, 4.4.3.5
Schatten, R. 6.1.2.1
Schauder, J.P. 3.1.1.22
Schmidt, E 1.1.1.2, 5.5.1.18
Schur, I. 1.2.3.11, 1.2.3.12, 1.3.6.11, 7.1.1.7
Schwartz, L. 2.4.6.5, 2.4.6.8
Segal, I.E. 4.2.6.2, 4.2.6.5, 4.2.8.2, 5.4.1.2, 6.3.6.2
Shirali, S. 2.4.2.4
Sierpiński, W. 1.1.2.17
Šilow, G. 2.2.4.27
Šmulian, V. 1.3.7.3, 1.3.7.15
Steinhaus, H.D. 1.4.1.2
Stone, M.H. 1.3.4.10, 1.3.5.16, 2.3.3.12, 4.1.2.5
Størmer, E. 4.2.6.3, 7.2.1.9
Sturm, C. 6.2.2.5

Takeda, Z. 6.3.2.1
Takesaki, M. 6.3.2.2
Toeplitz, O. 1.2.3.4, 2.3.1.3, 5.2.5.4
Vaught, R.L. 4.2.1.1
Vigier, J.P. 6.3.1.4
Vitushkin, A.G. 2.4.3.7
Volterra, V. 2.2.4.22
Wedderburn, J.M. 6.3.6.5
Weierstrass, K. 1.3.5.16
Wene, G.P. 7.2.1.10
Weyl, H. 2.2.1.15, 6.1.3.26, 6.1.7.23
Wielandt, H. 2.2.5.8
Wiener, N. 2.4.5.7
Yood, B. 3.1.3.11, 3.1.3.12, 4.1.1.13
Żelazko, W. 2.2.5.6

Subject Index

NT means Notation and Terminology

$(\mathcal{A},\mathcal{B},\mathcal{C})$–multiplication 1.5.1.1
absolute value of a number 1.1.1.1
absolute value of a measure NT
absolutely convex 1.2.7.1
absolutely convex closed hull 1.2.7.6
absolutely convex hull 1.2.7.4
absolutely summable family 1.1.6.9
acts irreducibly 5.3.2.19
acts non–degenerately 5.3.2.19
additive group NT
adherence, point of NT
adherent point NT
adjoint 2.3.1.1, 5.6.1.8
adjoint differential operator 3.2.2.3
adjoint kernel 3.1.6.5
adjoint operator 5.3.1.4
adjoint sesquilinear form 2.3.3.1
adjoint sesquilinear map 2.3.3.1
adjointable 5.6.1.7
algebra 2.1.1.1
algebra, Calkin 3.1.1.13
algebra, complex 2.1.1.1
algebra, degenerate 2.1.1.1
algebra, division 2.1.2.1
algebra, Gelfand 2.4.1.1
algebra, Gelfand unital 2.4.2.1
algebra, involutive 2.3.1.3
algebra, involutive Gelfand 2.4.2.1
algebra, involutive unital Gelfand 2.4.2.1
algebra, normed 2.2.1.1
algebra, real 2.1.1.1
algebra, semi–simple 2.1.3.18
algebra, strongly symmetric 2.3.1.26

algebra, symmetric 2.3.1.26
algebra, unital 2.1.1.3
algebra, unital Gelfand 2.4.1.1
algebra homomorphism 2.1.1.6
algebra homomorphism associated to A 5.4.2.3
algebra homomorphism associated to x' 5.4.1.2
algebra homomorphism associated to $(x'_\iota)_{\iota\in I}$ 5.4.2.2
algebra homomorphism, unital 2.1.1.6
algebra isomorphism 2.1.1.6
algebra isomorphism, unital 2.1.1.6
algebraic dimension 1.1.2.18
algebraic dual 1.1.1.1
algebraic eigenspace 5.3.3.20
algebraic eigenvector 5.3.3.20
algebraic isomorphism, associated 1.2.4.6
algebraic multiplicity 5.3.3.20
algebras, isomorphism of involutive 2.3.1.3
analytic function 1.3.10.1
approximate unit 2.2.1.15
approximate unit of a C^*–algebra, canonical 4.2.8.2
Arens multiplication, left 1.5.2.10, 2.2.7.13
Arens mutliplication, right 1.5.2.10, 2.2.7.13
associated algebraic isomorphism 1.2.4.6
associated quadratic form 2.3.3.1
associated quadratic map 2.3.3.1
associated unital C^*–algebra 4.1.1.13
atom 4.3.2.20
atomic 4.3.2.20
atomic part 6.3.3.7
atomless 4.3.2.20
atomless part 6.3.3.7
Baire function 1.7.2.12
Baire set 1.7.2.12
ball, unit 1.1.1.2
Banach algebra 2.2.1.1
Banach algebra, involutive 2.3.2.1
Banach algebra, quasiunital 2.2.1.15
Banach algebra, unital 2.2.1.1

Banach categories, functor of 1.5.2.1
Banach categories, functor of unital 1.5.2.1
Banach category 1.5.1.1
Banach category, unital 1.5.1.1
Banach space 1.1.1.2
Banach space, complex 1.1.1.2
Banach space, involutive 2.3.2.1
Banach space, ordered 1.7.1.4
Banach space, real 1.1.1.2
Banach subalgebra generated by 2.2.1.9
Banach system 1.5.1.1
Banach system, bidual of a 1.5.1.9
Banach system, dual of a 1.5.1.9
Banach systems, isometric 1.5.2.1
band 1.7.2.1
basis, Fourier 5.6.3.11
basis, orthonormal 5.5.1.1
basis of $\ell^2(T)$, canonical orthonormal 5.5.3.1
Bergman kernel 5.2.5.9
Bessel's identity 5.5.1.7
Bessel's inequality 5.5.1.8
bicommutant 2.1.1.16
bidual of a Banach system 1.5.1.9
bidual of a normed space 1.3.6.1
bijective NT
bilinear map 1.2.9.1
binomial theorem 2.2.3.12
bitranspose 1.3.6.15
bound, lower 1.7.2.1
bound, upper 1.7.2.1
bounded map 1.1.1.2
bounded operator 1.2.1.3
bounded operator, lower 1.2.1.18
bounded sequence 1.1.1.2
bounded set 1.1.1.2
boundedness, principle of uniform 1.4.1.2
boundedness theorem, Nikodym's 4.3.2.14
C^*–algebra 4.1.1.1

C^*–algebra, canonical approximate unit of a 4.2.8.2
C^*–algebra, canonical order of a 4.2.1.2
C^*–algebra, complex 4.1.1.1
C^*–algebra, complex unital 4.1.1.1
C^*–algebra, Gelfand 4.1.1.1
C^*–algebra, purely real 4.1.1.8
C^*–algebra, real 4.1.1.1
C^*–algebra, real unital 4.1.1.1
C^*–algebra, simple 4.3.5.1
C^*–algebra, unital 4.1.1.1
C^*–algebra associated, unital 4.1.1.13
Calkin algebra 3.1.1.13
Calkin category 3.1.1.12
canonical approximate unit of a C^*–algebra 4.2.8.2
canonical involution of E^F 2.3.1.1
canonical metric of a normed space 1.1.1.2
canonical norm of a pre–Hilbert space 5.1.1.2
canonical norm of $\mathcal{L}(E,F)$ 1.2.1.9
canonical order of a C^*–algebra 4.2.1.2
canonical orthonormal basis of $\ell^2(T)$ 5.5.3.1
canonical projection of the tridual of E 1.3.6.19
canonical scalar product of $\mathbb{K}^n$ 5.1.2.4
cardinal number NT
cardinality, topological NT
carrier 4.3.3.1
carrier, left 4.3.3.1
carrier, right 4.3.3.1
carrier of a function NT
carrier of a Radon measure NT
category, Banach 1.5.1.1
C^*–direct product 4.1.1.6
C^*–direct sum 4.1.1.6
character 2.4.1.1
characteristic family 6.1.2.1
characteristic family of eigenvalues 6.1.1.1
characteristic function of a set 1.1.2.1
characteristic number, n–th 6.1.2.1
characteristic sequence 6.1.2.1

C^*–hull 4.1.1.22
class NT
Clifford algebra associated to ρ 7.2.1.2
Clifford algebra generated by T 7.2.1.2
Clifford algebra of degree p 7.2.2.1
Clifford C^*–algebra associated to ρ 7.2.1.3
Clifford C^*–algebra of T 7.2.1.3
Clifford W^*–algebra associated to ρ 7.2.1.3
Clifford W^*–algebra of T 7.2.1.3
closed graph theorem 1.4.2.19
closed involutive subalgebra generated by 2.3.2.14, 2.3.2.15
closed involutive unital subalgebra generated by 2.3.2.14, 2.3.2.15
closed subalgebra generated by 2.2.1.9
closed unital subalgebra generated by 2.2.1.9
closed vector subspace generated by 1.1.5.5
C^*–module, Hilbert right 5.6.1.4
codimension 1.2.4.1
codomain NT
cokernel of a linear map 1.2.4.5
commutant 2.1.1.16
commutative 2.1.1.1
commutative monoid E 2.1.1
compact, relatively 1.1.2.9
compact operator 3.1.1.1
compatible, simultaneously 1.5.1.1
compatible (left and right) multiplications 1.5.1.1
complement of a subspace 1.2.5.3
complemented subspace 1.2.5.3
complete, $\mathbb{C}$–order 4.3.2.3
complete, order 1.7.2.1
complete norm 1.1.1.2
complete normed space 1.1.1.2
complete ordered set 1.7.2.1
completion of a normed algebra 2.2.1.13
completion of a normed space 1.3.9.1
completion of a pre–Hilbert space 5.1.1.7
complex algebra 2.1.1.1
complex Banach space 1.1.1.2

complex C^*–algebra 4.1.1.1
complex C^*–algebra, unital 4.1.1.1
complex Hilbert space 5.1.1.2
complex normed algebra 2.2.1.1
complex normed space 1.1.1.2
complex pre–Hilbert space 5.1.1.1
complex unital C^*–algebra 4.1.1.1
complex universal representation 5.4.2.6
complex W^*–algebra 4.4.1.1
complexification of algebras 2.1.5.7
complexification of Banach algebras 2.2.1.19
complexification of Hilbert spaces 5.3.1.8
complexification of involutive algebras 2.3.1.40
complexification of involutive vector spaces 2.3.1.38
complexification of right C^*–modules 5.6.1.6
complexification of vector spaces 2.1.5.1
component of x on $\mathcal{M}_\iota$ 6.3.3.6
composition of functors 1.5.2.1
composition of maps NT
compression of a representation 5.4.2.8
cone 1.3.7.4
cone, sharp 1.3.7.4
conjugacy class 2.2.2.7
conjugate exponent of 1.2.2.1
conjugate exponents 1.2.2.1
conjugate exponents, weakly 1.2.2.1
conjugate involution 2.3.1.3
conjugate linear map 1.3.7.10
conjugate number 1.1.1.1
continuous, order 1.7.2.3
convergence, radius of 1.1.6.22
convex 1.2.7.1
convex, absolutely 1.2.7.1
convex closed hull 1.2.7.6
convex closed hull, absolutely 1.2.7.6
convex hull 1.2.7.4
convex hull, absolutely 1.2.7.4
convolution 2.2.2.7, 2.2.2.10

$\mathbb{C}$–order complete 4.3.2.3
$\mathbb{C}$–order σ–complete 4.3.2.3
C^*–subalgebra 4.1.1.1
C^*–subalgebra, unital 4.1.1.1
C^*–subalgebra generated by 4.1.1.1
C^*–subalgebra generated by, hereditary 4.3.4.1
C^*–subalgebra generated by, unital 4.1.1.1
cyclic element 5.4.1.1
cyclic representation 5.4.1.1
cyclic vector 5.3.2.19, 5.4.1.1
cyclic vector associated to x' 5.4.1.2
decomposition, Schatten 6.1.3.4
decomposition, spectral 4.3.2.19, 5.3.4.7
degenerate algebra 2.1.1.1
derivative 1.1.6.24
diagonalization of u 5.5.6.1
differentiable 1.1.6.24
differential operator, adjoint 3.2.2.3
differential operator, selfadjoint 3.2.2.3
dimension, algebraic 1.1.2.18
dimension, Hilbert 5.5.2.2
Dirac measure 1.2.7.14
direct integral of E with respect to μ 5.5.2.19
direct sum 1.2.5.3
directed, downward 1.1.6.1
directed, upward 1.1.6.1
disjoint family of sets 1.2.3.9
distance of a point from a set 1.1.4.1
division algebra 2.1.2.1
domain NT
downward directed 1.1.6.1
dual, algebraic 1.1.1.1
dual of a Banach system 1.5.1.9
dual of a normed space 1.2.1.3
dual space 1.3.1.11
E–algebra 2.2.7.1
E–algebra, involutive 2.3.6.1
E–algebra, involutive unital 2.3.6.1

E–algebra, unital 2.2.7.1
E–algebras, homomorphism of 2.2.7.1
E–algebras, homomorphism of involutive 2.3.6.1
E–algebras, homomorphism of involutive unital 2.3.6.1
E–algebras, homomorphism of unital 2.2.7.1
E–C^*–algebra 5.6.1.10
E–C^*–algebra, unital 5.6.1.10
E–C^*–algebras, isomorphism of 5.6.1.10
eigenspace 3.1.4.1
eigenspace, algebraic 5.3.3.20
eigenvalue 3.1.4.1
eigenvalues, characteristic family of 6.1.1.1
eigenvector 3.1.4.1
eigenvector, algebraic 5.3.3.20
E–module 2.2.7.1
E–module, Hilbert 5.6.1.4
E–module, Hilbert right 5.6.1.4
E–module, inner–product right 5.6.1.1
E–module, involutive 2.3.6.1
E–module, involutive unital 2.3.6.1
E–module, semi–inner–product right 5.6.1.1
E–module, weak semi–inner-product right 5.6.1.1
E–module, unital 2.2.7.1
E–module, unital Hilbert 5.6.1.4
E–module, von Neumann (right) 5.6.3.2
E–modules, homomorphism of 2.2.7.1
E–modules, homomorphism of involutive 2.3.6.1
empty word 7.2.1.1
equicontinuous 1.1.2.14
equivalence class NT
equivalence class of a point NT
equivalence of GNS–triples 5.4.1.2
equivalence of representations 5.4.1.1
equivalence relation NT
equivalent GNS–triples 5.4.1.2
equivalent norms 1.1.1.2
equivalent representations 5.4.1.1
essential spectrum 3.1.3.24

E–submodule 2.2.7.1
Euclidean norm 1.1.5.2
evaluation 1.2.1.8
evaluation functor 1.5.2.1
evaluation operator of a normed space 1.3.6.3
E–valued spectral measure 4.3.2.16
exact set 1.7.2.12
expansion, Fourier 5.5.1.15
exponential function 2.2.3.5
exponents, conjugate 1.2.2.1
exponents, weakly conjugate 1.2.2.1
extreme Fourier set 6.3.9.4
extreme point 1.2.7.9
face of a convex set 1.2.7.9
factorization of a linear map 1.2.4.6
faithful, order 4.2.2.18
faithful representation 5.4.1.1
family NT
family, absolutely summable 1.1.6.9
family, sum of a 1.1.6.2
family, summable 1.1.6.2
family of sets, disjoint 1.2.3.9
Fatou's Lemma 6.1.3.15
filter, lower section 1.1.6.1
filter, upper section 1.1.6.1
filter of cofinite subsets NT
finite–dimensional 1.1.2.18
$\mathcal{F}$–invariant 3.1.4.4
Fourier basis 5.6.3.11
Fourier expansion 5.5.1.15
Fourier integral 2.4.6.2
Fourier-Plancherel operator 5.5.4.1
Fourier set 5.6.3.11
Fourier set, extreme 6.3.9.4
Fourier transform 2.4.6.2
Fréchet–Riesz Theorem 5.2.5.2
Fredholm alternative 3.1.6.23
Fredholm operator 3.1.3.1

Fredholm operator, index of a 3.1.3.1
free ultrafilter NT
function NT
function, Baire 1.7.2.12
function, step NT
functional calculus 4.1.3
functor 1.5.2.1
functor, identity 1.5.2.1
functor, inclusion 1.5.2.16
functor, isometric 1.5.2.1
functor, quotient 1.5.2.17
functor, transpose of a 1.5.2.3
functor of (unital) Banach categories 1.5.2.1
functor of (unital) A–categories 1.5.2.1
functor of (left, right) A–modules 1.5.2.1
functors, composition of 1.5.2.1
Gelfand, Theorem of 2.2.5.4
Gelfand algebra 2.4.1.1
Gelfand algebra, involutive 2.4.2.1
Gelfand algebra, involutive unital 2.4.2.1
Gelfand algebra, spectrum of a 2.4.1.1
Gelfand algebra, unital 2.4.1.1
Gelfand C^*–algebra 4.1.1.1
Gelfand–Mazur, Theorem of 2.2.5.5
Gelfand transform 2.4.1.2
generators of $\mathcal{C}\ell(T)$ 7.2.1.2
GNS–construction 5.4.1.2
GNS–triple of E associated to x' 5.4.1.2
GNS–triples, equivalence of 5.4.1.2
GNS–triples, equivalent 5.4.1.2
Gram–Schmidt orthonormalization 5.5.1.18
graph NT, 1.4.2.18
Green function 3.2.1.2
group, additive NT
Hahn–Banach Theorem 1.3.3.1
hereditary 4.3.4.1
hereditary C^*–subalgebra generated by 4.3.4.1
Hermitian sesquilinear map 2.3.3.3

Hilbert dimension 5.5.2.2
Hilbert E–module 5.6.1.4
Hilbert E–module, unital 5.6.1.4
Hilbert right C^*–module 5.6.1.4
Hilbert right E–module 5.6.1.4
Hilbert right E–modules, isomorphic 5.6.1.7
Hilbert-Schmidt operator 6.1.4.1
Hilbert space 5.1.1.2
Hilbert space, complex 5.1.1.2
Hilbert space, complexification 5.3.1.8
Hilbert space, involutive 5.5.7.1
Hilbert space, real 5.1.1.2
Hilbert space associated to A 5.4.2.3
Hilbert space associated to x' 5.4.1.2
Hilbert space associated to $(x'_\iota)_{\iota\in I}$ 5.4.2.2
Hilbert space of square summable sequences 5.1.2.3
Hilbert sum of a family of Hilbert spaces 5.1.3.1
Hilbert sum of a family of representations 5.4.2.1
Hölder inequality 1.2.2.5, 6.1.3.21, 6.1.5.11
homomogeneous W^*–algebra 5.6.7.8
homomorphism of C^*–algebras 4.1.1.20
homomorphism of E–algebras 2.2.7.1
homomorphism of E–modules 2.2.7.1
homomorphism of involutive E–algebras 2.3.6.1
homomorphism of involutive E–modules 2.3.6.1
homomorphism of involutive unital E–algebras 2.3.6.1
homomorphism of unital E–algebras 2.2.7.1
hyperstonian space 1.7.2.12
ideal 2.1.1.1
ideal, left 2.1.1.1
ideal, maximal proper 2.1.1.1
ideal, maximal proper left 2.1.1.1
ideal, maximal proper right 2.1.1.1
ideal, proper 2.1.1.1
ideal, proper left 2.1.1.1
ideal, proper right 2.1.1.1
ideal, regular maximal proper 2.1.3.17
ideal, regular maximal proper left 2.1.3.17

ideal, regular maximal proper right 2.1.3.17
ideal, right 2.1.1.1
ideal generated by 2.1.1.2
idempotent 2.1.3.6
identical representation 5.5.1.23
identity functor 1.5.2.1
identity map NT
identity operator 1.2.1.3
iff NT
image of a linear map 1.2.4.5
imaginary part 1.1.1.1, 2.3.1.22
inclusion functor 1.5.2.16
inclusion map NT
index of a Fredholm operator 3.1.3.1
index of U 3.1.3.21
induced norm 1.1.1.2
infimum 1.7.2.1
infinite–dimensional 1.1.2.18
infinite matrix 1.2.3.1
initial segment of $\mathbb{N}$ 6.1.2.1
injective NT
inner multiplication 1.5.1.1
inner–product 5.6.1.1
inner–product right E–module 5.6.1.1
integral of E with respect to μ, direct 5.5.2.19
interior point NT
invariant vector subspace 3.1.4.4
inverse of a bijective map NT
inverse of a morphism 1.5.1.6
inverse of an element in a unital algebra 2.1.2.4
inverse operators, principle of 1.4.2.4
invertible 1.5.1.5, 2.1.2.1
invertible, left 1.5.1.5
invertible, right 1.5.1.5
involution 2.3.1.1
involution, conjugate 2.3.1.3
involution of E^F, canonical 2.3.1.1
involutive algebra 2.3.1.3

involutive algebra, complexification of an 2.3.1.40
involutive algebra, strongly symmetric 2.3.1.26
involutive algebra, symmetric 2.3.1.26
involutive algebras, isomorphism of 2.3.1.3
involutive Banach algebra 2.3.2.1
involutive Banach space 2.3.2.1
involutive Banach unital algebra associated to 2.3.2.9
involutive E–algebra 2.3.6.1
involutive E–module 2.3.6.1
involutive Gelfand algebra 2.4.2.1
involutive Hilbert space 5.5.7.1
involutive map 2.3.1.1
involutive normed algebra 2.3.2.1
involutive normed space 2.3.2.1
involutive normed unital algebra associated to 2.3.2.9
involutive set 2.3.1.1
involutive space 2.3.1.1
involutive subalgebra generated by 2.3.1.18
involutive unital algebra associated to 2.3.1.9
involutive unital E–algebra 2.3.6.1
involutive unital E–module 2.3.6.1
involutive unital Gelfand algebra 2.4.2.1
involutive unital subalgebra generated by 2.3.1.18
involutive vector space 2.3.1.3
involutive vector spaces, isomorphism of 2.3.1.3
involutive vector subspace generated by 2.3.1.18
irreducible representation 5.4.1.1
irreducibly, acts 5.3.2.19
isometric Banach systems 1.5.2.1
isometric functor 1.5.2.1
isometric normed algebras 2.2.1.1
isometric normed spaces 1.2.1.12
isometric normed unital algebras 2.2.1.1
isometry of Hilbert spaces, partial 5.3.2.25
isometry of normed algebras 2.2.1.1
isometry of W^*–algebras 4.4.4.5
isometry of normed spaces 1.2.1.12
isometry of normed unital algebras 2.2.1.1

isomorphic algebras 2.1.1.6
isomorphic Hilbert right E–modules 5.6.1.7
isomorphic normed algebras 2.2.1.1
isomorphic normed spaces 1.2.1.12
isomorphic normed unital algebras 2.2.1.1
isomorphic unital algebras 2.1.1.6
isomorphism, algebra 2.1.1.6
isomorphism associated to a linear map, algebraic 1.2.4.6
isomorphism of E–C^*–algebras 5.6.1.10
isomorphism of involutive algebras 2.3.1.3
isomorphism of involutive vector spaces 2.3.1.3
isomorphism of normed algebras 2.2.1.1
isomorphism of normed spaces 1.2.1.12
isomorphism of normed unital algebras 2.2.1.1
kernel, Bergman 5.2.5.9
kernel of a linear map 1.2.4.5
Kronecker's symbol 1.2.2.6
lattice 1.7.2.1
lattice, vector 1.7.2.1
Laurent series 1.3.10.8, 1.3.10.9
Lax–Milgram Theorem 5.3.1.3
Lebesgue's Dominated Convergence Theorem 6.1.3.20
left Arens multiplication 1.5.2.10, 2.2.7.13
left carrier 4.3.3.1
left ideal 2.1.1.1
left ideal, maximal proper 2.1.1.1
left ideal, proper 2.1.1.1
left ideal, regular maximal proper 2.1.3.17
left ideal generated by 2.1.1.2
left invertible 1.5.1.5
left multiplication 1.5.1.1, 5.6.1.4
left shift 1.2.2.9, E 1.2.11
left (unital) A–module 1.5.1.10
linear form 1.1.1.1
linear form, positive 1.7.1.9
linear map, conjugate 1.3.7.10
locally finite 7.1.1.2
lower bound 1.7.2.1

lower bounded operator 1.2.1.18
lower section filter 1.1.6.1
L^2–distributions, in the sense of 3.2.2.3
map NT
map, bilinear 1.2.9.1
map, bounded 1.1.1.2
map, conjugate linear 1.3.7.10
map, identity NT
map, inclusion NT
map, inverse of a bijective NT
map, involutive 2.3.1.1
map, nuclear 1.6.1.1
map, quotient 1.2.4.1
maps, composition of NT
matrix, infinite 1.2.3.1
maximal proper ideal 2.1.1.1
maximal proper ideal, regular 2.1.3.17
maximal proper left ideal 2.1.1.1
maximal proper left ideal, regular 2.1.3.17
maximal proper right ideal 2.1.1.1
maximal proper right ideal, regular 2.1.3.17
mean ergodic theorem 5.2.4.3
measure, Dirac 1.2.7.14
measure, E–valued spectral 4.3.2.19
measure, Radon NT
measure of x, spectral 4.3.2.19
measure space, σ–finite 3.1.6.14
metric of a normed space, canonical 1.1.1.2
module 2.2.7.1
module, involutive 2.3.6.1
module, involutive unital 2.3.6.1
module, unital 2.2.7.1
modules, homomorphism of 2.2.7.1
modules, homomorphism of involutive 2.3.6.1
modulo NT
modulus 4.2.5.1, 4.4.3.5
monoid E 2.1.1
monoid, commutative E 2.1.1

morphism 1.5.1.1
morphism, inverse of a 1.5.1.6
multipliable sequence 2.2.4.33
multiplication 2.1.1.1
multiplication, $(\mathcal{A},\mathcal{B},\mathcal{C})$– 1.5.1.1
multiplication, compatible (left and right) 1.5.1.1
multiplication, inner 1.5.1.1
multiplication, left 1.5.1.1, 5.6.1.4
multiplication, left (right) Arens 1.5.2.10, 2.2.7.13
multiplication, right 1.5.1.1, 5.6.1.1
multiplication operator 2.2.2.22
multiplication operators, von Neumann algebra of 6.3.7.2
multiplicity 3.1.4.1
multiplicity, algebraic 5.3.3.20
negative 1.7.1.1
negative part 4.2.2.9, 4.2.8.13
Nikodym's boundedness theorem 4.3.2.14
nilpotent 2.1.1.1
non–degenerate representation 5.4.1.1
non–degenerately, acts 5.3.2.19
norm 1.1.1.2
norm, complete 1.1.1.2
norm, Euclidean 1.1.5.2
norm, induced 1.1.1.2
norm, quotient 1.2.4.2
norm, supremum 1.1.2.2, 1.1.5.2
norm of an operator 1.2.1.3
norm of a pre–Hilbert space, canonical 5.1.1.2
norm of $\mathcal{L}(E,F)$, canonical 1.2.1.9
norm topology 1.1.1.2
normal 2.3.1.3
normed algebra 2.2.1.1
normed algebra, completion of a 2.2.1.13
normed algebra, complex 2.2.1.1
normed algebra, involutive 2.3.2.1
normed algebra, quasiunital 2.2.1.15
normed algebra, real 2.2.1.1
normed algebras, isometric 2.2.1.1

normed algebras, isometry of 2.2.1.1
normed algebras, isomorphic 2.2.1.1
normed algebras, isomorphism of 2.2.1.1
normed space 1.1.1.2
normed space, bidual of a 1.3.6.1
normed space, complete 1.1.1.2
normed space, completion of a 1.3.9.1
normed space, complex 1.1.1.2
normed space, involutive 2.3.2.1
normed space, ordered 1.7.1.4
normed space, real 1.1.1.2
normed spaces, isometric 1.2.1.12
normed spaces, isometry of 1.2.1.12
normed spaces, isomorphic 1.2.1.12
normed spaces, isomorphism of 1.2.1.12
normed unital algebra 2.2.1.1
normed unital algebras, isometric 2.2.1.1
normed unital algebras, isometry of 2.2.1.1
normed unital algebras, isomorphic 2.2.1.1
normed unital algebras, isomorphism of 2.2.1.1
norms, equivalent 1.1.1.2
n–th characteristic number 6.1.2.1
nuclear map 1.6.1.1
number, cardinal NT
number, ordinal NT
object of a Banach system 1.5.1.1
onto NT
open mapping principle 1.4.2.3
operator 1.2.1.3
operator, adjoint 5.3.1.4
operator, adjoint differential 3.2.2.3
operator, bounded 1.2.1.3
operator, compact 3.1.1.1
operator, Fourier–Plancherel 5.5.4.1
operator, Fredholm 3.1.3.1
operator, identity 1.2.1.3
operator, index of a Fredholm 3.1.3.1
operator, lower bounded 1.2.1.18

operator, multiplication 2.2.2.22
operator, order of an 3.1.3.18
operator, selfadjoint differential 3.2.2.3
operators, principle of inverse 1.4.2.4
order complete 1.7.2.1
order continuous 1.7.2.3
order faithful 4.2.2.18
order of a pole 1.3.10.9
order relation of a C^*–algebra, canonical 4.2.1.2
order summable 1.7.2.10
order σ–complete 1.7.2.1
order σ–continuous 1.7.2.3
order σ–faithful 4.2.2.18
ordered Banach space 1.7.1.4
ordered normed space 1.7.1.4
ordered set, complete 1.7.2.1
ordered set, totally NT
ordered set, σ–complete 1.7.2.1
ordered vector space 1.7.1.1
ordinal number NT
orthogonal 5.2.2.1
orthogonal projection 4.1.2.18, 5.2.3.2
orthogonal set of A 5.2.2.1
orthogonal sets 5.2.2.1
orthogonal vectors 5.2.2.1
orthonormal basis 5.5.1.1
orthonormal basis of $\ell^2(T)$, canonical 5.5.3.1
orthonormal family 5.5.1.1
orthonormal set 5.5.1.1
orthonormalization, Gram–Schmidt 5.5.1.18
parallelogram law 2.3.3.2
Parseval's Equation 5.5.1.15
partial isometry of Hilbert spaces 5.3.2.25
partition of a set NT
p–norm 1.1.2.5, 1.1.5.2, 6.1.2.1
point, adherent NT
point, extreme 1.2.7.9
point, interior NT

point of adherence NT
point spectrum 3.1.4.1
polar 1.3.5.1
polar representation 4.2.6.9, 4.4.3.1, 4.4.3.5
polarization identity 2.3.3.2
pole (of order) 1.3.10.9
positive 1.7.1.1, 2.3.3.3, 2.3.4.1
positive linear form 1.7.1.9, 2.3.4.1
positive part 4.2.2.9, 4.2.8.13
power series 1.1.6.22
precompact 1.1.2.9
predual of a Banach space 1.3.1.11
predual of a W^*–algebra 4.4.1.1, 4.4.4.4
pre–Hilbert space 5.1.1.1
pre–Hilbert space, canonical norm of a 5.1.1.2
pre–Hilbert space, completion of a 5.1.1.7
pre–Hilbert space, complex 5.1.1.1
pre–Hilbert space, real 5.1.1.1
prepolar 1.3.5.1
pretranspose of an operator 1.3.4.9, 4.4.4.8
principal part 1.3.10.8, 1.3.10.9
principle of inverse operators 1.4.2.4
principle of open mapping 1.4.2.3
principle of uniform boundedness 1.4.1.2
product 2.1.1.1, 7.2.1.1
product, C^*–direct 4.1.1.6
product of a family of sets NT
product of a sequence 2.2.4.33
product, scalar 5.1.1.1
product associated to f, scalar 5.1.2.9
projection 1.2.5.7
projection, orthogonal 4.1.2.18, 5.2.3.2
projection of the tridual of E, canonical 1.3.6.19
proper ideal 2.1.1.1
proper ideal, maximal 2.1.1.1
proper ideal, regular maximal 2.1.3.17
proper left ideal 2.1.1.1
proper left ideal, maximal 2.1.1.1

proper left ideal, reguar maximal 2.1.3.17
proper right ideal 2.1.1.1
proper right ideal, maximal 2.1.1.1
proper right ideal, regular maximal 2.1.3.17
pure state 2.3.5.1
pure state space 2.3.5.1
purely real C^*–algebra 4.1.1.8
Pythagoras' Theorem 5.2.2.3
quadratic form, associated 2.3.3.1
quadratic map, associated 2.3.3.1
quasinilpotent 2.2.4.20
quasiunital 2.2.1.15
quaternion 2.1.4.17
quotient functor 1.5.2.17
quotient map NT, 1.2.4.1
quotient norm 1.2.4.2
quotient space 1.2.4.2
quotient A–category 1.5.2.17
quotient A–module 1.5.2.17
Raabe's ratio test 2.2.3.11
radical 2.1.3.18
radius of convergence 1.1.6.22
Radon measure NT
Radon–Nikodym Theorem 4.4.3.15
range of values NT
real algebra 2.1.1.1
real Banach space 1.1.1.2
real C^*–algebra 4.1.1.1
real C^*–algebra, purely 4.1.1.8
real C^*–algebra, unital 4.1.1.1
real Hilbert space 5.1.1.2
real normed space 1.1.1.2
real part 1.1.1.1, 2.3.1.3
real pre–Hilbert space 5.1.1.1
real W^*–algebra 4.4.1.1
reduces u 5.2.3.11
reflexive 1.3.8.1
regular maximal proper ideal 2.1.3.17

regular maximal proper left ideal 2.1.3.17
regular maximal proper right ideal 2.1.3.17
relatively compact 1.1.2.9
representation 5.4.1.1
representation, associated to A 5.4.2.3
representation, associated to x' 5.4.1.2
representation, associated to $(x_\iota)_{\iota\in I}$ 5.4.2.2
representation, complex universal 5.4.2.6
representation, compression of a 5.4.2.8
representation, cyclic 5.4.1.1
representation, faithful 5.4.1.1
representation, identical 5.5.1.23
representation, irreducible 5.4.1.1
representation, non–degenerate 5.4.1.1
representation, unital 5.4.1.1
representation, universal 5.4.2.3
representation, 0– 5.4.1.1
representations, equivalence of 5.4.1.1
representations, equivalent 5.4.1.1
representations, Hilbert sum of 5.4.2.1
residue 1.3.10.8, 1.3.10.9
resolvent 2.1.3.1
resolvent equation 2.1.3.9
Riesz, theorem of 2.2.5.1
right Arens multiplication 1.5.2.10, 2.2.7.13
right carrier 4.3.3.1
right C^*–module, Hilbert 5.6.1.4
right E–module, Hilbert 5.6.1.4
right E–module, inner–product 5.6.1.1
right E–module, semi–inner–product 5.6.1.1
right E–module, von Neumann 5.6.3.2
right E–module, weak semi–inner–product 5.6.1.1
right ideal 2.1.1.1
right ideal, maximal proper 2.1.1.1
right ideal, proper 2.1.1.1
right ideal, regular maximal proper 2.1.3.17
right ideal generated by 2.1.1.2
right invertible 1.5.1.5

right multiplication 1.5.1.1, 5.6.1.1
right shift 1.2.2.9, E1.2.11
right (unital) A–module, 1.5.1.10
right W^*–module, von Neumann 5.6.3.2
scalar 1.1.1.1
scalar product 5.1.1.1
scalar product associated to f 5.1.2.9
scalar product of $\mathbb{K}^n$, canonical 5.1.2.4
Schatten decomposition 6.1.3.4
Schur function 7.1.1.7
Schur function associated to ρ 7.2.1.2
Schwartz space of rapidly decreasing $\mathcal{C}^\infty$–functions 2.4.6.5
Schwarz inequality 2.3.3.9, 5.1.1.2
section filter, lower 1.1.6.1
section filter, upper 1.1.6.1
selfadjoint 2.3.1.1
selfadjoint differential operator 3.2.2.3
self–dual 5.6.2.2
self–normal 2.3.1.3
semi–inner–product right E–module 5.6.1.1
semi–inner–product right E–module, weak 5.6.1.1
seminorm 1.1.1.2
semi–simple algebra 2.1.3.18
separating vector 5.3.2.19, 5.4.4.1
sequence NT
series, Laurent 1.3.10.8, 1.3.10.9
series, power 1.1.6.22
sesquilinear form 2.3.3.1
sesquilinear form, adjoint 2.3.3.1
sesquilinear map 2.3.3.1
sesquilinear map, adjoint 2.3.3.1
sesquilinear map, Hermitian 2.3.3.3
set, Baire 1.7.2.12
set, bounded 1.1.1.2
set, complete ordered 1.7.2.1
set, exact 1.7.2.12
set, partition of a NT
set, totally ordered NT

set, μ–null NT
set, σ–complete ordered 1.7.2.1
sharp cone 1.3.7.4
shift, left 1.2.2.9
shift, right 1.2.2.9
simple C^*–algebra 4.3.5.1
simultaneously compatible 1.5.1.1
space, Banach 1.1.1.2
space, bidual of a normed 1.3.6.1
space, complete normed 1.1.1.2
space, completion of a normed 1.3.9.1
space, complex Banach 1.1.1.2
space, complex Hilbert 5.1.1.2
space, complex normed 1.1.1.2
space, complex pre–Hilbert 5.1.1.1
space, dual 1.3.1.11
space, Hilbert 5.1.1.2
space, hyperstonian 1.7.2.12
space, involutive 2.3.1.1
space, involutive Banach 2.3.2.1
space, involutive normed 2.3.2.1
space, involutive vector 2.3.1.3
space, normed 1.1.1.2
space, ordered Banach 1.7.1.4
space, ordered normed 1.7.1.4
space, ordered vector 1.7.1.1
space, pre–Hilbert 5.1.1.1
space, pure state 2.3.5.1
space, quotient 1.2.4.2
space, real Banach 1.1.1.2
space, real Hilbert 5.1.1.2
space, real normed 1.1.1.2
space, real pre–Hilbert 5.1.1.1
space, state 2.3.5.1
space, Stone 1.7.2.12
space, subspace of a normed 1.1.1.2
space, vector 1.1.1.1
space, σ–Stone 1.7.2.12

space of square summable sequences, Hilbert 5.1.2.3
spaces, isometric normed 1.2.1.12
spaces, isometry of normed 1.2.1.12
spaces, isomorphic normed 1.2.1.12
spaces, isomorphism of involutive vector 2.3.1.3
spaces, isomorphism of normed 1.2.1.12
spectral decomposition 4.3.2.19, 5.3.4.7
spectral measure, E–valued 4.3.2.16
spectral measure of x 4.3.2.19
spectral radius 2.1.3.1
spectrum, essential 3.1.3.24
spectrum, point 3.1.4.1
spectrum of an element 2.1.3.1
spectrum of a Gelfand algebra 2.4.1.1
square summable sequences, Hilbert space of 5.1.2.3
state 2.3.5.1
state, pure 2.3.5.1
state space 2.3.5.1
state space, pure 2.3.5.1
step function NT
Stone space 1.7.2.12
strong topology 6.3
strongly symmetric involutive algebra 2.3.1.26
subalgebra 2.1.1.1
subalgebra, unital 2.1.1.3
subalgebra generated by 2.1.1.4
subalgebra generated by, involutive 2.3.1.18
subspace, complemented 1.2.5.3
subspace generated by, closed vector 1.1.5.5
subspace of a normed space 1.1.1.2
sum, C^*–direct 4.1.1.6
sum, direct 1.2.5.3
sum of a family 1.1.6.2
sum of a family in F_A 5.6.3.6
sum of representations, Hilbert 5.4.2.1
summable, absolutely 1.1.6.9
summable, order 1.7.2.10
summable family 1.1.6.2

summable in F_A 5.6.3.6
support of a function NT
support of a Radon measure NT
support of x 6.3.3.4
supremum 1.7.2.1
supremum norm 1.1.2.2, 1.1.5.2
surjective NT
symbol, Kronecker's 1.2.2.6
symmetric involutive algebra 2.3.1.26
symmetric involutive algebra, strongly 2.3.1.26
theorem, mean ergodic 5.2.4.3
Theorem of Alaoglu–Bourbaki 1.2.8.1
Theorem of Banach 1.3.1.2
Theorem of Banach–Steinhaus 1.4.1.2
Theorem of closed graph 1.4.2.19
Theorem of Fréchet–Riesz 5.2.5.2
Theorem of Gelfand 2.2.5.4
Theorem of Gelfand–Mazur 2.2.5.5
Theorem of Hahn–Banach 1.3.3.1
Theorem of Laurent 1.3.10.8
Theorem of Lax–Milgram 5.3.1.3
Theorem of Liouville 1.3.10.6
Theorem of Krein–Milman 1.3.1.10
Theorem of Krein–Šmulian 1.3.7.3
Theorem of Minkowski 1.1.3.4
Theorem of Murray 1.2.5.8
Theorem of Pythagoras 5.2.2.3
Theorem of Radon–Nikodym 4.4.3.15
Theorem of Riesz 2.2.5.1
Theorem of Weierstrass–Stone 1.3.5.16
topological cardinality NT
topological zero–divisor 2.2.4.24
topology, norm 1.1.1.2
topology, weak 1.3.6.9
totally ordered set NT
trace 6.1.5.1
trace operator 6.1.5.1
transpose kernel 3.1.6.5

transpose of a functor 1.5.2.3
transpose of an operator 1.3.4.1
transpose unital category of $\mathcal{L}$ 1.5.2.2
transposition functor of $\mathcal{L}$ 1.5.2.2
triangle inequality 1.1.1.2
tridual of a Banach system 1.5.1.9
tridual of a normed space 1.3.6.1
type I W^*–algebra 5.6.7.11
u–invariant 3.1.4.4
ultrafilter, free NT
ultraweak operator topology 6.3
uniform boundedness, principle of 1.4.1.2
unit 1.5.1.1, 1.5.1.4, 2.1.1.1
unit, approximate 2.2.1.15
unit ball 1.1.1.2
unit of an inner multiplication 1.5.1.1
unital algebra 2.1.1.3
unital algebra, normed 2.2.1.1
unital algebra associated to 2.1.1.8
unital algebra associated to, involutive 2.3.1.9
unital algebra homomorphism 2.1.1.6
unital algebra isomorphism 2.1.1.6
unital algebras, isometric normed 2.2.1.1
unital algebras, isometry of normed 2.2.1.1
unital algebras, isomorphic 2.1.1.6
unital algebras, isomorphic normed 2.2.1.1
unital algebras, isomorphism of normed 2.2.1.1
unital Banach algebra 2.2.1.1
unital Banach algebra associated to 2.2.1.4
unital Banach algebras, isomorphism of 2.2.1.1
unital Banach category 1.5.1.1
unital Banach subalgebra generated by 2.2.1.9
unital C^*–algebra 4.1.1.1
unital C^*–algebra, complex 4.1.1.1
unital C^*–algebra, real 4.1.1.1
unital C^*–algebra, associated to a C^*–algebra 4.1.1.13
unital C^*–subalgebra 4.1.1.1
unital C^*–subalgebra generated by 4.1.1.1

unital E–algebra 2.2.7.1
unital E–algebra, involutive 2.3.6.1
unital E–C^*–algebra 5.6.1.10
unital E–module 2.2.7.1
unital E–module, involutive 2.3.6.1
unital Gelfand algebra 2.4.1.1
unital Gelfand algebra, involutive 2.4.2.1
unital Hilbert E–module 5.6.1.4
unital involutive algebra associated to 2.3.1.9
unital involutive Banach algebra associated to 2.3.2.9
unital involutive normed algebra associated to 2.3.2.9
unital left Λ–module 1.5.1.10
unital normed algebra associated to 2.2.1.4
unital normed algebras, isomorphism of 2.2.1.1
unital representation 5.4.1.1
unital right Λ–module 1.5.1.10
unital subalgebra 2.1.1.3
unital subalgebra generated by 2.1.1.4
unital subalgebra generated by, involutive 2.3.1.18
unital von Neumann algebra 6.3.4.1
unital W^*–subalgebra 4.4.4.5
unital W^*–subalgebra generated by 4.4.4.5
unital Λ–category 1.5.1.14
unital Λ–module 1.5.1.12
unital (Λ, Δ)–module 1.5.1.12
unitary 2.3.1.3
universal representation 5.4.2.3
universal representation, complex 5.4.2.6
upper bound 1.7.2.1
upper section filter 1.1.6.1
upward directed 1.1.6.1
vector, cyclic 5.3.2.19, 5.4.1.1
vector, separating 5.3.2.19, 5.4.1.1
vector associated to x', cyclic 5.4.1.2
vector lattice 1.7.2.1
vector space 1.1.1.1
vector space, involutive 2.3.1.3
vector spaces, isomorphisms of involutive 2.3.1.3

Volterra integral equation 2.2.4.22
von Neumann algebra 6.3.4.1
von Neumann algebra, unital 6.3.4.1
von Neumann algebra of multiplication operators 6.3.7.2
von Neumann (right) E–module 5.6.3.2
von Neumann (right) W^*–module 5.6.3.2
W^*–algebra 4.4.1.1
W^*–algebra, complex 4.4.1.1
W^*–algebra, homogeneous 5.6.7.8
W^*–algebra, predual of a 4.4.1.1, 4.4.4.4
W^*–algebra, real 4.4.1.1
W^*–algebra, type I 5.6.7.11
W^*–algebras, isometry of 4.4.4.5
W^*–homomorphism 4.4.4.5
W^*–module, von Neumann (right) 5.6.3.2
W^*–subalgebra 4.4.4.5
W^*–subalgebra, unital 4.4.4.5
W^*–subalgebra generated by 4.4.4.5
W^*–subalgebra generated by, unital 4.4.4.5
weak operator topology 6.3
weak semi–inner–product right E–module 5.6.1.1
weak topology 1.3.6.9
weakly conjugate exponents 1.2.2.1
word 7.2.1.1
word, empty 7.2.1.1
zero–divisor 2.1.1.1
zero–divisor, topological 2.2.4.24
Λ–categories, functor of (unital) 1.5.2.1
Λ–category 1.5.1.14
Λ–category, quotient 1.5.2.17
Λ–category, unital 1.5.1.14
Λ–module 1.5.1.12
Λ–module, left (right) 1.5.1.10
Λ–module, quotient 1.5.2.17
Λ–module, unital 1.5.1.12
Λ–module, unital left (right) 1.5.1.10
Λ–modules, functor of left (right) 1.5.2.1
Λ–subcategory 1.5.2.16

Λ–submodule 1.5.2.16
(Λ, Δ)–module 1.5.1.12
(Λ, Δ)–module, unital 1.5.1.12
μ–null set NT
σ–complete, $\mathbb{C}$–order 4.3.2.3
σ–complete order 1.7.2.1
σ–complete ordered set 1.7.2.1
σ–continuous, order 1.7.2.3
σ–faithful, order 4.2.2.18
σ–finite measure space 3.1.6.14
σ–Stone space 1.7.2.12
0–representation 5.4.1.1

Symbol Index

NT means Notation and Terminology

$|a|$ 4.4.3.5
a^* 2.3.1.30
A^* 2.3.1.1
$A^\perp$ 5.2.2.1
$\overline{A}$ NT
$\vec{A}$ 7.1.1.7
$\mathring{A}$ NT
$\mathfrak{A}, A^\circ$ 1.3.5.1
A^c, A^{cc}, A^{ccc} 2.1.1.16
$\mathcal{A}', \mathcal{A}'', \mathcal{A}'''$ 1.5.1.9
$A_{A'}$ 1.3.6.9
aa', $a'a$ 2.2.7.8
ab 2.1.4.23, 7.2.1.1
ax 2.2.7.23
$a'x''$ 2.2.7.11
$a''x'$ 1.5.2.8
$\widetilde{(a, \xi)}$ 5.6.3.2
$\widetilde{(a, \xi, \eta)}$ 5.6.3.2
$\mathcal{A}/\mathcal{B}$ 1.5.2.17
$A + B$ 1.2.4.1
$A \backslash B$ NT
$A \triangle B$ NT
$A \times B$ NT
$A + z$ 1.2.4.1
$\mathcal{B}$ 1.1.2.4
$\mathbb{C}$ NT
c 1.1.2.3, 2.1.4.3
c_0 1.1.2.3, 2.1.4.3
$c(T)$ 1.1.2.3, 2.1.4.3
$c_0(T)$ 1.1.2.3, 2.1.4.3
$\mathcal{C}(T)$ 1.1.2.4, 2.1.4.4
$\mathcal{C}(T, E)$ 1.1.2.8

$\mathcal{C}_0(T)$ 1.2.2.10, 2.1.4.4
Card NT
$\mathcal{C}\ell_p^{\mathbb{K}}$ 7.2.2.1
$\mathcal{C}\ell_{p,q}^{\mathbb{K}}$ 7.2.2.1
$\mathcal{C}\ell_{p,q}^{\mathbb{K},+}$ 7.2.2.1
$\mathcal{C}\ell^{\mathbb{K}}(T)$ 7.2.1.2
$\mathcal{C}\ell_c^{\mathbb{K}}(T)$ 7.2.1.3
$\mathcal{C}\ell_w^{\mathbb{K}}(T)$ 7.2.1.3
$\mathcal{C}\ell^{\mathbb{K}}(\rho)$ 7.2.1.2
$\mathcal{C}\ell^{\mathbb{K},+}(\rho)$ 7.2.1.9
$\mathcal{C}\ell_c^{\mathbb{K}}(\rho)$ 7.2.1.3
$\mathcal{C}\ell_w^{\mathbb{K}}(\rho)$ 7.2.1.3
Coker 1.2.4.5
d_A 1.1.4.1
Det NT
Dim 1.1.2.18
$\mathcal{D}(k,\mu,\nu)$ 3.1.6.15
$\mathcal{D}_0(k,\nu)$ 3.1.6.1
E' 1.2.1.3
E'' 1.3.6.1
E''' 1.3.6.1
$\overset{\circ}{E}$ 2.1.5.1, 2.1.5.7, 2.3.1.38, 2.3.1.40, 5.3.1.8
e_A 1.1.2.1
e_t 1.1.2.1
e_A^T 1.1.2.1
e_t^T 1.1.2.1
e^x 2.2.3.5
$E_a(u)$ 3.1.3.18
$E_b(u)$ 3.1.3.18
$E_{m,n}$ 2.1.4.23, 2.3.1.30
$E_{n,n}$ 2.1.4.24, 2.3.1.31, 5.6.6.1
E^T 1.1.2.1
$E^{(T)}$ 1.1.2.1
E^π 1.7.2.3
E^σ 1.7.2.3
$\ddot{E}$ 4.4.4.4
E_+ 1.7.1.1, 4.2.1.1
$E^\#$ 1.1.1.2

$E_+^{\#}$ 1.7.1.4
$E''_{\dashv}$, $E''_{\vdash}$ 2.2.7.15
$\mathfrak{E}_n$ 6.1.2.1
$E(x)$ 2.3.2.15
$E(x,1)$ 2.3.2.15
$E \overset{u}{\to} F$ 6.1.7.1
$E \underset{p}{\overset{u'}{\to}} F$ 6.1.7.1
$E \overset{x}{\to} F$ 1.5.1.1
$E \underset{A}{\overset{x}{\to}} F$ 1.5.1.1
E/F 1.2.4.1, 2.1.1.13, 2.3.1.42
$\dot{F}$ 4.3.1.7
$F(E)$ 5.5.7.10
f' 1.1.6.24
$\widetilde{f}$ 2.3.3.1, 7.1.1.7
$\mathcal{F}_A$ 1.2.6.1
$\mathcal{F}(E)$ 3.1.3.1
$\mathcal{F}(E,F)$ 3.1.3.1
$\mathfrak{F}_I$ 1.1.6.1
$f|S$ NT
$f(a,\cdot)$ NT
$f(\cdot,b)$ NT
$f(A)$ NT
$f(x)$ NT, 4.1.3.1, 4.1.3.2, 4.3.2.5, 6.3.3.1, 6.3.3.4
f^{-1} NT
$\overset{-1}{f}(B)$ NT
$\overset{-1}{f}(y)$ NT
$f : X \to Y$ NT
$f : X \to Y,\ x \mapsto T(x)$ NT
$F[s,t]$ NT
$F[t]$ NT
$F \oplus G$ 1.2.5.3
$\{f = g\}$ NT
$\{f \neq g\}$ NT
$\{f > \alpha\}$ NT
$g \circ f$ NT, 1.5.2.1
$\mathfrak{G}_T$ 7.1.1.2

$\mathbb{H}$ 2.1.4.17, 2.3.1.46, 4.1.1.31
$\overset{\circ}{H}$ 5.6.1.6
$\widehat{H}$ 5.6.2.2
$\ddot{H}$ 5.6.3.2
$\dddot{H}$ 5.6.3.2
$\mathfrak{H}_A$ 1.7.2.3
im 1.1.1.1, 2.3.1.22
Im 1.2.4.5
Ind u 3.1.3.1
Ind U 3.1.3.21
j_E 1.3.6.3, 1.5.2.1
j_{EF} 1.5.2.1
$\mathbb{K}$ 1.1.1.1
$\mathbb{K}1$ 2.1.1.3
$\mathbb{K}[\cdot], \mathbb{K}[\cdot,\cdot]$ 1.1.1.1
$\mathcal{K}(E)$ 3.1.1.1
$\mathcal{K}(E,F)$ 3.1.1.1
$\mathcal{K}_E(G,H)$ 5.6.5.3
$\mathcal{K}_E(H)$ 5.6.5.3
kx 3.1.6.1, 3.1.6.15
k' 3.1.6.5
k^* 3.1.6.5
$k^\nu x$ 3.1.6.1, 3.1.6.15
$\overset{\cap}{k}$ 1.2.3.1
$\overset{\cup}{k}$ 1.2.3.1
Ker 1.2.4.5
$\mathcal{L}$ 1.2.1.3, 1.5.1.1, 2.1.4.6
$\mathcal{L}_\Omega$ 1.5.1.1
$\mathcal{L}_f$ 1.2.1.3
$\mathcal{L}^1$ 1.6.1.1, 1.6.1.3
$\mathcal{L}^1_\Omega$ 1.6.1.13
$\mathcal{L}^p(E)$ 6.1.2.1
$\mathcal{L}^0(E)$ 6.1.2.1
$\mathcal{L}^p(E,F)$ 6.1.2.1
$\mathcal{L}^0(E,F)$ 6.1.2.1
$\mathcal{L}_E(G,H)$ 5.6.1.7
$\widehat{\mathcal{L}}_E(G,H)$ 5.6.2.2

$\mathcal{L}_E(H)$ 5.6.1.7
$\mathcal{L}_E^p(G,H)$ 6.3.9.12
$\mathcal{L}_E^p(H)$ 6.3.9.12
$\mathcal{L}(H)_H$ 6.3
$\mathcal{L}(H)_{\mathcal{L}_f(H)}$ 6.3
$\mathcal{L}(H)_{\mathcal{L}^1(H)}$ 6.3
ℓ^p 1.1.2.5
$\ell^2(I,F)$ 5.6.4.2
$\ell^2(T)$ 5.5.7.1
$\ell^p(T)$ 1.1.2.5
ℓ^0 1.1.2.3
$\ell^0(T)$ 1.1.2.3
ℓ^∞ 1.1.2.2, 2.1.4.3
$\ell^\infty(T)$ 1.1.2.2, 2.1.4.3
$\ell^{p,q}(S,T)$ 1.2.3.2
$\ell_0^{\infty,q}(S,T)$ 1.2.3.2
log 2.2.3.9, 4.2.4.4
$\mathcal{M}_b$ 1.1.2.26
$\mathcal{M}_{F,F}$ 5.6.4.20
$\mathcal{M}_{F,G}$ 5.6.4.19
$\mathbb{N}$ NT
$\mathbb{N}_n$ 1.1.3.3
$N_{x'}$ 2.3.4.1
No 2.3.1.3
$\mathcal{N}_{F,F}$ 5.6.4.16
$\mathcal{N}_{F,G}$ 5.6.4.15, 5.6.4.19
$N(u)$ 6.1.2.1
Pr 4.1.2.18
$\mathfrak{P}$ 1.1.2.1
$\mathfrak{P}_f$ 1.1.2.1
$\mathbb{Q}$ NT
$\mathbb{R}$ NT
$\overline{\mathbb{R}}$ NT
re 1.1.1.1, 2.3.1.3
Re 2.3.1.1
Re $E^\#$ 2.3.2.1
$r(x)$, $r_E(x)$ 2.1.3.1
Sn 2.3.1.3

$\mathcal{S}(x)$ 6.3.3.4
$\mathcal{S}^{\mathbb{K}}(f)$ 7.1.2.3
$\mathcal{S}_c^{\mathbb{K}}(f)$ 7.1.1.7
$\mathcal{S}_w^{\mathbb{K}}(f)$ 7.1.1.7
$\mathcal{S}(\mathbb{R}^n)$ 2.4.6.5
$\mathfrak{S}_T$ 7.1.1.2
Supp f NT
Supp μ NT
tr 6.1.5.1
$\mathbf{T}$ 2.4.4.1
u' 1.3.4.1
u'^* 6.1.7.1
u'' 1.3.6.15
u^* 2.3.1.1, 3.2.2.3, 5.3.1.4, 5.6.1.8, 5.6.4.12, 5.6.4.14
$\widetilde{u}$ 6.1.6.1
$\overset{p}{\widetilde{u}}$ 6.1.6.1
$\overset{\circ}{u}$ 2.1.5.11, 2.3.1.41
$\overrightarrow{u}$ 6.3.9.15
$\|u\|_p$ 6.1.6.1
$\|u\|_0$ 6.1.6.1
$u'w$ 6.1.7.1
Un 2.3.1.3
$U_\alpha(t)$ 1.1.1.2
$U_\alpha^T(t)$ 1.1.1.2
vu 5.6.4.12, 5.6.4.14
vu' 6.1.7.1
$X/\sim$ NT
x^n 2.1.1.1
x^α 4.2.4.1, 4.2.4.4
x^{-n} 2.1.2.5
x^0 2.1.1.3
x^{-1} 1.5.1.6, 2.1.2.4
x^* 2.3.1.1, 7.1.1.7
$\overline{x}$ 5.5.7.1
$\overrightarrow{x}$ 7.1.1.7
$\widehat{x}$, $\overset{E}{\widehat{x}}$ 2.4.1.1
$\overset{\vee}{x}$ 2.3.5.1

$|x|$ 4.2.5.1
x^+, x^- 4.2.2.9
x'^+, x'^- 4.2.8.13
x_S 7.2.1.3
$(x_\iota)_{\iota\in I}$ NT
xa 2.2.7.23
$x_{\infty,A}$ 7.2.2.3
$x_{\infty,j}$ 7.2.2.3
$x'a''$ 1.5.2.8
$x''a'$ 2.2.7.11
xx', $x'x$ 1.5.2.5
$x\xi$ 5.6.1.4
$\langle x, x'\rangle$, $\langle x', x\rangle$ 1.2.1.3
$\langle\cdot|x\rangle y$ 5.3.2.12
xy 7.2.1.3
$x * y$ 2.2.2.7, 7.1.1.7
$x'' \dashv y''$, $x'' \vdash y''$ 1.5.2.10, 2.2.7.13
$x \otimes y$ 3.1.1.25
$\overbrace{(x', \xi, \eta)}$ 5.6.3.1
$\{x \mid P(x)\}$ NT
$\{x \in X \mid P(x)\}$ NT
$\langle\cdot, x'\rangle y$ 1.3.3.3
$\mathbb{Z}$ NT
$z + A$ 1.2.4.1
$\overline{\alpha}$ 1.1.1.1
$|\alpha|$ 1.1.1.1
αA 1.2.4.1
$\binom{\alpha}{n}$ 2.2.3.10
$]\alpha, \beta[$, $]\alpha, \beta]$, $[\alpha, \beta[$, $[\alpha, \beta]$ NT
Δ NT
δ_{st} 1.2.2.6
$\delta(s, t)$ 1.2.2.6
δ_t 1.2.7.14
$\theta(u)$ 6.1.2.1
$\theta_n(u)$ 6.1.2.1
$|\mu|$ NT
$\mu * \nu$ 2.2.2.10

$\bar{\xi}$ 5.5.7.1
ξx 5.6.1.1
$\xi\langle\cdot|\eta\rangle$ 5.6.5.1
π_A 5.2.1.2

$\prod_{t\in E} E_t$ 2.1.4.1, 2.3.1.4
$\prod_{\iota\in I} X_\iota$ NT
$\prod_{n\in A} x_n$, $\prod_{n\in\mathbb{N}} x_n$ 2.2.4.33
ρ_E 6.1.8.1

$\rho_{E,F}$ 6.1.8.1
$\sigma(E)$, $\sigma_0(E)$ 2.4.1.1
$\sigma(T)$ 2.4.4.1
$\sigma(x)$, $\sigma_E(x)$ 2.1.3.1
$\sigma^e(u)$ 3.1.3.24
$\sigma_p(u)$ 3.1.4.1

$\sum_{t\in T} x(t)$ 1.1.2.1
$\sum_{n=p}^{q} x_n$ 1.1.6.2
$\sum_{\iota\in I} x_\iota$ 1.1.6.2
$\sum_{\iota\in I}\langle\cdot, x'_\iota\rangle y_\iota$ 1.3.3.3
$\sum_{n=0}^{\infty} \alpha^n x_n$ 1.1.6.22
$\sum_{\iota\in I}^{A} \xi_\iota$ 5.6.3.6
$\sum_{\iota\in I}^{\leq} x_\iota$ 1.7.2.10
$\tau(E)$, $\tau_0(E)$ 2.3.5.1
$\omega(\xi)$ 6.3.9.10
1 1.2.1.3, 2.1.1.3, 7.2.1.1
1_E 1.2.1.3, 1.5.1.5
$\frac{1}{\alpha - x}$ 2.2.3.5
$[1+x]^\alpha$ 2.2.3.13
0^0 2.1.1.3
$+$ 1.2.4.1

$\times$ NT
$\setminus$ NT
$\langle \cdot, \cdot \rangle$ 1.2.1.3
$\langle \cdot | \cdot \rangle$ 5.1.1.1, 5.6.1.1
$\{ \cdot | \cdot \}$ NT
$\{ \cdot = \cdot \}, \{ \cdot \neq \cdot \}, \{ \cdot > \cdot \}$ NT
$\equiv (\mathrm{mod}\, p)$ NT
$\dashv, \vdash$ 1.5.2.10, 2.2.7.13
$\perp$ 5.2.2.1
$\vee, \wedge$ 1.7.2.1
$\| \cdot \|$ 1.1.1.2, 1.2.1.3, 5.1.1.1, 5.6.1.4
$\| \cdot \|_p$ 1.1.2.5
$\| \cdot \|_1$ 1.6.1.1
$\| \cdot \|_0$ 1.1.2.3
$\| \cdot \|_\infty$ 1.1.2.2
$\forall, \exists, \exists!$ NT
$\circ$ NT, 1.5.2.1
$\overset{w}{\textcircled{1}}$ 5.1.3.1, 5.1.3.3, 5.6.4.1
$\overset{w}{\textcircled{1}}$ 5.6.4.6
$\oplus$ 1.2.5.3
$[\cdot, \cdot],]\cdot, \cdot[, [\cdot, \cdot[,]\cdot, \cdot]$ NT
$\int x d\mu$ 4.3.2.17